Rational Transfer Functions

When we write the transfer function of an arbitrary circuit, it always takes the following form. This is called a "rational transfer function." We also like to factor the numerator and denominator, so that they become easier to work with and plot:

$$H(\omega) = K\frac{(j\omega)^{N_{z0}}\left(1+j\frac{\omega}{\omega_{z1}}\right)\left(1+j\frac{\omega}{\omega_{z2}}\right)\cdots\left(1+j\frac{\omega}{\omega_{zn}}\right)}{(j\omega)^{N_{p0}}\left(1+j\frac{\omega}{\omega_{p1}}\right)\left(1+j\frac{\omega}{\omega_{p2}}\right)\cdots\left(1+j\frac{\omega}{\omega_{pm}}\right)}$$

Here, we define the constants ω_z as "zeros" and ω_p as "poles."

Bode Plots

Bode plots provide us with a simple and easy tool to plot these transfer functions by hand. Always remember that Bode plots are an approximation; if you want the precisely correct plots, you need to use numerical methods (like solving using MATLAB or IPython).

When we make Bode plots, we plot the frequency and magnitude on a logarithmic scale, and the angle in either degrees or radians. We use the logarithmic scale because it allows us to break up complex transfer functions into its constituent components.

For two transfer functions $H_1(\omega)$ and $H_2(\omega)$, if $H(\omega) = H_1(\omega) \cdot H_2(\omega)$,

$$\log\left|H(\omega)\right| = \log\left|H_1(\omega)\cdot H_2(\omega)\right| = \log\left|H_1(\omega)\right| + \log\left|H_2(\omega)\right| \tag{1}$$

$$\angle H(\omega) = \angle(H_1(\omega)\cdot H_2(\omega)) = \angle H_1(\omega) + \angle H_2(\omega) \tag{2}$$

Decibels

We define the decibel as the following:

$$20\log_{10}(|H(\omega)|) = |H(\omega)|\ [\mathrm{dB}]$$

This means that $20\,\mathrm{dB}$ per decade is equivalent to one order of magnitude. **We won't be using dB when plotting, but understanding the conversion to dB will help when reading the Bode plot sheet on the next page.**

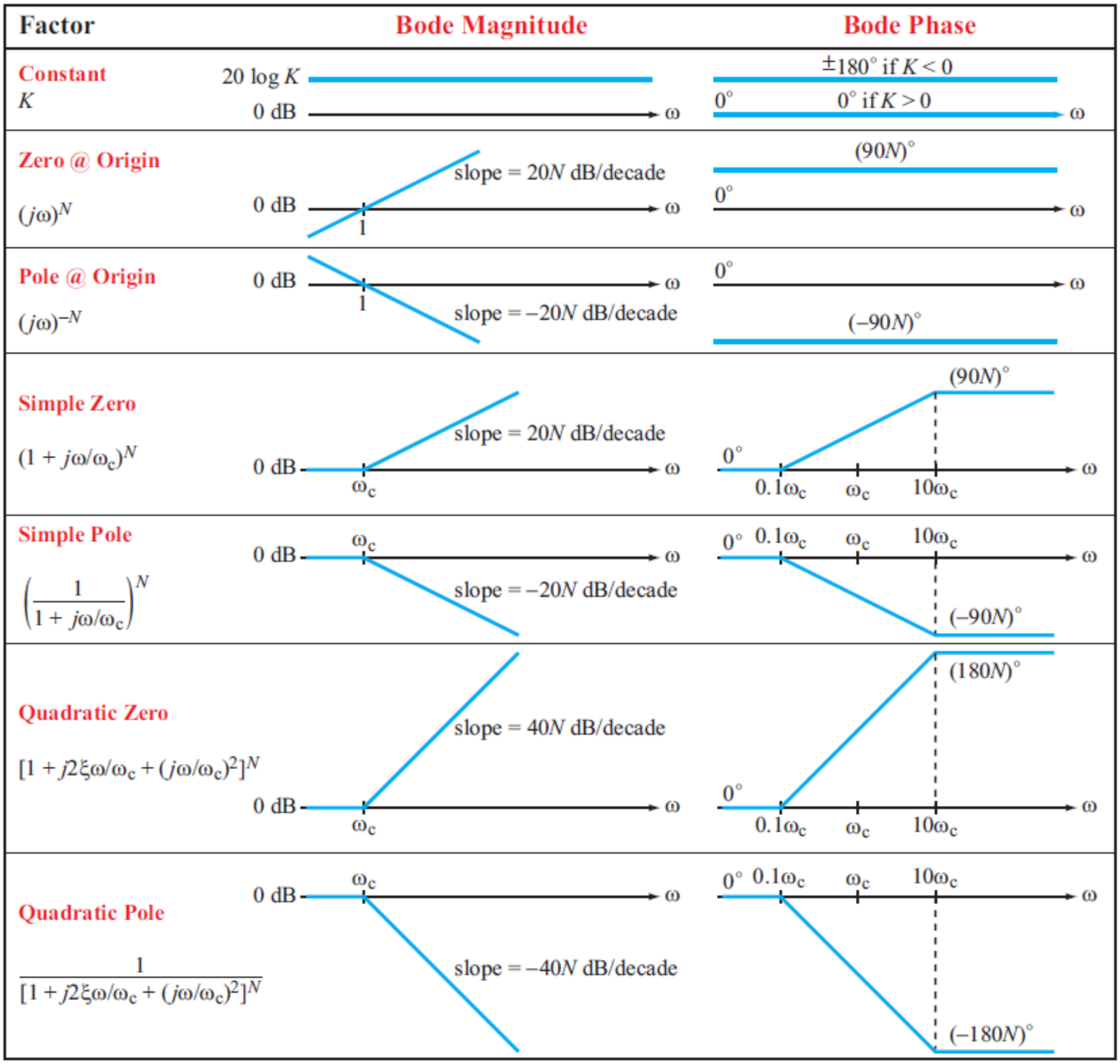
Algorithm

Given a frequency response $H(\omega)$,

a) Break $H(\omega)$ into a product of poles and zeros and put it in "rational transfer function" form. Appropriately divide terms to reduce $H(\omega)$ into one of the given forms. We determine ω_c by reducing a term into one of the above forms.

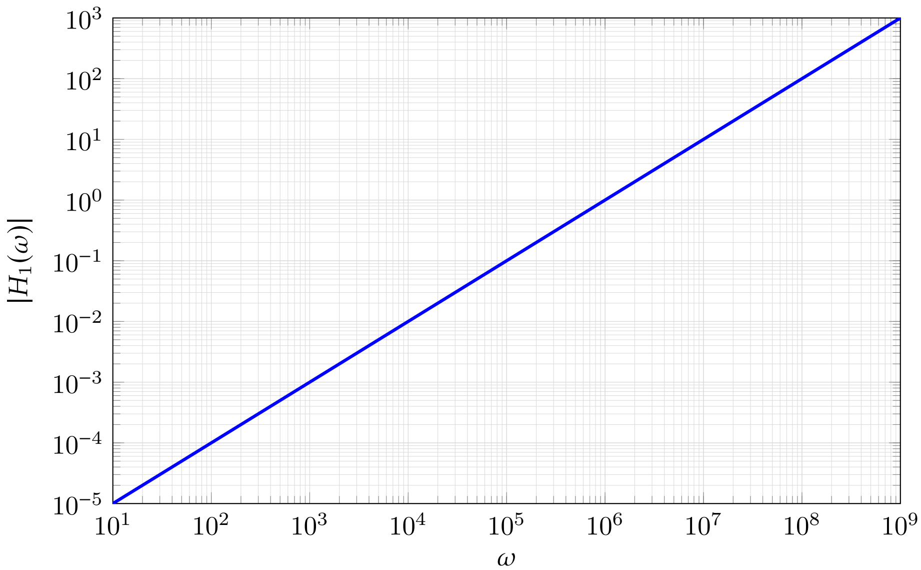
b) Draw out the Bode plot for each pole and zero in the product above.

c) Add the resulting plots to get the final Bode plot.


Factor
Bode Magnitude
Bode Phase
Constant
K
20 log K
0 dB
ω
±180° if K < 0
0°
0° if K > 0
ω
Zero @ Origin
(jω)^N
0 dB
1
slope = 20N dB/decade
ω
(90N)°
0°
ω
Pole @ Origin
(jω)^−N
0 dB
1
slope = −20N dB/decade
ω
0°
(−90N)°
ω
Simple Zero
(1 + jω/ωc)^N
0 dB
ωc
slope = 20N dB/decade
ω
(90N)°
0°
0.1ωc
ωc
10ωc
ω
Simple Pole
(1/(1 + jω/ωc))^N
0 dB
ωc
slope = −20N dB/decade
ω
0°
0.1ωc
ωc
10ωc
(−90N)°
ω
Quadratic Zero
[1 + j2ξω/ωc + (jω/ωc)²]^N
0 dB
ωc
slope = 40N dB/decade
ω
(180N)°
0°
0.1ωc
ωc
10ωc
ω
Quadratic Pole
1/[1 + j2ξω/ωc + (jω/ωc)²]^N
0 dB
ωc
slope = −40N dB/decade
ω
0°
0.1ωc
ωc
10ωc
(−180N)°
ω

1 Bode Plot Practice

a) **Identify the locations of the poles and zeroes in the following magnitude Bode plot. What transfer function $H_1(\omega)$ would result in this plot?**

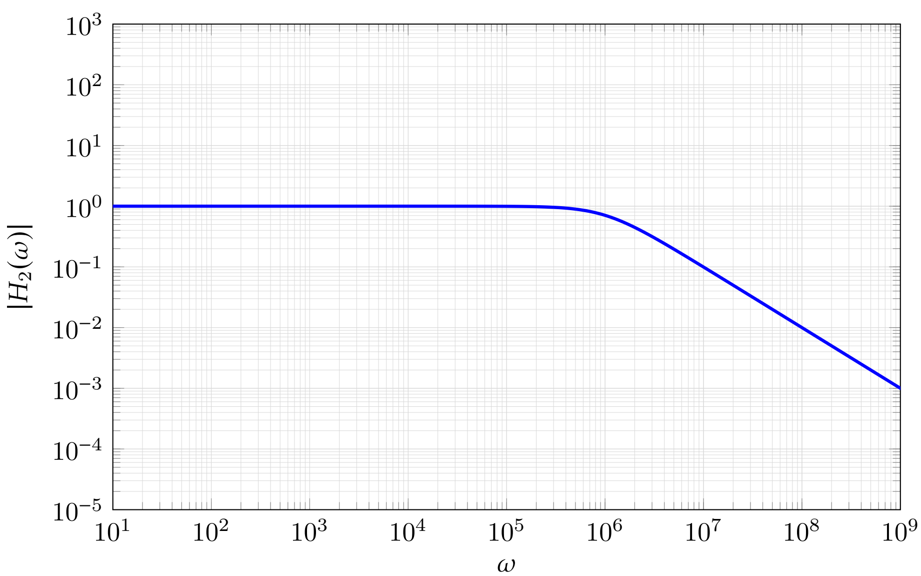


Answer

This is a zero at the origin, scaled by 10^{-6}. Notice how the scaling translated the original line instead of changing the slope, because of the logarithm property $\ln K\omega = \ln\omega + \ln K$.

$$H_1(\omega) = \frac{j\omega}{10^6}$$

b) **Identify the locations of the poles and zeroes in the following magnitude Bode plot. What transfer function $H_2(\omega)$ would result in this plot?**



Answer

There is a single pole at $\omega_p = 10^6 \frac{\text{rad}}{\text{s}}$. The transfer function is thus:

$$H_2(\omega) = \frac{1}{1 + \frac{j\omega}{10^6}}$$

This may also be recognized as the familiar form of a simple low pass filter.

c) **Identify the locations of the poles and zeroes in the following transfer function. Then sketch the magnitude Bode plot.**

$$H_3(\omega) = \frac{\frac{j\omega}{10^6}}{1 + \frac{j\omega}{10^6}}$$

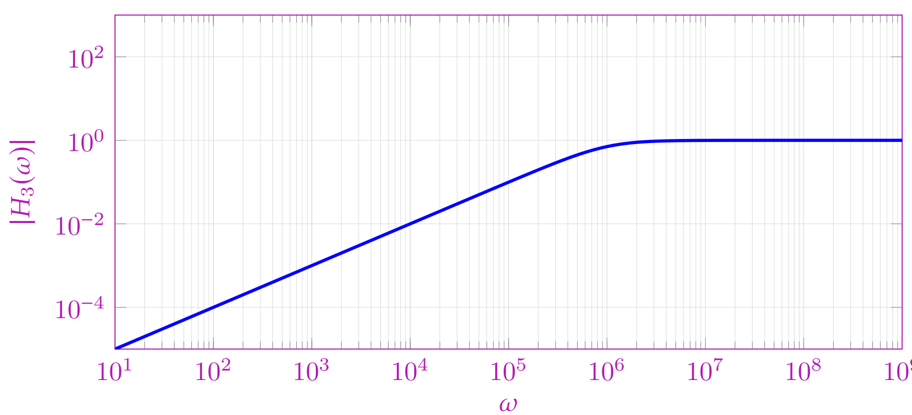
Answer

(1) Poles and Zeroes
There is a single zero at the origin ($\omega = 0$), and single pole at $\omega_{p1} = 10^6$.

(2) Constant K
Since $K = 10^{-6}$, and we have a zero at the origin, this means that our entire plot is shifted downward by 10^{-6}.

(3) Plotting
The plot should start with a slope of 1 since there is a zero at the origin. Since $K = 10^{-6}$, and there are no poles or zeros before 10^{-6}, we can say that $H(10^6) \approx 1$. Then we have a single pole at $\omega_{p1} = 10^6$ so this should cancel out the +1 slope from the zero and the magnitude plot will stay constant at 1.

There is a zero at the origin and a pole at $\omega_p = 10^6 \frac{\text{rad}}{\text{s}}$. The whole transfer function is scaled by 10^{-6} so that $|H_3(\omega >> 10^6)| \approx 1$ in the passband. This may also be recognized as the form of a simple high pass filter, created by cascading parts (a) and (b).



d) **Identify the locations of the poles and zeroes in the following magnitude Bode plot. Then sketch the magnitude Bode plot.**

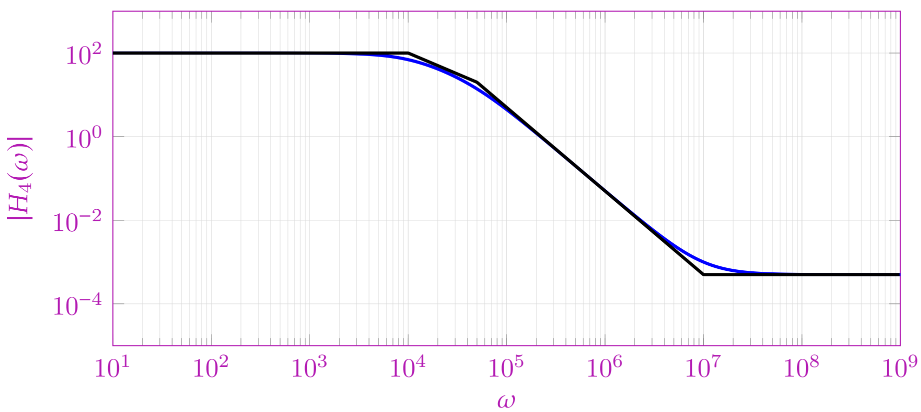
$$H_4(\omega) = 100\frac{(1 + \frac{j\omega}{10^7})^2}{(1 + \frac{j\omega}{10^4})(1 + \frac{j\omega}{5\times10^4})}$$

Answer

(1) Poles and Zeroes
There is a double zero at $\omega_{z1} = 10^7$ and two poles at $\omega_{p1} = 10^4$ and $\omega_{p2} = 5 \cdot 10^4$.

(2) Constant K
Since $K = 100$, the entire plot should be shifted up by a factor of 100. However, since there are no zeros at the origin, this means that $H(0) = K = 100$ and our Bode plot will start at 100

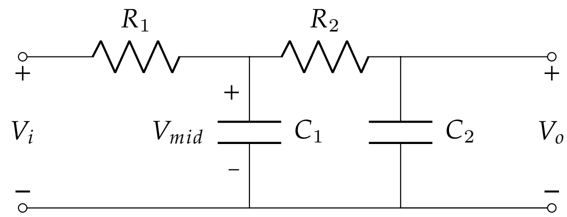
(3) Plotting
The plot should start at 100. Then at $\omega_{p1} = 10^4$, it drops off with slope -1 and at the second pole ω_{p2} it drops off even quicker with slope -2. then at $\omega_{z1} = 10^7$, the double zero cancels out the -2 slope and the magnitude plot stays constant.



Since the two poles are spaced closely, it can be difficult to determine exactly where the second pole is located. As a result, Bode plots are often drawn using straight line approximations (depicted in the above figure). For many practical applications, this is sufficient and has the added benefit that changes in slope are sharply delineated, making it easier to identify frequencies of interest. However, it is important to note that ultimately straight lines are an approximation and in certain cases they can fail to convey crucial information, with one particularly relevant example being resonant RLC circuits.

2 Transfer functions and why loading is annoying

Consider the circuit below.



The circuit has an input phasor voltage $\widetilde{V}_i$ at frequency ω rad/sec applied at the input terminals shown in the illustration above, causing an output phasor voltage $\widetilde{V}_o$ at output terminals.

a) We are going to construct the transfer function $H(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i}$ in two steps. We will compute two intermediate transfer functions, $H_1(\omega) = \frac{\widetilde{V}_{mid}}{\widetilde{V}_i}$ and $H_2(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_{mid}}$. Then, we will find the overall transfer function as the product of these two intermediate transfer functions, i.e. $H(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i} = H_1(\omega)H_2(\omega)$. This approach is valid since the $\widetilde{V}_{mid}$ cancel. For the first step, **find the intermediate transfer function** $H_2(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_{mid}}$**.** Have your expression be in terms of Z_{R2} and Z_{C2}, that is the impedances of R_2 and C_2.

Answer

V_{mid} and V_o are in a voltage divider configuration, with impedances Z_{R2} and Z_{C2}. This means

$$V_o = (V_{mid})\frac{Z_{C2}}{Z_{R2} + Z_{C2}}, \tag{3}$$

so we can say

$$H_2(\omega) = \frac{V_o}{V_{mid}} = \frac{Z_{C2}}{Z_{R2} + Z_{C2}} = \frac{1}{1 + j\omega R_2 C_2}. \tag{4}$$

b) Now, **compute the other intermediate transfer function** $H_1(\omega) = \frac{\widetilde{V}_{mid}}{\widetilde{V}_i}$. Have your expression be in terms of Z_{R1}, Z_{R2}, Z_{C1}, and Z_{C2}. (i.e. Don't forget to consider the impact of loading by R_2 and C_2 in this transfer function.)

Answer

The combined parallel impedance is

$$\frac{1}{j\omega C_1} \parallel \left(\frac{1}{j\omega C_2} + R_2\right) = \frac{\frac{1}{j\omega C_1}\left(\frac{1}{j\omega C_2} + R_2\right)}{\frac{1}{j\omega C_1} + \frac{1}{j\omega C_2} + R_2} = \frac{1 + j\omega R_2 C_2}{-\omega^2 R_2 C_1 C_2 + j\omega(C_2 + C_1)}$$

Therefore, the transfer function from V_i to V_{mid} can be computed using a Voltage Divider

$$H_1(\omega) = \frac{\widetilde{V}_{mid}}{\widetilde{V}_i} = \frac{\left(\frac{1}{j\omega C_2} + R_2\right) \parallel \frac{1}{j\omega C_1}}{R_1 + \left(\frac{1}{j\omega C_2} + R_2\right) \parallel \frac{1}{j\omega C_1}} = \frac{1 + j\omega R_2 C_2}{1 - \omega^2 R_1 R_2 C_1 C_2 + j\omega(R_2 C_2 + R_1 C_1 + R_1 C_2)}$$

Alternatively, from KCL, we have

$$\frac{V_{mid} - V_i}{Z_{R1}} + \frac{V_{mid}}{Z_{C1}} + \frac{V_{mid}}{Z_{R2} + Z_{C2}} = 0 \tag{5}$$

$$V_{mid}\left(\frac{1}{Z_{R1}} + \frac{1}{Z_{C1}} + \frac{1}{Z_{R2} + Z_{C2}}\right) = V_i\left(\frac{1}{Z_{R1}}\right) \tag{6}$$

$$H_1(\omega) = \frac{V_{mid}}{V_i} = \frac{1}{Z_{R1}\left(\frac{1}{Z_{R1}} + \frac{1}{Z_{C1}} + \frac{1}{Z_{R2}+Z_{C2}}\right)} \tag{7}$$

c) Then, **use these two intermediate transfer functions to calculate the overall transfer function** as $H(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i} = H_1(\omega)H_2(\omega)$.

Answer

Combining the above formulas, we have

$$\begin{aligned} H(\omega) &= H_1(\omega) \cdot H_2(\omega) \\ &= \frac{1}{1 + j\omega R_2C_2} \cdot \frac{1 + j\omega R_2C_2}{1 - \omega^2 R_1R_2C_1C_2 + j\omega(R_2C_2 + R_1C_1 + R_1C_2)} \\ &= \frac{1}{1 - \omega^2 R_1R_2C_1C_2 + j\omega(R_2C_2 + R_1C_1 + R_1C_2)} \end{aligned}$$

Alternatively, if we used KCL to find $H_2(\omega)$,

$$\begin{aligned} H(\omega) &= \frac{Z_{C2}}{Z_{R2} + Z_{C2}} \frac{1}{Z_{R1}\left(\frac{1}{Z_{R1}} + \frac{1}{Z_{C1}} + \frac{1}{Z_{R2}+Z_{C2}}\right)} && (8) \\ &= \frac{Z_{C2}}{Z_{R2} + Z_{C2}} \frac{1}{1 + \frac{Z_{R1}}{Z_{C1}} + \frac{Z_{R1}}{Z_{R2}+Z_{C2}}} && (9) \\ &= \frac{Z_{C2}}{Z_{R2} + Z_{C2} + \frac{Z_{R2}+Z_{C2}}{Z_{C1}}Z_{R_1} + Z_{R1}} && (10) \\ &= \frac{Z_{C1}Z_{C2}}{Z_{R1}Z_{C1} + Z_{C1}Z_{C2} + Z_{R1}Z_{C2} + Z_{R2}Z_{C1} + Z_{R1}Z_{R2}} && (11) \\ &= \frac{\frac{1}{j\omega C_1}\frac{1}{j\omega C_2}}{\frac{R_1}{j\omega C_1} + \frac{1}{j\omega C_1}\frac{1}{j\omega C_2} + \frac{R_1}{j\omega C_2} + \frac{R_2}{j\omega C_1} + R_1R_2} && (12) \\ &= \frac{1}{1 + j\omega\big(R_1C_1 + (R_1 + R_2)C_2\big) + (j\omega)^2 R_1R_2C_1C_2}. && (13) \end{aligned}$$

d) Sometimes it is useful to collect all the frequency dependence into one place and to figure out how to think about what scale might be somewhat natural for the frequency. **Obtain an expression for** $H(\omega) = \widetilde{V}_o/\widetilde{V}_i$ **in the form**

$$H(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i} = \frac{1}{1 + 2\xi\frac{j\omega}{\omega_c} + \frac{(j\omega)^2}{\omega_c^2}},$$

given that $R_1 = 2\,\Omega$, $R_2 = 4\,\Omega$, $C_1 = \frac{9}{2}\,\text{F}$, and $C_2 = 1\,\text{F}$. What are the values of ξ and ω_c?

Answer

Using the result in part (d)

$$H(\omega) = \frac{1}{1 + j\omega\big(R_1C_1 + (R_1 + R_2)C_2\big) + (j\omega)^2 R_1R_2C_1C_2} \tag{14}$$

$$= \frac{1}{1 + j\omega(15) + (j\omega)^2(36)} \tag{15}$$

$$= \frac{1}{1 + 2(15/12)\frac{j\omega}{1/6} + (\frac{j\omega}{1/6})^2}, \tag{16}$$

Which means $\xi = \frac{15}{12} = \frac{5}{4}$ and $\omega_c = \frac{1}{6}$.

e) **For the previous case, what is the magnitude of the transfer function at the $\omega = \omega_c$ you calculated?**

This is here so that you can see that just because we called it ω_c doesn't mean that the amplitude here is $\frac{1}{\sqrt{2}}$.

Answer

At $\omega = \omega_c$, the transfer function becomes:

$$H(\omega_c) = \frac{1}{2\xi j}$$

Thus using $\xi = \frac{5}{4}$, the magnitude of the transfer function is

$$|H(\omega_c)| = \frac{1}{2\xi} = \frac{1}{2 \cdot \frac{5}{4}} = \frac{4}{10} = 0.4$$

f) We can express the transfer function $H(\omega)$ in the polar form. That is,

$$H(\omega) = M(\omega)e^{j\phi(\omega)}$$

The functions $M(\omega)$ and $\phi(\omega)$ are the magnitude and the phase angle of $H(\omega)$, respectively. **Write down $M(\omega)$ and $\phi(\omega)$ using the transfer function you derived in part (b).**

Answer

Rewriting the result in part (b), we have

$$H(\omega) = \frac{1}{1 - 36\omega^2 + 15j\omega} \tag{17}$$

Taking the magnitude, we have

$$M(\omega) = \frac{1}{\sqrt{(1 - 36\omega^2)^2 + (15\omega)^2}} \tag{18}$$

We can find the phase of the transfer function as the negative of the phase of the denominator, i.e.

$$\angle H(\omega) = -\angle(1 - 36\omega^2 + 15j\omega) = -\text{atan2}(15\omega, 1 - 36\omega^2). \tag{19}$$

A solution using $\tan^{-1}(\cdot)$ is also acceptable, but you have to be careful: it is only correct for complex numbers in the first or the fourth quadrants of the complex plane. For complex numbers in the second and the third quadrants, we need to shift their $\tan^{-1}(\cdot)$ by π. In other words, we need

$$\phi(\omega) = \begin{cases} -\tan^{-1}\left(\frac{15\omega}{1-36\omega^2}\right) & \text{when } 0 \leq \omega \leq \frac{1}{6} \\ -\tan^{-1}\left(\frac{15\omega}{1-36\omega^2}\right) - \pi & \text{when } \omega > \frac{1}{6} \end{cases} \tag{20}$$

Note that $1 - 36\omega^2 + j15\omega$ is in the first quadrant when $\omega < 1/6$ and in the second quadrant when $\omega > 1/6$.

g) Use a computer and then **draw Bode Plots of $|H(\omega)|$ and $\angle H(\omega)$.**

Answer

The plots for $M(\omega)$ and $\phi(\omega)$ are shown below in black. We've also showed the Bode plots if there were a unity gain buffer in between the high-pass and low-pass as a reference in blue.