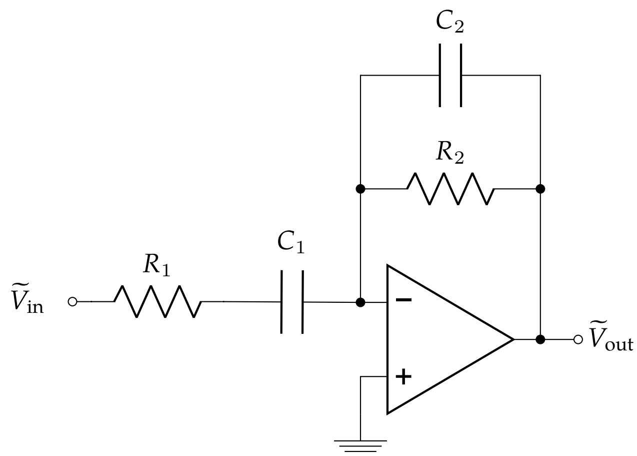
1 Differentiator Circuit

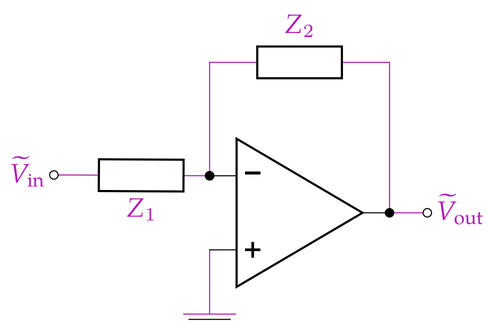
Consider the following circuit



1. What is the transfer function $H(j\omega)$?

Answer

This is a non-inverting amplifier with impedances $Z_1 = R_1 + \frac{1}{j\omega C_1}$ and $Z_2 = R_2 \parallel \frac{1}{j\omega C_2}$



Therefore, the transfer function will be $H(j\omega) = -\frac{z_2}{z_1}$.

$$H(j\omega) = -\frac{Z_2}{Z_1} = -\frac{j\omega R_2 C_1}{(1 + j\omega R_2 C_2)(1 + j\omega R_1 C_1)}$$

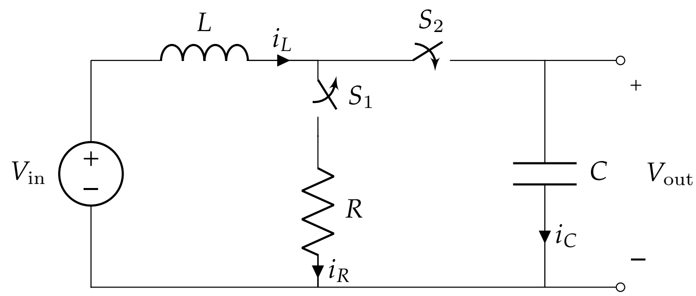
2 Parallel RLC

Consider the circuit shown below.

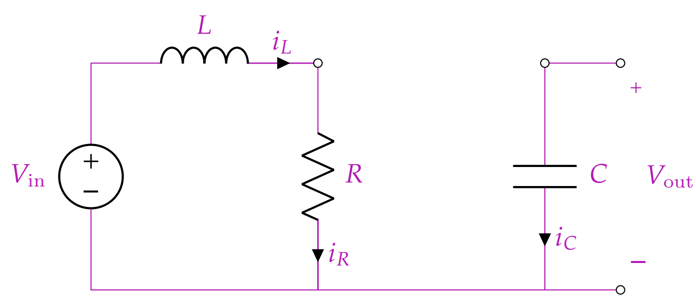
1. Right after the switches change state (i.e., at $t = 0$), what is the value of i_L?

Answer

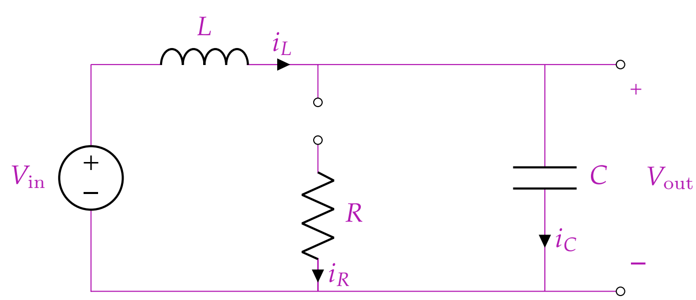
$V_L = L\frac{di_L}{dt} = 0, i_c = C\frac{dV_{\text{out}}}{dt} = 0 \implies V_{\text{out}} = V_{\text{in}}, i_L = i_R$, so $i_L(0^-) = \frac{V_{\text{in}}}{R}$.
Right after switches change state, the inductor current cannot change instantaneously (since this could require infinite voltage across it), so $i_L(0^+) = \frac{V_{\text{in}}}{R}$.



At $t < 0$, S_1 is on (short-circuited), and S_2 is off (open-circuited).
At $t \geq 0$, S_1 is off (open-circuited), and S_2 is on (short-circuited).



$t < 0, t < 0.$



At $t < 0$, S_1 is on (short-circuited), and S_2 is off (open-circuited).
At $t \geq 0$, S_1 is off (open-circuited), and S_2 is on (short-circuited).

2. Choosing the state variables as $\vec{x}(t) = \begin{bmatrix} V_{\text{out}}(t) \\ i_L(t) \end{bmatrix}$, derive the $\mathbf{A}$ matrix that captures the behavior of this circuit for $t \geq 0$ with the matrix differential equation $\frac{d\vec{x}(t)}{dt} = \mathbf{A}\vec{x}(t) + \vec{b}$, where $\vec{b}$ is a vector of constants.

Answer

$$
\begin{aligned}
V &= L\frac{di_L}{dt} \\
V_{\text{in}} - V_{\text{out}} &= L\frac{di_L}{dt} \\
\frac{di_L}{dt} &= -\frac{V_{\text{out}}}{L} + \frac{V_{\text{in}}}{L} \\
i_C = i_L &= C\frac{dV_{out}}{dt} \\
\frac{dV_{out}}{dt} &= \frac{1}{C}i_L \\
\frac{d}{dt}\begin{bmatrix} V_{\text{out}} \\ i_L \end{bmatrix} &= \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}\begin{bmatrix} V_{\text{out}} \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{V_{\text{in}}}{L} \end{bmatrix} \\
\mathbf{A} &= \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 0 \\ \frac{V_{\text{in}}}{L} \end{bmatrix}
\end{aligned}
$$

3. Assuming that $V_{\text{out}}(0) = 0\,\text{V}$, derive an expression for $V_{\text{out}}(t)$ for $t \geq 0$.

Answer

We perform a change of variables, so that $\hat{V}_{\text{out}}(t) = V_{\text{out}} - V_{\text{in}}$. The $\mathbf{A}$ matrix will not change, but the $\vec{b}$ vector goes to zero.

$$
\begin{aligned}
\det(\mathbf{A} - \lambda\mathbf{I}) &= 0 \\
\lambda^2 + \frac{1}{LC} &= 0 \\
\lambda &= \pm j\frac{1}{\sqrt{LC}}
\end{aligned}
$$

At this point, there are two possible solutions.
Method 1:

$$
\begin{aligned}
\hat{V}_{\text{out}}(t) &= k_1\cos\left(\frac{t}{\sqrt{LC}}\right) + k_2\sin\left(\frac{t}{\sqrt{LC}}\right) \\
V_{\text{out}}(t) &= k_1\cos\left(\frac{t}{\sqrt{LC}}\right) + k_2\sin\left(\frac{t}{\sqrt{LC}}\right) + V_{\text{in}} \\
V_{\text{out}}(0) = 0 &= k_1\cos(0) + k_2\sin(0) + V_{\text{in}} \\
k_1 &= -V_{\text{in}} \\
i_L(0) = \frac{V_{\text{in}}}{R} &= C\frac{dV_{\text{out}}}{dt}(0) \\
\frac{V_{\text{in}}}{RC} &= -\frac{k_1}{\sqrt{LC}}\sin(0) + \frac{k_2}{\sqrt{LC}}\cos(0) \\
k_2 &= \frac{\sqrt{L/C}}{R}V_{\text{in}} \\
V_{\text{out}} &= V_{\text{in}}\left(1 - \cos\left(\frac{t}{\sqrt{LC}}\right) + \frac{\sqrt{L/C}}{R}\sin\left(\frac{t}{\sqrt{LC}}\right)\right)
\end{aligned}
$$

Method 2:

$$
\begin{aligned}
\hat{V}_{\text{out}}(t) &= k_1 e^{j\frac{1}{\sqrt{LC}}t} + k_2 e^{-j\frac{1}{\sqrt{LC}}t} \\
V_{\text{out}}(t) &= k_1 e^{j\frac{1}{\sqrt{LC}}t} + k_2 e^{-j\frac{1}{\sqrt{LC}}t} + V_{\text{in}} \\
V_{\text{out}}(0) = 0 &= k_1 e^0 + k_2 e^0 + V_{\text{in}} \\
k_1 + k_2 &= -V_{\text{in}} \\
i_L(0) = \frac{V_{\text{in}}}{R} &= C\frac{dV_{\text{out}}}{dt}(0) \\
\frac{V_{\text{in}}}{RC} &= j\frac{k_1}{\sqrt{LC}} - j\frac{k_2}{\sqrt{LC}} \\
k_1 - k_2 &= -j\frac{\sqrt{L/C}}{R}V_{\text{in}} \\
k_1 = -V_{in}\left(\frac{1}{2} + j\frac{\sqrt{L/C}}{2R}\right), k_2 &= -V_{in}\left(\frac{1}{2} - j\frac{\sqrt{L/C}}{2R}\right) \\
V_{\text{out}}(t) &= V_{in}\left(-\left(\frac{1}{2} + j\frac{\sqrt{L/C}}{2R}\right)e^{j\frac{1}{\sqrt{LC}}t} - \left(\frac{1}{2} - j\frac{\sqrt{L/C}}{2R}\right)e^{-j\frac{1}{\sqrt{LC}}t} + 1\right)
\end{aligned}
$$

At this point, let $\frac{1}{2} + j\frac{\sqrt{L/C}}{2R} = a + bj$, $e^{j\frac{1}{\sqrt{LC}}t} = \cos\left(\frac{1}{\sqrt{LC}}t\right) + j\sin\left(\frac{1}{\sqrt{LC}}t\right) = c + dj \implies e^{-j\frac{1}{\sqrt{LC}}t} = \cos\left(-\frac{1}{\sqrt{LC}}t\right) + j\sin\left(-\frac{1}{\sqrt{LC}}t\right) = \cos\left(\frac{1}{\sqrt{LC}}t\right) - j\sin\left(\frac{1}{\sqrt{LC}}t\right) = c - dj$

$$
\begin{aligned}
V_{\text{out}}(t) &= V_{in}\left(-\left(\frac{1}{2} + j\frac{\sqrt{L/C}}{2R}\right)e^{j\frac{1}{\sqrt{LC}}t} - \left(\frac{1}{2} - j\frac{\sqrt{L/C}}{2R}\right)e^{-j\frac{1}{\sqrt{LC}}t} + 1\right) \\
&= V_{in}(1 - (a+bj)(c+dj) - (a-bj)(c-dj)) \\
&= V_{in}(1 - ac + bd - (ad+bc)j - ac + bd + (ad+bc)j) \\
&= V_{in}(1 - 2ac + 2bd) \\
&= V_{in}\left(1 - 2\frac{1}{2}\cos\left(\frac{1}{\sqrt{LC}}t\right) + 2\frac{\sqrt{L/C}}{2R}\sin\left(\frac{1}{\sqrt{LC}}t\right)\right) \\
&= V_{in}\left(1 - \cos\left(\frac{1}{\sqrt{LC}}t\right) + \frac{\sqrt{L/C}}{R}\sin\left(\frac{1}{\sqrt{LC}}t\right)\right)
\end{aligned}
$$

3 Diagonalizability and Invertibility

1. Given an example of a matrix A, or prove that no such example can exist.
   - Can be diagonalized and is invertible.
   - Cannot be diagonalized but is invertible.
   - Can be diagonalized but is non-invertible.
   - Cannot be diagonalized and is non-invertible.

   Answer

   - $$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$
   - $$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$
   - $$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
   - $$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

4 Eigenvalue Decomposition and Singular Value Decomposition

We define Eigenvalue Decomposition as follows:

If a matrix $A \in \mathbb{R}^{n \times n}$ has n linearly independent eigenvectors $\vec{p}_1, \dots, \vec{p}_n$ with eigenvalues $\lambda_i, \dots, \lambda_n$, then we can write:

$$A = P \Lambda P^{-1}$$

Where columns of P consist of $\vec{p}_1, \dots, \vec{p}_n$, and Λ is a diagonal matrix with diagonal entries $\lambda_i, \dots, \lambda_n$.

Consider a matrix $A \in \mathbb{S}^n$, that is, $A = A^T \in \mathbb{R}^{n \times n}$. This is a symmetric matrix and has orthorgonal eigenvectors. Therefore its eigenvalue decomposition can be written as,

$$A = P \Lambda P^T$$

1. First, assume $\lambda_i \geq 0, \forall i$. Find the SVD of A.

   Answer

   Observe that,

   $$A^T A = P \Lambda^2 P^T$$

   This means that,

   $$\sigma_i = \lambda_i \text{ and } V = P$$

   We have,

   $$A v_i = \lambda_i v_i = \sigma_i v_i$$

Plugging into our SVD condition $Av_i = \sigma_i u_i$:

$$\sigma_i v_i = \sigma_i u_i$$

This means that,

$$U = V = P$$

Therefore, in this case, the eigenvalue decomposition is the same as the singular value decompositions.

2. Let one particular eigenvalue λ_j be negative, with the associated eigenvector being p_j. Succinctly,

$$Ap_j = \lambda_j p_j \text{ with } \lambda_j < 0$$

We are still assuming that,

$$A = P\Lambda P^T$$

a) What is the singular value σ_j associated to λ_j?

b) What is the relationship between the left singular vector u_j, the right singular vector v_j and the eigenvector p_j?

Answer

a)

$$\sigma_j = |\lambda_j|$$

b) Either,

$$u_j = p_j \text{ and } v_j = -p_j$$

or,

$$u_j = -p_j \text{ and } v_j = p_j$$

This is because the diagonal entries of Σ MUST be non-negative.