Complex Inner Product

The **complex inner product** $\langle \cdot, \cdot \rangle$ on a vector space V over $\mathbb{C}$ is a function that takes in two vectors and outputs a scalar, such that $\langle \cdot, \cdot \rangle$ is symmetric, linear, and positive-definite.

- Conjugate Symmetry: $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$
- Scaling: $\langle c\vec{u}, \vec{v} \rangle = c\langle \vec{u}, \vec{v} \rangle$ and $\langle \vec{u}, c\vec{v} \rangle = \overline{c}\langle \vec{u}, \vec{v} \rangle$
- Additivity: $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ and $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$
- Positive-definite: $\langle \vec{u}, \vec{u} \rangle \geq 0$ with $\langle \vec{u}, \vec{u} \rangle = 0$ if and only if $\vec{u} = \vec{0}$

For two vectors, $\vec{u}, \vec{v} \in \mathbb{C}^n$, we usually define their inner product $\langle \vec{u}, \vec{v} \rangle$ to be $\langle \vec{u}, \vec{v} \rangle = \vec{v}^* \vec{u}$. We define the **norm**, or the magnitude, of a vector $\vec{v}$ to be $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{\vec{v}^* \vec{v}}$. For any non-zero vector, we can *normalize*, i.e., set its magnitude to 1 while preserving its direction, by dividing the vector by its norm $\frac{\vec{v}}{\|\vec{v}\|}$.

Adjoint of a Matrix

The **adjoint** or **conjugate-transpose** of a matrix A is the matrix A^* such that $A^*_{ij} = \overline{A_{ji}}$. From the complex inner product, one can show that

$$\langle A\vec{x}, \vec{y} \rangle = \langle \vec{x}, A^* \vec{y} \rangle \tag{1}$$

A matrix is **self-adjoint** or **Hermitian** if $A = A^*$.

Orthogonality and Orthonormality

We know that the angle between two vectors is given by this equation $\langle \vec{u}, \vec{v} \rangle = \|\vec{u}\| \|\vec{v}\| \cos \theta$. Notice that if $\theta = \pm 90°$, the right hand side is 0.

Therefore, we define two vectors $\vec{u}$ and $\vec{v}$ to be **orthogonal** to each other if $\langle \vec{u}, \vec{v} \rangle = \vec{v}^* \vec{u} = 0$. A set of vectors is orthogonal if any two vectors in this set are orthogonal to each other.

Furthermore, we define two vectors $\vec{u}$ and $\vec{v}$ to be **orthonormal** to each other if they are orthogonal to each other and their norms are 1. A set of vectors is orthonormal if any two vectors in this set are orthogonal to each other and every vector has a norm of 1. In fact, for any two vectors $\vec{u}$ and $\vec{v}$ in an orthonormal set,

$$\langle \vec{u}, \vec{v} \rangle = \vec{v}^* \vec{u} = \begin{cases} 1, & \text{if } \vec{u} = \vec{v} \\ 0, & \text{otherwise} \end{cases}.$$

A **unitary** matrix is a square matrix whose columns are orthonormal with respect to the complex inner product.

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{bmatrix}, \qquad \vec{u}_j^* \vec{u}_i = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

Note that $U^* U = UU^* = I$, so the inverse of a unitary matrix is its conjugate transpose $U^{-1} = U^*$.

Since the columns of a unitary matrix are orthonormal vectors, we can interpret these matrices as "rotation" and "reflection" matrices of the standard axes. This also implies that $\|U\vec{v}\| = \|\vec{v}\|$ for any vector $\vec{v}$.

1 Spectral Theorem

For a complex $n \times n$ Hermitian matrix A,

a) All eigenvalues of A are real.

b) A has n linearly independent eigenvectors $\in \mathbb{C}^n$.

c) A has orthogonal eigenvectors, i.e., $A = V\Lambda V^{-1} = V\Lambda V^*$, where Λ is a diagonal matrix and V is a unitary matrix. We say that A is orthogonally diagonalizable.

Recall that a matrix A is Hermitian if $A = A^*$. Furthermore, if A is of the form B^*B for some arbitrary matrix B, all of its eigenvalues are non-negative, i.e., $\lambda \geq 0$.

a) Prove the following: All eigenvalues of a Hermitian matrix A are real.

Hint: Let $(\lambda, \vec{v})$ be an eigenvalue/vector pair and use the definition of an eigenvalue to show that $\lambda\langle\vec{v}, \vec{v}\rangle = \overline{\lambda}\langle\vec{v}, \vec{v}\rangle$.

Answer

Let λ be an eigenvalue of A with corresponding eigenvector $\vec{v}$.

$$A\vec{v} = \lambda\vec{v}$$

Then if we look at $\lambda\langle\vec{v}, \vec{v}\rangle$ and use the definition of the Complex Inner Product,

$$\lambda\langle\vec{v}, \vec{v}\rangle = \lambda\vec{v}^*\vec{v} = \vec{v}^*\lambda\vec{v} = \vec{v}^*A\vec{v}$$

Since A is Hermitian, we can replace A with A^* to get

$$\vec{v}^*A\vec{v} = \vec{v}^*A^*\vec{v} = (A\vec{v})^*\vec{v} = (\lambda\vec{v})^*\vec{v}$$

The adjoint of a scalar λ is its complex conjugate $\overline{\lambda}$ so

$$(\lambda\vec{v})^*\vec{v} = \lambda^*\vec{v}^*\vec{v} = \overline{\lambda}\langle\vec{v}, \vec{v}\rangle$$

Since $\lambda\langle\vec{v}, \vec{v}\rangle = \overline{\lambda}\langle\vec{v}, \vec{v}\rangle$ and $\langle\vec{v}, \vec{v}\rangle > 0$, $\lambda = \overline{\lambda}$ which implies that λ is real.

b) Prove the following: For any Hermitian matrix A, any two eigenvectors corresponding to distinct eigenvalues of A are orthogonal.

Hint: Use the definition of an eigenvalue to show that $\lambda_1\langle\vec{v}_1, \vec{v}_2\rangle = \lambda_2\langle\vec{v}_1, \vec{v}_2\rangle$.

Answer

Since A is Hermitian, we can say that $\langle A\vec{x}, \vec{y}\rangle = \langle\vec{x}, A\vec{y}\rangle$.

$$\begin{aligned}\lambda_1\langle\vec{v}_1, \vec{v}_2\rangle &= \langle\lambda_1\vec{v}_1, \vec{v}_2\rangle = \langle A\vec{v}_1, \vec{v}_2\rangle \\ &= \langle\vec{v}_1, A\vec{v}_2\rangle = \langle\vec{v}_1, \lambda_2\vec{v}_2\rangle \\ &= \overline{\lambda_2}\langle\vec{v}_1, \vec{v}_2\rangle\end{aligned}$$

We showed the eigenvalues of A are real so $\overline{\lambda_2} = \lambda_2$ so it follows that $\lambda_1\langle\vec{v}_1, \vec{v}_2\rangle = \lambda_2\langle\vec{v}_1, \vec{v}_2\rangle$. This implies that

$$(\lambda_1 - \lambda_2)\langle\vec{v}_1, \vec{v}_2\rangle = 0$$

The only way this equation can be satisfied when $\lambda_1 \neq \lambda_2$ is for $\langle\vec{v}_1, \vec{v}_2\rangle$ to be zero. Therefore, $\vec{v}_1$ and $\vec{v}_2$ must be orthogonal to each other

c) Prove the following: For any matrix A, A^*A is Hermitian and only has non-negative eigenvalues.

Answer

A^*A is Hermitian because its adjoint $(A^*A)^* = A^*A^{**} = A^*A$ equals itself.
Let λ be an eigenvalue of A^*A with corresponding eigenvector $\vec{v}$.

$$A^*A\vec{v} = \lambda\vec{v}$$

We left-multiply $\vec{v}^*$ on both sides.

$$\vec{v}^*A^*A\vec{v} = \vec{v}^*\lambda\vec{v}$$
$$\left(A\vec{v}\right)^* A\vec{v} = \lambda\vec{v}^*\vec{v}$$
$$\left\|A\vec{v}\right\|^2 = \lambda\|\vec{v}\|^2$$
$$\lambda = \frac{\left\|A\vec{v}\right\|^2}{\|\vec{v}\|^2} \geq 0$$

Note that $\|\vec{v}\| \neq 0$ because we assumed that $\vec{v}$ is an eigenvector corresponding to λ.

2 Fundamental Theorem of Linear Algebra

a) Let $\vec{v}$ be an eigenvector of nonzero eigenvalue of A^*A. Show that $\vec{v} \in \text{Col}(A^*)$.

Answer

Since $\vec{v}$ is an eigenvector of nonzero eigenvalue, we know that

$$A^*A\vec{v} = \lambda\vec{v} \neq \vec{0}$$

Therefore, since $\lambda\vec{v} \in \text{Col}(A^*)$, $\vec{v}$ must also be in $\text{Col}(A^*)$.

b) Show that the two subspaces $\text{Nul}(A)$ and $\text{Nul}(A^*A)$ are equal.

Answer

One way to show that two sets are equal are to show that they are subsets of each other.

Let $\vec{x} \in \text{Nul}(A)$. Then $A\vec{x} = \vec{0}$. Left multiplying by A^*, it follows that

$$A^*A\vec{x} = A^*\vec{0} = \vec{0} \implies \vec{x} \in \text{Nul}(A^*A)$$

Now let $\vec{x} \in \text{Nul}A^*A$. Then $A^*A\vec{x} = \vec{0}$. Left multiplying by $\vec{x}^*$, it follows that

$$\vec{x}^*A^*A\vec{x} = \left\|A\vec{x}\right\|^2 = 0$$

By the positive-definiteness of norms, we see that $A\vec{x}$ must be $\vec{0}$ so $\vec{x} \in \text{Nul}(A)$.

c) Let $\vec{u}$ be an eigenvector of eigenvalue 0 of A^*A. Show that $\vec{u} \in \text{Nul}(A)$.

Answer

Since $\vec{u}$ is an eigenvector of zero eigenvalue, we know that

$$A^*A\vec{u} = \vec{0}$$

This implies that $\vec{u} \in \text{Nul}(A^*A)$. However, since the two vector spaces $\text{Nul}(A^*A)$ and $\text{Nul}(A)$ are equal, this implies that $\vec{u} \in \text{Nul}(A)$.

d) If A is a $m \times n$ matrix of rank k what are the dimensions of $\text{Col}(A^*)$ and $\text{Nul}(A)$?

Answer

If A is a matrix of rank k then the rank of A^* will also be k. By the Rank-Nullity Theorem, we see that $\text{Rank}(A) + \dim \text{Nul}(A) = n$ so $\dim \text{Nul}(A) = n - k$.

e) Use parts (a)-(d) to show that $\text{Col}(A^*)$ is the orthogonal complement of $\text{Nul}(A)$.
Use the spectral theorem on the matrix A^*A to create an orthonormal eigenbasis of $\mathbb{C}^n$

Answer

Since the matrix A^*A is Hermitian, we know that it has a full basis of orthonormal eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$. If A is of rank k then A^*A must also be of rank k by applying the rank nullity theorem and part (b).

We showed in part (a) that vectors of nonzero eigenvalues were in the $\text{Col}(A^*)$, so we can pick the k eigenvectors of A^*A that have nonzero eigenvalue to form a basis for $\text{Col}(A^*)$. The remaining $n - k$ eigenvectors will form a basis for $\text{Nul}(A)$ by part (c)

Therefore, we conclude that since the basis vectors of $\text{Col}(A^*)$ are orthogonal to the basis vectors of $\text{Nul}(A)$, the two subspaces must be orthogonal complements.