1 Complex Inner Product

For the complex vector space $\mathbb{C}^n$, we can no longer use our conventional real dot product as a valid inner product for $\mathbb{C}^n$. This is because the real dot product is no longer positive-definite for complex vectors.

For example, let $\vec{v} = \begin{bmatrix} j \\ j \end{bmatrix}$. Then, $\vec{v} \cdot \vec{v} = j^2 + j^2 = -2 < 0$.

Therefore, for two vectors $\vec{u}, \vec{v} \in \mathbb{C}^n$, we define the complex inner product to be:

$$\langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^{n} u_i \overline{v}_i = \vec{u}^T \overline{\vec{v}} = \vec{v}^* \vec{u}$$

where $\overline{\cdot}$ denotes the complex conjugate and $\vec{\cdot}^*$ denotes the complex conjugate transpose. Recall that the complex conjugate of a complex number $z = a + jb = re^{j\theta}$ is equal to $\overline{z} = a - jb = re^{-j\theta}$. The conjugate transpose of a vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is $\vec{v}^* = \begin{bmatrix} \overline{v}_1 & \overline{v}_2 & \cdots & \overline{v}_n \end{bmatrix}$.

Note that this inner product is no longer symmetric but conjugate-symmetric, i.e., $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$.

2 Gram-Schmidt Process

Gram-Schmidt is an algorithm that takes a set of linearly independent vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ and generates an orthonormal set of vectors $\{\vec{u}_1, \ldots, \vec{u}_n\}$ that span the same vector space as the original set. Concretely, $\{\vec{u}_1, \ldots, \vec{u}_n\}$ satisfy the following:

- $\forall 0 < k \leq n$, $\text{span}(\{\vec{v}_1, \ldots, \vec{v}_k\}) = \text{span}(\{\vec{u}_1, \ldots, \vec{u}_k\})$
- $\{\vec{u}_1, \ldots, \vec{u}_n\}$ is an orthonormal set of vectors

Orthonormal Vectors

A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is **orthonormal** if all the vectors are mutually orthogonal to each other and all are of unit length. That is:

- Orthogonal: For all pairs of vectors $\vec{v}_i, \vec{v}_j$ where $i \neq j$, $\langle \vec{v}_i, \vec{v}_j \rangle = 0$. For real vectors, this means $\vec{v}_i^T \vec{v}_j = 0$.
- Normalized: For all i, $\|\vec{v}_i\| = 1$. (This implies that $\|\vec{v}_i\| = \langle \vec{v}_i, \vec{v}_i \rangle = 1$.)

Projection of a Vector

The projection of a vector $\vec{w}$ onto another vector v is denoted as $\text{proj}_{\vec{v}}(\vec{w})$

$$\text{proj}_{\vec{v}}(\vec{w}) = \frac{\langle \vec{w}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v} = \frac{\langle \vec{w}, \vec{v} \rangle}{\left\|\vec{v}\right\|^2} \vec{v} \tag{1}$$

If we project onto a vector $\vec{v}$ that is normalized, $\left\|\vec{v}\right\| = 1$, then the projection will be

$$\text{proj}_{\vec{v}}(\vec{w}) = \langle \vec{w}, \vec{v} \rangle \vec{v} \tag{2}$$

Gram-Schmidt Algorithm

The Gram-Schmidt algorithm works by first finding the unit vector $\vec{u}_1$ such that span($\{\vec{u}_1\}$) = span($\{\vec{v}_1\}$). Subsequently, the unit vector $\vec{u}_2$ is calculated such that $\langle \vec{u}_1, \vec{u}_2 \rangle = 0$ and span($\{\vec{u}_1, \vec{u}_2\}$) = span($\{\vec{v}_1, \vec{v}_2\}$). This is continued through n vectors, resulting in the orthonormal set of vectors $\{\vec{u}_1, \ldots, \vec{u}_n\}$ that span the same vector space as $\{\vec{v}_1, \ldots, \vec{v}_n\}$.

How is this done? Finding $\vec{u}_1$ is straightforward, since it is the first vector in our new set, and therefore we must only satisfy $\|\vec{u}_1\| = 1$ and span($\{\vec{u}_1\}$) = span($\{\vec{v}_1\}$). Since span($\{\vec{v}_1\}$) is a one dimensional vector space, the unit vector that spans the same vector space would just be the unit vector in the same direction as $\vec{v}_1$. We have

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}. \tag{3}$$

Calculating $\vec{u}_2$ requires that we satisfy:

1. Spanning the same vector space as original set: span($\{\vec{u}_1, \vec{u}_2\}$) = span($\{\vec{v}_1, \vec{v}_2\}$)
2. Orthogonal to previous vectors: $\langle \vec{u}_1, \vec{u}_2 \rangle = 0$
3. Normalized: $\|\vec{u}_2\| = 1$

Using the vector $\vec{u}_1$ that we calculated above, we notice that

$$\text{span}(\{\vec{u}_1, \vec{v}_2\}) = \text{span}(\{\vec{v}_1, \vec{v}_2\}),$$

satisfying the first condition. However, $\vec{u}_1$ and $\vec{v}_2$ are not necessarily orthogonal. Recall from 16A/54 that the following subspaces are equivalent for any pair of linearly independent vectors $\vec{v}_1, \vec{v}_2$:

- span($\vec{v}_1$, $\vec{v}_2$)
- span($\vec{v}_1$, $\alpha\vec{v}_2$)
- span($\vec{v}_1$, $\vec{v}_1 + \vec{v}_2$)
- span($\vec{v}_1$, $\vec{v}_1 - \vec{v}_2$)
- span($\vec{v}_1$, $\vec{v}_2 - \alpha\vec{v}_1$)

Therefore, let us pick

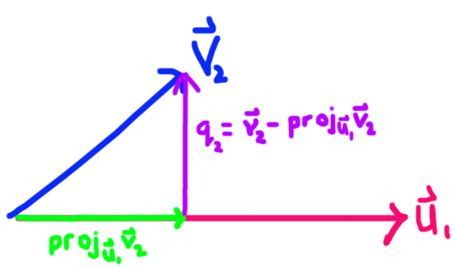
$$\vec{q}_2 = \vec{v}_2 - \alpha\vec{u}_1,$$

so that span($\vec{u}_1, \vec{q}_2$) will have the same span as $\{\vec{u}_1, \vec{v}_2\}$ = span($\vec{v}_1, \vec{v}_2$).

What should α be if we would like $\vec{u}_1$ and $\vec{q}_2 = \vec{v}_2 - \alpha\vec{u}_1$ to be orthogonal to each other? We can recall from 16A, that subtracting the projection $\text{proj}_{\vec{u}_1}(\vec{v}_2)$ will create a vector orthogonal to $\vec{u}_1$, where

$$\text{proj}_{\vec{u}_1}(\vec{v}_2) = \langle \vec{v}_2, \vec{u}_1 \rangle \vec{u}_1$$

is the projection of $\vec{v}_2$ onto $\vec{u}_1$.



This makes sense because the projection of $\vec{v}_2$ onto $\vec{u}_1$ provides the component of $\vec{v}_2$ that is along $\vec{u}_1$. Subtracting off this component from $\vec{v}_2$ will only leave components of $\vec{v}_2$ that are orthogonal to $\vec{u}_1$.

Therefore, if we set

$$\alpha \vec{u}_1 = \text{proj}_{\vec{u}_1}(\vec{v}_2),$$

the resulting

$$\vec{q}_2 = \vec{v}_2 - \text{proj}_{\vec{u}_1}(\vec{v}_2) = \vec{v}_2 - \langle \vec{v}_2, \vec{u}_1 \rangle \vec{u}_1$$

will be orthogonal to $\vec{u}_1$.

Finally, we normalize $\vec{q}_2$ to complete the process of finding the $\vec{u}_2$ which satisfies all three conditions above:

$$\vec{u}_2 = \frac{\vec{q}_2}{\|\vec{q}_2\|}$$

In the question below, you will work through how this methodology leads to the Gram-Schmidt algorithm for calculating the orthonormal set $\{\vec{u}_1, \ldots, \vec{u}_n\}$ from n linearly independent vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$. Note that the Gram-Schmidt process can be done for the real or complex inner product so we will leave all of our results in general inner product form.

3 Gram-Schmidt Algorithm

Let's apply Gram-Schmidt orthonormalization to a set of three linearly independent vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

a) Find unit vector $\vec{u}_1$ such that $\text{span}(\{\vec{u}_1\}) = \text{span}(\{\vec{v}_1\})$.

Answer

Since $\text{span}(\{\vec{v}_1\})$ is a one dimensional vector space, the unit vector that span the same vector space would just be the normalized vector point at the same direction as $\vec{v}_1$. We have

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}. \tag{4}$$

b) Given $\vec{u}_1$ from the previous step, find $\vec{u}_2$ such that $\text{span}(\{\vec{u}_1, \vec{u}_2\}) = \text{span}(\{\vec{v}_1, \vec{v}_2\})$ and $\vec{u}_2$ is orthogonal to $\vec{u}_1$.

Answer

We know that $\vec{v}_2-$ (the projection of $\vec{v}_2$ on $\vec{u}_1$) would be orthogonal to $\vec{u}_1$ as seen earlier. Hence, a vector $\vec{q}_2$ orthogonal to $\vec{u}_1$ where $\text{span}(\{\vec{u}_1, \vec{q}_2\}) = \text{span}(\{\vec{v}_1, \vec{v}_2\})$ is

$$\vec{q}_2 = \vec{v}_2 - \langle \vec{v}_2, \vec{u}_1 \rangle \vec{u}_1. \tag{5}$$

Normalizing, we have $\vec{u}_2 = \frac{\vec{q}_2}{\|\vec{q}_2\|}$.

c) Now given $\vec{u}_1$ and $\vec{u}_2$ in the previous steps, find $\vec{u}_3$ such that $\text{span}(\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}) = \text{span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\})$.

Answer

We know that the projection of $\vec{v}_3$ onto the subspace spanned by $\vec{u}_1, \vec{u}_2$ is

$$\langle \vec{v}_3, \vec{u}_2 \rangle \vec{u}_2 + \langle \vec{v}_3, \vec{u}_1 \rangle \vec{u}_1. \tag{6}$$

We know that

$$\vec{q}_3 = \vec{v}_3 - \langle \vec{v}_3, \vec{u}_2 \rangle \vec{u}_2 - \langle \vec{v}_3, \vec{u}_1 \rangle \vec{u}_1 \tag{7}$$

is orthogonal to $\vec{u}_1$ and $\vec{u}_2$. Normalizing, we have $\vec{u}_3 = \frac{\vec{q}_3}{\|\vec{q}_3\|}$.

d) Let's extend this algorithm to n linearly independent vectors. That is, given an input $\{\vec{v}_1, \ldots, \vec{v}_n\}$, write the algorithm to calculate the orthonormal set of vectors $\{\vec{u}_1, \ldots, \vec{u}_n\}$, where $\text{span}(\{\vec{v}_1, \ldots, \vec{v}_n\}) = \text{span}(\{\vec{u}_1, \ldots, \vec{u}_n\})$. *Hint: How would you calculate the i^{th} vector, $\vec{q}_i$?*

Answer

Inputs

- A set of linearly independent vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$.

Outputs

- An orthonormal set of vectors $\{\vec{u}_1, \ldots, \vec{u}_n\}$, where $\text{span}(\{\vec{v}_1, \ldots, \vec{v}_n\}) = \text{span}(\{\vec{u}_1, \ldots, \vec{u}_n\})$.

Gram Schmidt Procedure

- compute $\vec{q}_1 : \vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$
- for $(i = 2 \ldots n)$:
  a) Compute the vector $\vec{q}_i$, such that $\text{span}(\{\vec{u}_1, \ldots, \vec{q}_i\}) = \text{span}(\{\vec{v}_1, \ldots, \vec{v}_i\})$:
  $\vec{q}_i = \vec{v}_i - \sum_{j=1}^{i-1} \langle \vec{v}_i, \vec{u}_j \rangle \vec{u}_j$
  b) Normalize to compute $\vec{q}_i : \vec{u}_i = \frac{\vec{q}_i}{\|\vec{q}_i\|}$

4 The Order of Gram-Schmidt

a) If we are performing the Gram-Schmidt method on a set of vectors, does the order in which we take the vectors matter? Consider the set of vectors

$$\left\{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\right\} = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\} \tag{8}$$

Perform Gram-Schmidt on these vectors first in the order $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$.

Answer

If we start with $\vec{v}_1$ we get the basis

$$\left\{\vec{q}_1 = \vec{v}_1,\ \vec{q}_2 = \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{q}_1\rangle}{||\vec{q}_1||^2}\vec{q}_1,\ \vec{v}_3 - \frac{\langle \vec{v}_3, \vec{q}_1\rangle}{||\vec{q}_1||^2}\vec{q}_1 - \frac{\langle \vec{v}_3, \vec{q}_2\rangle}{||\vec{q}_2||^2}\vec{q}_2\right\} \tag{9}$$

$$\left\{\vec{q}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \vec{q}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{1}{1}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}, \vec{q}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{1}{1}\begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1}{1}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} \tag{10}$$

Since all of these vectors are normalized, we can say that $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$ is an orthonormal basis

b) Now perform Gram-Schmidt on these vectors in the order $\vec{v}_3,\ \vec{v}_2,\ \vec{v}_1$. Do you get the same result?

Answer

If we write the basis starting with $\vec{v}_3$.

$$\left\{\vec{q}_1 = \vec{v}_3,\ \vec{q}_2 = \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{q}_1\rangle}{||\vec{q}_1||^2}\vec{q}_1,\ \vec{v}_1 - \frac{\langle \vec{v}_1, \vec{q}_1\rangle}{||\vec{q}_1||^2}\vec{q}_1 - \frac{\langle \vec{v}_1, \vec{q}_2\rangle}{||\vec{q}_2||^2}\vec{q}_2\right\} \tag{11}$$

$$\tag{12}$$

$$\left\{\vec{q}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \vec{q}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1/3\\1/3\\-2/3\end{bmatrix}, \vec{q}_3 = \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1/3\\1/3\\-2/3\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\0\end{bmatrix}\right\} \tag{13}$$

Normalized this is

$$\left\{\vec{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \vec{u}_2 = \sqrt{\frac{3}{2}}\begin{bmatrix}1/3\\1/3\\-2/3\end{bmatrix}, \vec{u}_3 = \sqrt{2}\begin{bmatrix}1/2\\-1/2\\0\end{bmatrix}\right\} \tag{14}$$