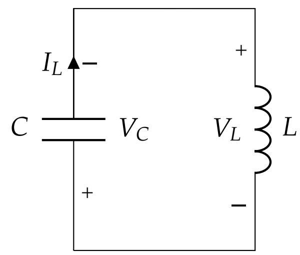
## 1 LC Tank

Consider the following circuit like you saw in lecture:




This is sometimes called an $LC$ tank and we will derive its response in this problem. Assume at $t = 0$ we have $V_C(0) = V_S = 1$ V and $\frac{dV_C}{dt}(t = 0) = 0$. Also suppose $L = 9\,\text{nH}$ and $C = 1\,\text{nF}$.

a) **Write the system of differential equations in terms of state variables $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ that describes this circuit for $t \geq 0$. Leave the system symbolic in terms of $V_s$, $L$, and $C$.**

### Answer

For this part, we need to find two differential equations, each including a derivative of one of the state variables.

First, let's consider the capacitor equation $I_C(t) = C\frac{d}{dt}V_C(t)$. In this circuit, $I_C(t) = I_L(t)$, so we can write

$$I_C(t) = C\frac{d}{dt}V_C(t) = I_L(t) \tag{1}$$

$$\frac{d}{dt}V_C(t) = \frac{1}{C}I_L(t). \tag{2}$$

If we use the state variable names, we can write this as

$$\frac{d}{dt}x_2(t) = \frac{1}{C}x_1(t), \tag{3}$$

so now we have one differential equation.

For the other differential equation, we can apply KVL around the single loop in this circuit. (Alternatively, we could just solve it directly and substitute in for the desired voltage on the capacitor, which is a state variable.) Going clockwise, we have

$$V_C(t) + V_L(t) = 0. \tag{4}$$

Using the inductor equation $V_L = L\frac{d}{dt}I_L(t)$, we can write this as

$$V_C(t) + L\frac{d}{dt}I_L(t) = 0, \tag{5}$$

which we can rewrite as

$$\frac{d}{dt}I_L(t) = -\frac{1}{L}V_C(t). \tag{6}$$

If we use the state variable names, this becomes

$$\frac{d}{dt}x_1(t) = -\frac{1}{L}x_2(t), \tag{7}$$

and we have a second differential equation.

To summarize the final system is

$$\frac{d}{dt}x_1(t) = -\frac{1}{L}x_2(t) \tag{8}$$
$$\frac{d}{dt}x_2(t) = \frac{1}{C}x_1(t). \tag{9}$$

b) **Write the system of equations in vector/matrix form with the vector state variable $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.** This should be in the form $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ with a $2 \times 2$ matrix $A$.

**Find the initial conditions $\vec{x}(0)$.**

### Answer

By inspection from the previous part, we have

$$\begin{bmatrix} \frac{d}{dt}x_1(t) \\ \frac{d}{dt}x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{10}$$

which is in the form $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$, with

$$A = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}. \tag{11}$$

We know that $V_C(0) = V_S$ and we know that $i_L(0) = C\frac{d}{dt}V_C(0) = 0$, thus $\vec{x}(0) = \begin{bmatrix} 0 \\ V_S \end{bmatrix}$

c) **Find the eigenvalues of the $A$ matrix symbolically.**

### Answer

To find the eigenvalues, we'll solve $\det(A - \lambda I) = 0$. In other words, we want to find $\lambda$ such that

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & -\frac{1}{L} \\ \frac{1}{C} & -\lambda \end{bmatrix}\right) \tag{12}$$
$$= \lambda^2 + \frac{1}{LC} = 0. \tag{13}$$

Solving for $\lambda$ we see

$$\lambda = \pm\frac{j}{\sqrt{LC}}. \tag{14}$$

d) Recall from yesterday's discussion that solutions for $x_i(t)$ will all be of the form

$$x_i(t) = \sum_k c_k e^{\lambda_k t}$$

where $\lambda_k$ is an eigenvalue of our differential equation relation matrix $A$. Thus, we make the following guess for $\vec{x}(t)$:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_3 e^{\lambda_1 t} + c_4 e^{\lambda_2 t} \end{bmatrix}$$

where $c_1,\ c_2,\ c_3,\ c_4$ are all constants.

**Evaluate $\vec{x}(t)$ and $\frac{d\vec{x}}{dt}(t)$ at time $t = 0$ in order to obtain four equations in four unknowns.**

### Answer

Evaluating $\vec{x}(0)$ and equating it to our result from part (b), we obtain

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_3 + c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ V_s \end{bmatrix}$$

which provides us two equations. Taking the derivative of our guess for $\vec{x}(t)$,

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1\lambda_1 e^{\lambda_1 t} + c_2\lambda_2 e^{\lambda_2 t} \\ c_3\lambda_1 e^{\lambda_1 t} + c_4\lambda_2 e^{\lambda_2 t} \end{bmatrix}$$

Which becomes at $t = 0$

$$\frac{d}{dt}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} c_1\lambda_1 + c_2\lambda_2 \\ c_3\lambda_1 + c_4\lambda_2 \end{bmatrix}$$

We can equate this to $\frac{d}{dt}\vec{x}(0) = A\vec{x}(0)$ from part b:

$$\begin{bmatrix} c_1\lambda_1 + c_2\lambda_2 \\ c_3\lambda_1 + c_4\lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -\frac{x_2(0)}{L} \\ \frac{x_1(0)}{C} \end{bmatrix} = \begin{bmatrix} -\frac{V_s}{L} \\ 0 \end{bmatrix}$$

thus providing our other two equations.

e) **Solve those equations for $c_1,\ c_2,\ c_3,\ c_4$ and plug them into your guess for $\vec{x}(t)$.** What do you notice about the solutions? Are they complex functions? HINT: Remember $e^{j\theta} = \cos(\theta) + j\sin(\theta)$.

### Answer

Solving the four equations, we obtain

$$c_1 = -\sqrt{\frac{C}{L}}\frac{V_s}{2j} \qquad c_2 = \sqrt{\frac{C}{L}}\frac{V_s}{2j}$$

$$c_3 = \frac{V_s}{2} \qquad c_4 = \frac{V_s}{2}$$

Plugging these into our guess,

$$\vec{x}(t) = \begin{bmatrix} \frac{V_s}{2j}\sqrt{\frac{C}{L}}\left(e^{-\frac{j}{\sqrt{LC}}t} - e^{\frac{j}{\sqrt{LC}}t}\right) \\ \frac{V_S}{2}\left(e^{-\frac{j}{\sqrt{LC}}t} + e^{\frac{j}{\sqrt{LC}}t}\right) \end{bmatrix}$$

$$= \begin{bmatrix} -V_s\sqrt{\frac{C}{L}}\sin\left(\frac{t}{\sqrt{LC}}\right) \\ V_s\cos\left(\frac{t}{\sqrt{LC}}\right) \end{bmatrix} \qquad \text{Since} \qquad \begin{aligned} \sin(j\theta) &= \tfrac{1}{2j}(e^{j\theta} - e^{-j\theta}) \\ \cos(j\theta) &= \tfrac{1}{2}(e^{j\theta} + e^{-j\theta}) \end{aligned}$$

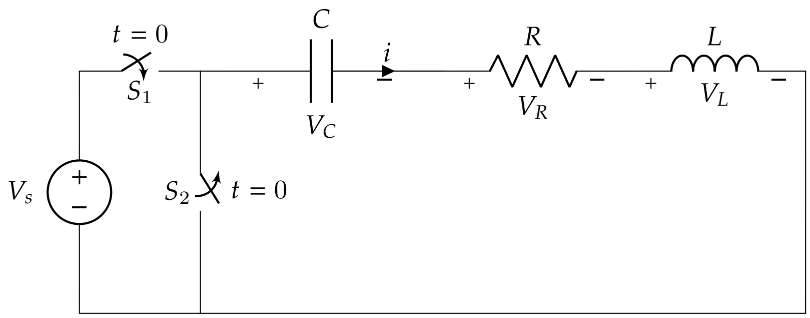
Using the provided values:

$$I_L(t) = -I_{\max}\sin\left(\frac{t}{3\,\mathrm{ns}}\right), \qquad I_{\max} = 333.33\,\mathrm{mA}$$

$$V_C = V_S\cos\left(\frac{t}{3\,\mathrm{ns}}\right), \qquad V_S = 1\,\mathrm{V}$$

## 2 Charging RLC Circuit

Consider the following circuit. Before $t = 0$, switch $S_1$ is off while $S_2$ is on. At $t = 0$, both switches flip state ($S_1$ turns on and $S_2$ turns off):




a) Write out the differential equation describing this circuit for $t \geq 0$ in the form:

$$\frac{d^2V_c}{dt^2} + a_1\frac{dV_c}{dt} + a_0V_c = b$$

**Answer**

From KVL:

$$V_s = V_c + V_R + V_L$$

In terms of the current through the series RLC circuit $i$, this becomes

$$V_s = V_c + iR + L\frac{di}{dt}$$

Noting that the current can be related to $V_c$ via $i = C\frac{dV_c}{dt}$,

$$V_s = V_c + RC\frac{dV_c}{dt} + LC\frac{d^2V_c}{dt^2}$$

Which can be written in the desired form as:

$$\frac{d^2V_c}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{1}{LC}V_c = \frac{V_s}{LC}$$

b) Find a $\tilde{V}_c$ and substitute it to the previous equation such that

$$\frac{d^2\tilde{V}_c}{dt^2} + a_1\frac{d\tilde{V}_c}{dt} + a_0\tilde{V}_c = 0$$

**Answer**

$\tilde{V}_c = V_c - V_s$

c) Solve for $V_c(t)$ for $t \geq 0$. Use component values $V_s = 4\text{V}$, $C = 2\text{fF}$, $R = 60\text{k}\Omega$, and $L = 1\mu\text{H}$.

**Answer**

$$\frac{d}{dt}\begin{bmatrix} \tilde{V}_c \\ \frac{d\tilde{V}_c}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} \tilde{V}_c \\ \frac{d\tilde{V}_c}{dt} \end{bmatrix}$$

Then, we need to solve the following quadratic equation for the eigenvalues:

$$\lambda^2 + \lambda\frac{R}{L} + \frac{1}{LC} = 0$$

Solving for the eigenvalues and plugging in component values:

$$\lambda = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$\lambda_1 = -1 \times 10^{10}$$

$$\lambda_2 = -5 \times 10^{10}$$

$$\tilde{V}_c = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

Substituting back in $V_c(t)$:

$$V_c - V_s = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$V_c = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + V_s$$

Now we can use initial conditions to solve for $c_1$ and $c_2$. Before the switches change states, all voltages and currents are 0. Immediately after the switch closes, the voltage across the capacitor cannot change instantaneously, so:

$$V_c(0) = 0 = c_1 + c_2 + 4 \tag{1}$$

Just like the voltage across the capacitor, the current through the inductor cannot change instantaneously, so:

$$C\frac{dV_c(0)}{dt} = i_c(0) = i_L(0) = 0$$

$$-c_1 - 5c_2 = 0 \tag{2}$$

Using equations 1 and 2, we can solve for $c_1$ and $c_2$:

$$c_1 = -5$$

$$c_2 = 1$$

This gives us:

$$V_c(t) = 4 - 5e^{-10^{10}t} + e^{-5\times 10^{10}t}$$