1 Diagonalization

Consider an $n \times n$ matrix A that has n linearly independent eigenvalue/eigenvector pairs $(\lambda_1, \vec{v}_1), \dots, (\lambda_n, \vec{v}_n)$ that can be put into a matrices V and Λ.

$$V = \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & & | \end{bmatrix} \quad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

a) Show that $AV = V\Lambda$.

Answer

Since $(\lambda_i, \vec{v}_i)$ are eigenvalue/vector pairs, we know that

$$\begin{aligned} A\vec{v}_1 &= \lambda_1 \vec{v}_1 \\ &\vdots \\ A\vec{v}_n &= \lambda_n \vec{v}_n \end{aligned}$$

However, we can also write out $A\vec{v}_1$ as a linear combination of the columns of V to get

$$\begin{aligned} A\vec{v}_1 &= \lambda_1 \cdot \vec{v}_1 + \dots + 0 \cdot \vec{v}_n \\ &= \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ 0 \end{bmatrix} = V \begin{bmatrix} \lambda_1 \\ \vdots \\ 0 \end{bmatrix} \end{aligned}$$

Generalizing this for all $i = 1...n$, we can say that $A\vec{v}_i = V\vec{\lambda}_i$ where $\vec{\lambda}_i$ is the i^{th} column of the matrix Λ that is a vector of 0's except for the i^{th} entry which is equal to λ_i. We can aggregate our results to conclude that

$$AV = \begin{bmatrix} | & & | \\ A\vec{v}_1 & \dots & A\vec{v}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ V\vec{\lambda}_1 & \dots & V\vec{\lambda}_n \\ | & & | \end{bmatrix} = V\Lambda$$

b) Use the fact in part (a) to conclude that $A = V\Lambda V^{-1}$.

Answer

From the previous part, we know that $AV = V\Lambda$. In addition, we know the matrix V is invertible since it is square and all of the eigenvectors of A are linearly independent. Therefore can right multiply by V^{-1} to conclude by saying $A = V\Lambda V^{-1}$.

2 Systems of Differential Equations

Consider a system of differential equations (valid for $t \geq 0$)

$$\frac{d}{dt}x_1(t) = -4x_1(t) + x_2(t) \tag{1}$$

$$\frac{d}{dt}x_2(t) = 2x_1(t) - 3x_2(t) \tag{2}$$

with initial conditions $x_1(0) = 3$ and $x_2(0) = 3$.

a) Write out the differential equations and initial conditions in matrix/vector form.

Answer

$$\begin{bmatrix} \frac{d}{dt}x_1(t) \\ \frac{d}{dt}x_2(t) \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

We will define the differential matrix as A, where

$$A = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix}$$

b) Find the eigenvalues $\lambda_1,\ \lambda_2$ and eigenspaces for the differential equation matrix above.

Answer

Eigenvalues λ and eigenvectors v of matrix A are given by

$$Av = \lambda v.$$

In order to find the eigenvalues, we take the determinant:

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} -4-\lambda & 1 \\ 2 & -3-\lambda \end{bmatrix}\right) = 0$$

$$\begin{aligned} (-4-\lambda)(-3-\lambda) - 2 &= 0 \\ 12 + 7\lambda + \lambda^2 - 2 &= 0 \\ \lambda^2 + 7\lambda + 10 &= 0 \\ (\lambda+5)(\lambda+2) &= 0 \end{aligned}$$

Giving:

$$\lambda = -5, -2$$

The eigenspace associated with $\lambda_1 = -5$ is given by:

$$\begin{bmatrix} -4+5 & 1 \\ 2 & -3+5 \end{bmatrix} \vec{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \vec{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\vec{v}_1 = \alpha_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The eigenspace associated with $\lambda_2 = -2$ is given by:

$$\begin{bmatrix} -4+2 & 1 \\ 2 & -3+2 \end{bmatrix} \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\vec{v}_2 = \alpha_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

c) Use the diagonalization of $A = V\Lambda V^{-1}$ to express the differential equation in terms of a new variables $z_1(t),\ z_2(t)$. Remember to find the new initial conditions $z_1(0), z_2(0)$. (These variables represent eigenbasis-aligned coordinates.)

Answer

The original differential equation was

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$$

Substituting in the diagonalization of A, we get

$$\frac{d}{dt}\vec{x}(t) = V\Lambda V^{-1}\vec{x}(t)$$

Applying V^{-1} to both sides, we see that

$$V^{-1}\frac{d}{dt}\vec{x}(t) = \Lambda V^{-1}\vec{x}(t)$$

The derivative is linear so $V^{-1}\frac{d}{dt}\vec{x}(t) = \frac{d}{dt}V^{-1}\vec{x}(t)$

$$\frac{d}{dt}V^{-1}\vec{x}(t) = \Lambda V^{-1}\vec{x}(t)$$

Finally, we define the vector $\vec{z}(t) = V^{-1}\vec{x}(t)$ to say

$$\frac{d}{dt}\vec{z}(t) = \Lambda\vec{z}(t)$$

Uncoupling the matrix, we can express this as two differential equations in terms of $z_1(t)$ and $z_2(t)$

$$\frac{d}{dt}z_1(t) = \lambda_1 z_1(t) = -5z_1(t)$$
$$\frac{d}{dt}z_2(t) = \lambda_2 z_2(t) = -2z_2(t)$$

d) Solve the differential equation for $z_i(t)$ in the eigenbasis.

Answer

Our initial condition is current in x_i so we must represent it using z_i :

$$\vec{z}(0) = V^{-1}\vec{x}(0) = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Then we solve based on the form of the problem and our previous differential equation experience:

$$\vec{z}(t) = \begin{bmatrix} K_1 e^{-5t} \\ K_2 e^{-2t} \end{bmatrix}$$

Plugging in for the initial condition gives:

$$\vec{z}(t) = \begin{bmatrix} e^{-5t} \\ 2e^{-2t} \end{bmatrix}$$

e) Convert your solution back into the original coordinates to find $x_i(t)$.

Answer

$$\vec{x}(t) = V\vec{z}(t) = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} e^{-5t} \\ 2e^{-2t} \end{bmatrix} = \begin{bmatrix} e^{-5t} + 2e^{-2t} \\ -e^{-5t} + 4e^{-2t} \end{bmatrix}$$

f) We can solve this equation using a slightly shorter approach by observing that the solutions for $x_i(t)$ will all be of the form

$$x_i(t) = \sum_k c_k e^{\lambda_k t}$$

where λ_k is an eigenvalue of our differential equation relation matrix A.

Since we have observed that the solutions will include $e^{\lambda_i t}$ terms, once we have found the eigenvalues for our differential equation matrix, we can guess the forms of the $x_i(t)$ as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t} \\ \beta_1 e^{\lambda_1 t} + \beta_2 e^{\lambda_2 t} \end{bmatrix}$$

where $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2$ are all constants.

Take the derivative to write out

$$\begin{bmatrix} \frac{d}{dt} x_1(t) \\ \frac{d}{dt} x_2(t) \end{bmatrix}.$$

and connect this to the given differential equation.

Solve for $x_i(t)$ from this form of the derivative.

Answer

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \alpha_1 e^{-5t} + \alpha_2 e^{-2t} \\ \beta_1 e^{-5t} + \beta_2 e^{-2t} \end{bmatrix}$$

With initial condition

$$\vec{x}(0) = \begin{bmatrix} \alpha_1 + \alpha_2 \\ \beta_1 + \beta_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Taking the derivative $\frac{d}{dt}\vec{x}(t)$ will be

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} -5\alpha_1 e^{-5t} - 2\alpha_2 e^{-2t} \\ -5\beta_1 e^{-5t} - 2\beta_2 e^{-2t} \end{bmatrix}$$

At $t = 0$, the derivative will be

$$\frac{d}{dt}\vec{x}(0) = \begin{bmatrix} -5\alpha_1 - 2\alpha_2 \\ -5\beta_1 - 2\beta_2 \end{bmatrix}$$

We also have that:

$$\frac{d}{dt}\vec{x}(0) = A\vec{x}(0) = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -4x_1(0) + x_2(0) \\ 2x_1(0) - 3x_2(0) \end{bmatrix} = \begin{bmatrix} -9 \\ -3 \end{bmatrix}$$

Equating terms:

$$\begin{aligned} \alpha_1 + \alpha_2 &= 3 \\ -5\alpha_1 - 2\alpha_2 &= -9 \\ \beta_1 + \beta_2 &= 3 \\ -5\beta_1 - 2\beta_2 &= -3 \end{aligned}$$

This gives:

$$\begin{aligned} \alpha_1 &= 1, \alpha_2 = 2 \\ \beta_1 &= -1, \beta_2 = 4 \end{aligned}$$

Therefore, we can write out $x_1(t)$ and $x_2(t)$ as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-5t} + 2e^{-2t} \\ -e^{-5t} + 4e^{-2t} \end{bmatrix}$$