1 Transfer Function

When we write the transfer function of an arbitrary circuit, it always takes the following form. This is called a "rational transfer function." We also like to factor the numerator and denominator, so that they become easier to work with and plot:

$$\begin{aligned} H(\omega) = \frac{z(\omega)}{p(\omega)} &= \frac{(j\omega)^{N_{z0}}}{(j\omega)^{N_{p0}}} \left(\frac{(j\omega)^n \alpha_n + (j\omega)^{n-1}\alpha_{n-1} + \cdots + j\omega\alpha_1 + \alpha_0}{(j\omega)^m \beta_m + (j\omega)^{m-1}\beta_{m-1} + \cdots + j\omega\beta_1 + \beta_0} \right) \\ &= K \frac{(j\omega)^{N_{z0}} \left(1 + j\frac{\omega}{\omega_{z1}}\right)\left(1 + j\frac{\omega}{\omega_{z2}}\right)\cdots\left(1 + j\frac{\omega}{\omega_{zn}}\right)}{(j\omega)^{N_{p0}} \left(1 + j\frac{\omega}{\omega_{p1}}\right)\left(1 + j\frac{\omega}{\omega_{p2}}\right)\cdots\left(1 + j\frac{\omega}{\omega_{pm}}\right)} \end{aligned}$$

Here, we define the constants ω_z as "zeros" and ω_p as "poles", and N_{z0}, N_{p0} are the number of zeros and poles at $\omega = 0$

2 Low-pass Filter

You have a 1 kΩ resistor and a 1 µF capacitor wired up as a low-pass filter.

a) **Draw the filter circuit, labeling the input node, output node, and ground.**

Answer

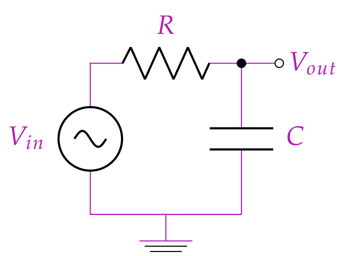


Figure 1: A simple RC circuit

b) **Write down the transfer function of the filter, $H(j\omega)$ that relates the output voltage phasor to the input voltage phasor.** Be sure to use the given values for the components.

Answer

First, we convert everything into the phasor domain. We have,

$$Z_R = R = 1 \times 10^3 \Omega \tag{1}$$

$$Z_C = \frac{1}{j\omega C} = \frac{1}{j\omega \times 10^{-6}} \text{F} \tag{2}$$

In phasor domain, we can treat these impedances essentially like we treat resistors and recognize the voltage divider. Hence,

$$\widetilde{V}_{out} = \frac{Z_C}{Z_C + Z_R}\widetilde{V}_{in} \tag{3}$$

$$\frac{\widetilde{V}_{out}}{\widetilde{V}_{in}} = H(\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \tag{4}$$

$$= \frac{1}{1 + j\omega RC} \tag{5}$$

$$= \frac{1}{1 + j\omega / \frac{1}{RC}} \tag{6}$$

$$= \frac{1}{1 + j\omega \times 10^{-3}} \tag{7}$$

$$\tag{8}$$

Hence, the corner frequency $\omega_C = \frac{1}{RC} = \frac{1}{10^3 \cdot 10^{-6}} = 10^3$ rad/sec.

c) **Write an exact expression for the *magnitude* of $H(j\omega = j10^2)$, and give an approximate numerical answer.**

Answer

We obtain this expression for the transfer function's magnitude:

$$|H(\omega)| = \frac{1}{\sqrt{1 + \omega^2/\omega_c^2}} = \frac{\sqrt{1 + \omega^2/\omega_c^2}}{1 + \omega^2/\omega_c^2}$$

Plugging in for $\omega = 10^2$:

$$|H(\omega = 10^2)| = \frac{\sqrt{1 + (10^2)^2/(10^3)^2}}{1 + (10^2)^2/(10^3)^2}$$

$$|H(\omega = 10^2)| = \frac{\sqrt{1 + 10^{-2}}}{1 + 10^{-2}}$$

Approximately:

$$|H(\omega = 10^2)| \approx \frac{\sqrt{1}}{1} = 1$$

$$|H(\omega = 10^2)| \approx 1$$

Note that for frequencies $\omega << \omega_c$, $(\omega/\omega_c)^2 \approx 0$ so the magnitude is approximately $|H(\omega << \omega_c)| \approx \frac{1}{\sqrt{1+0}} = 1$.

d) **Write an exact expression for the *magnitude* of $H(j\omega = j10^6)$, and give an approximate numerical answer.**

Answer

We obtain this expression for the transfer function's magnitude:

$$|H(\omega)| = \frac{1}{\sqrt{1 + \omega^2/\omega_c^2}} = \frac{\sqrt{1 + \omega^2/\omega_c^2}}{1 + \omega^2/\omega_c^2}$$

Plugging in for $\omega = 10^6$:

$$|H(\omega = 10^6)| = \frac{\sqrt{1 + (10^6)^2/(10^3)^2}}{1 + (10^6)^2/(10^3)^2}$$

$$|H(\omega = 10^6)| = \frac{\sqrt{1 + 10^6}}{1 + 10^6}$$

Approximately:

$$|H(\omega = 10^6)| \approx \frac{\sqrt{10^6}}{10^6} = \frac{10^3}{10^6} = 10^{-3}$$

$$|H(\omega = 10^6)| \approx 10^{-3}$$

Note that for frequencies $\omega >> \omega_c$, $(\omega/\omega_c)^2 >> 1$ so the magnitude is approximately $|H(\omega >> \omega_c)| \approx \frac{1}{\sqrt{\omega^2/\omega_c^2}} = \frac{\omega_c}{\omega}$

e) **Write an exact expression for the *phase* of $H(j\omega = j1)$, and give an approximate numerical answer.**

Answer

We obtain this expression for the transfer function's phase:

$$\angle H(\omega) = -\text{atan2}(\frac{\omega}{\omega_c}, 1)$$

Plugging in for $\omega = 1$:

$$\angle H(\omega = 1) = -\text{atan2}(\frac{10^0}{10^3}, 1) = -\tan^{-1}(10^{-3})$$

By the small angle approximation, this is:

$$\angle H(\omega = 1) \approx -10^{-3}\text{rad}$$

f) **Write an exact expression for the *phase* of $H(j\omega = j10^6)$, and give an approximate numerical answer.**

Answer

We obtain this expression for the transfer function's phase:

$$\angle H(\omega) = -\text{atan2}(\frac{\omega}{\omega_c}, 1)$$

Plugging in for $\omega = 10^6$:

$$\angle H(\omega = 10^6) = -\text{atan2}(\frac{10^6}{10^3}, 1) = -\tan^{-1}(10^3)$$

Noting that $\tan^{-1}(10^3) = \frac{\pi}{2}$:

$$\angle H(\omega = 10^6) \approx -\frac{\pi}{2}\text{rad}$$

g) **Write down an expression for the *corner frequency* ω_c of this circuit. Evaluate the *magnitude* and *phase* of $H(j\omega = j\omega_c)$.**

Answer

Reading off the transfer function, we obtain the corner frequency:

$$\omega_C = \frac{1}{RC} = \frac{1}{10^3 \cdot 10^{-6}} = 10^3 \frac{\text{rad}}{\text{s}}$$

Plugging in for $\omega = \omega_c$, the magnitude is exactly:

$$|H(\omega = \omega_c)| = \frac{1}{\sqrt{1 + \omega_c^2/\omega_c^2}} = \frac{1}{\sqrt{1+1^2}} = \frac{1}{\sqrt{2}}$$

The phase is exactly:

$$\angle H(\omega = \omega_c) = -\text{atan2}(\frac{\omega_c}{\omega_c}, 1) = -\tan^{-1}(1) = -\frac{\pi}{4}\text{rad}$$

h) **Write down an expression for the time-domain output waveform $V_{out}(t)$ of this filter if the input voltage is $V(t) = 1\sin(1000t)$ V.** You can assume that any transients have died out — we are interested in the steady-state waveform.

Answer

We have already evaluated the transfer function at the corner frequency $\omega_c = 10^3 \frac{\text{rad}}{\text{s}}$:

$$|H(\omega = 10^3)| = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

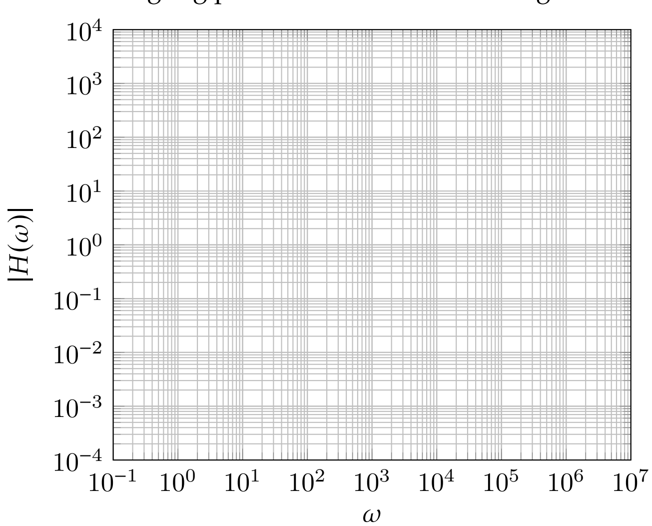
$$\angle H(\omega = 10^3) = \text{atan2}(-1, 1) = -\frac{\pi}{4}$$

Therefore the output will be:

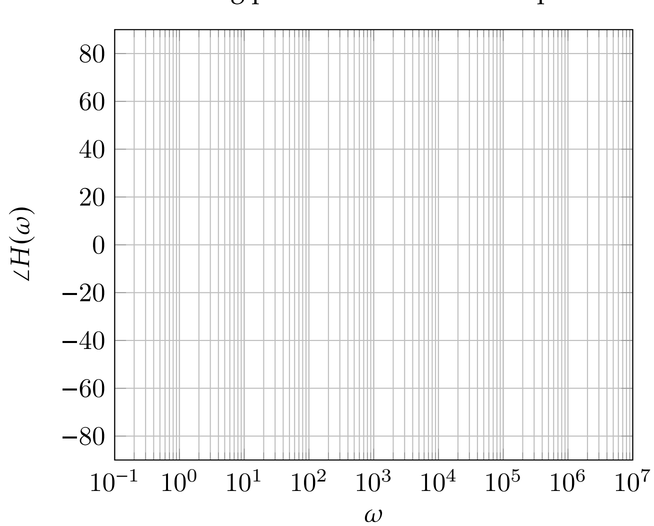
$$V_{out}(t) = \frac{1}{\sqrt{2}} \sin\left(1000t - \frac{\pi}{4}\right)$$

i) **Based off the values calculated in the previous parts, predict what the Bode plots (both magnitude and phase) of the filter will look like and sketch them on the graph paper below. You may use a straight line approximation.**

Log-log plot of transfer function magnitude



Semi-log plot of transfer function phase

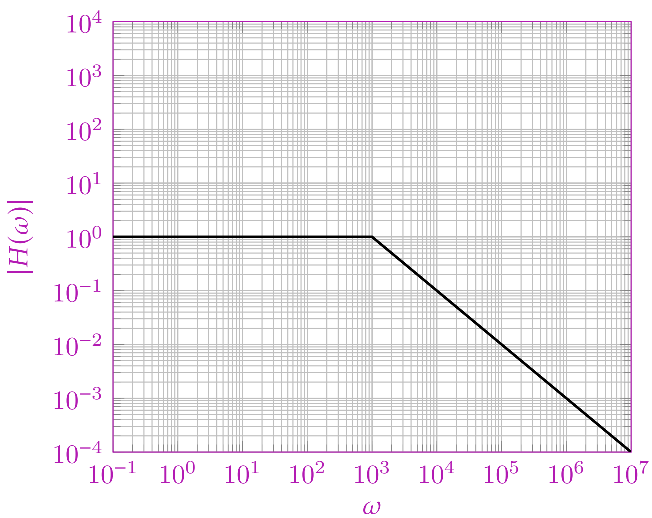


Answer

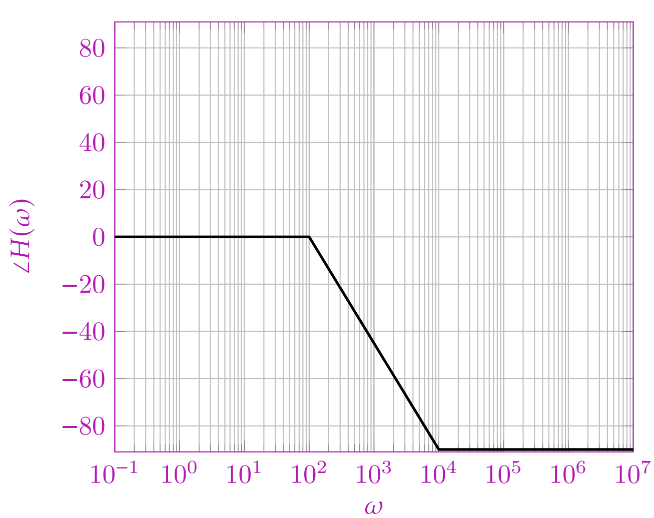
Magnitude: We calculated earlier that for frequencies $\omega << \omega_c$, the magnitude is approximately $|H(\omega << \omega_c)| \approx 1$. For frequencies $\omega >> \omega_c$, the magnitude is approximately 0 and in fact drops off at a slope of -1 (in a log-log plot): $|H(\omega >> \omega_c)| \approx \frac{\omega_c}{\omega}$. If we approximate the corner frequency ω_c as a sharp boundary between the passband and the stopband, then we can sketch the Bode plot as a flat line for $\omega < \omega_c$ and a line with slope -1 for $\omega > \omega_c$.

Phase: The phase plot is a bit trickier. We calculated earlier that for frequencies $\omega << \omega_c$, the phase $\angle H(\omega << \omega_c)$ is approximately 0 whereas for frequencies $\omega >> \omega_c$, the phase $\angle H(\omega >> \omega_c)$ is approximately $-90°$. Moreover, at the corner frequency ω_c, the phase is $-45°$. So we expect that the phase plot should be flat line at 0 for some time, decrease to $-90°$, and then be flat again at $-90°$. To determine when approximately the phase starts decreasing, we employ the small angle approximation used in part (e): for $\omega << \omega_c$, $\angle H(\omega) \approx -\frac{\omega}{\omega_c}$. Picking $\omega = \omega_c/10$ returns $\angle H(\omega = \omega_c/10) \approx -0.1\text{rad} \approx -6°$. Correspondingly, $\angle H(\omega = 10\omega_c) \approx -(90° - 6°) = -84°$. So $\omega = \omega_c/10$ is a good approximation for the threshold when the phase starts dropping, and $\omega = 10\omega_c$ is a good approximation for the threshold when the phase becomes flat again.

Log-log plot of transfer function magnitude

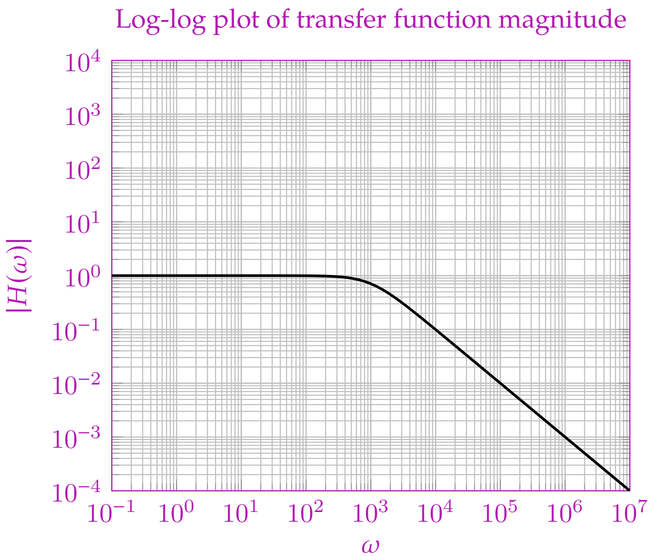


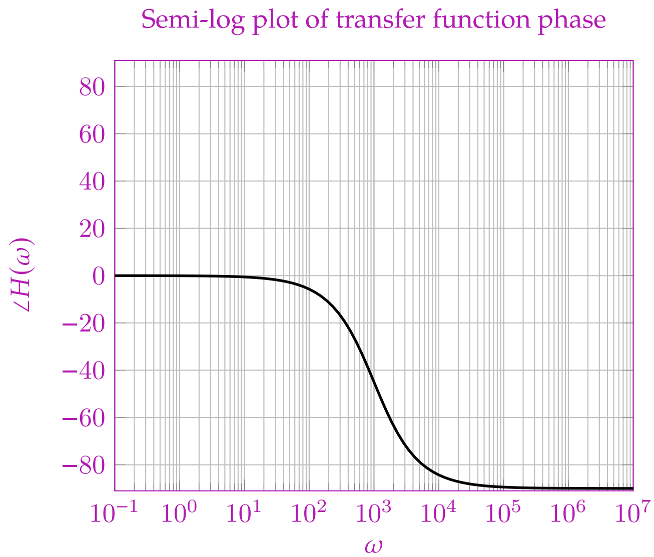
Semi-log plot of transfer function phase



j) **Use a computer to draw the Bode plots (both magnitude and phase) of the filter. Compare them to your sketch from before.**

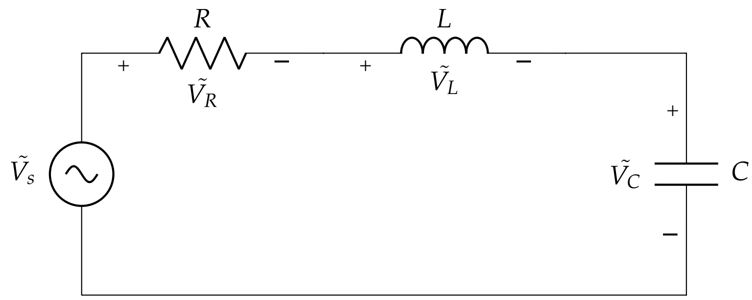
Answer


Log-log plot of transfer function magnitude
$|H(\omega)|$
10^{4} 10^{3} 10^{2} 10^{1} 10^{0} 10^{-1} 10^{-2} 10^{-3} 10^{-4}
10^{-1} 10^{0} 10^{1} 10^{2} 10^{3} 10^{4} 10^{5} 10^{6} 10^{7}
ω


Semi-log plot of transfer function phase
$\angle H(\omega)$
80 60 40 20 0 −20 −40 −60 −80
10^{-1} 10^{0} 10^{1} 10^{2} 10^{3} 10^{4} 10^{5} 10^{6} 10^{7}
ω

3 RLC Circuit

In this question, we will take a look at an electrical systems described by second-order differential equations and analyze it in the phasor domain. Consider the circuit below where $\tilde{V}_s$ is a sinusoidal signal, $L = 1\,\text{mH}$, and $C = 1\,\text{nF}$:



a) **Transform the circuit into the phasor domain.**

Answer

$$Z_R = R$$
$$Z_L = j\omega L$$
$$Z_C = \frac{1}{j\omega C}$$

b) **Solve for the transfer function $H_C(\omega) = \frac{\widetilde{V}_C}{\widetilde{V}_s}$ in terms of R, L, and C.**

Answer

$\widetilde{V}_C$ is a voltage divider where the output voltage is taken across the capacitor.

$$\widetilde{V}_C = \frac{Z_C}{Z_R + Z_L + Z_C}\widetilde{V}_s,$$
$$H_C(\omega) = \frac{Z_C}{Z_R + Z_L + Z_C} = \frac{\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}}.$$

Multiplying the numerator and denominator by $j\omega C$ gives

$$H_C(\omega) = \frac{1}{(j\omega)^2 LC + j\omega RC + 1}$$

c) **Solve for the transfer function $H_L(\omega) = \frac{\widetilde{V}_L}{\widetilde{V}_s}$ in terms of R, L, and C.**

Answer

$\widetilde{V}_L$ is a voltage divider where the output voltage is taken across the inductor.

$$\widetilde{V}_L = \frac{Z_L}{Z_R + Z_L + Z_C}\widetilde{V}_s,$$
$$H_L(\omega) = \frac{Z_L}{Z_R + Z_L + Z_C} = \frac{j\omega L}{R + j\omega L + \frac{1}{j\omega C}}.$$

Multiplying the numerator and denominator by $j\omega C$ gives

$$H_L(\omega) = \frac{(j\omega)^2 LC}{(j\omega)^2 LC + j\omega RC + 1}$$

d) **Solve for the transfer function $H_R(\omega) = \frac{\widetilde{V}_R}{\widetilde{V}_s}$ in terms of R, L, and C.**

Answer

$\widetilde{V}_R$ is a voltage divider where the output voltage is taken across the resistor.

$$\widetilde{V}_R = \frac{Z_R}{Z_R + Z_L + Z_C}\widetilde{V}_s,$$
$$H_R(\omega) = \frac{Z_R}{Z_R + Z_L + Z_C} = \frac{R}{R + j\omega L + \frac{1}{j\omega C}}.$$

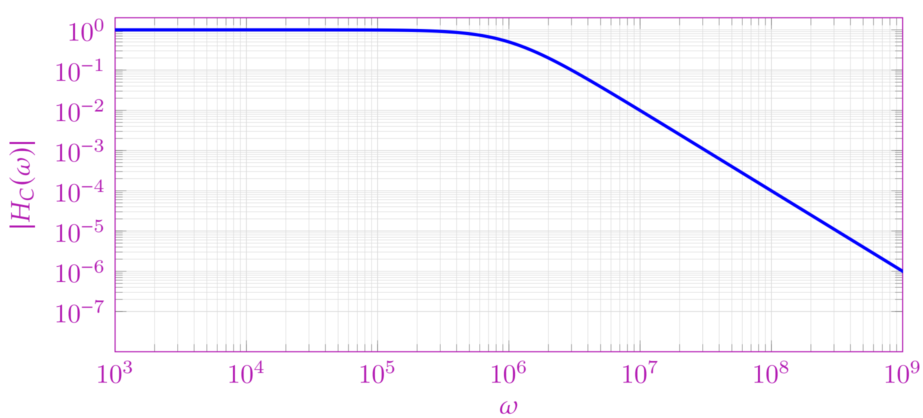
Multiplying the numerator and denominator by $j\omega C$ gives

$$H_R(\omega) = \frac{j\omega RC}{(j\omega)^2 LC + j\omega RC + 1}.$$

e) **Use a computer to draw the magnitude Bode plots of $H_C(\omega)$, $H_L(\omega)$, and $H_R(\omega)$ when $R = 2\,\mathrm{k\Omega}$.**

Answer

$$\begin{aligned} H_C(\omega) &= \frac{1}{(j\omega)^2 LC + j\omega RC + 1} \\ &= \frac{1}{(j\omega)^2(10^{-12}) + j\omega(2\times 10^{-6}) + 1} \\ |H_C(\omega)| &= \frac{1}{\sqrt{(1-\omega^2(10^{-12}))^2 + (\omega(2\times 10^{-6}))^2}} \end{aligned}$$

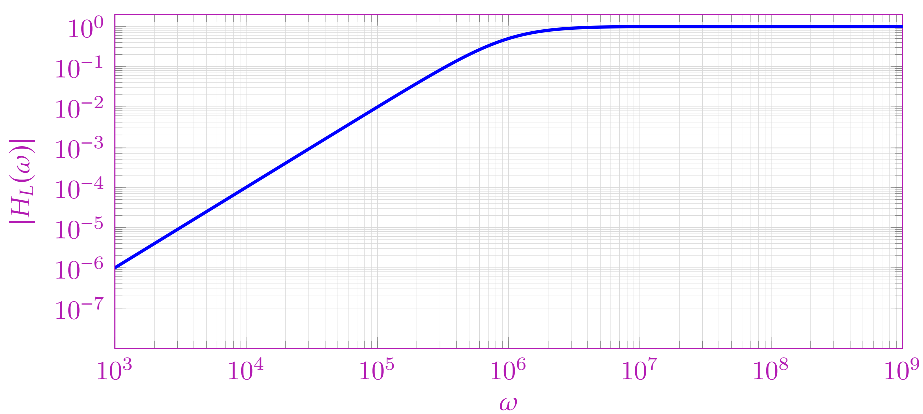


To build some intuition regarding this plot, let us look at how the transfer function approaches the limits $\omega \to 0$ and $\omega \to \infty$.

$$\lim_{\omega \to 0} H_C(\omega) = \frac{1}{1+0} = 1$$
$$\lim_{\omega \to \infty} H_C(\omega) = \frac{1}{(j\omega)^2 LC} = -\frac{10^{12}}{\omega^2}$$

To estimate the corner frequency, we see that $\frac{10^{12}}{\omega^2} = 1$ when $\omega = 10^6 \frac{\text{rad}}{\text{s}}$. Thus, we expect the Bode plot to look like a low pass filter that is flat at 1 up until the corner frequency $\omega_c = 10^6 \frac{\text{rad}}{\text{s}}$, at which point the plot looks approximately like $\frac{10^{12}}{\omega^2}$, which has a slope of -2 in log-log scale. Note that we have dropped negative signs because we are considering the magnitude plot.

$$H_L(\omega) = \frac{(j\omega)^2 LC}{(j\omega)^2 LC + j\omega RC + 1}$$
$$= \frac{(j\omega)^2(10^{-12})}{(j\omega)^2(10^{-12}) + j\omega(2 \times 10^{-6}) + 1}$$
$$|H_L(\omega)| = \frac{\omega^2(10^{-12})}{\sqrt{(1-\omega^2(10^{-12}))^2 + (\omega(2 \times 10^{-6}))^2}}$$

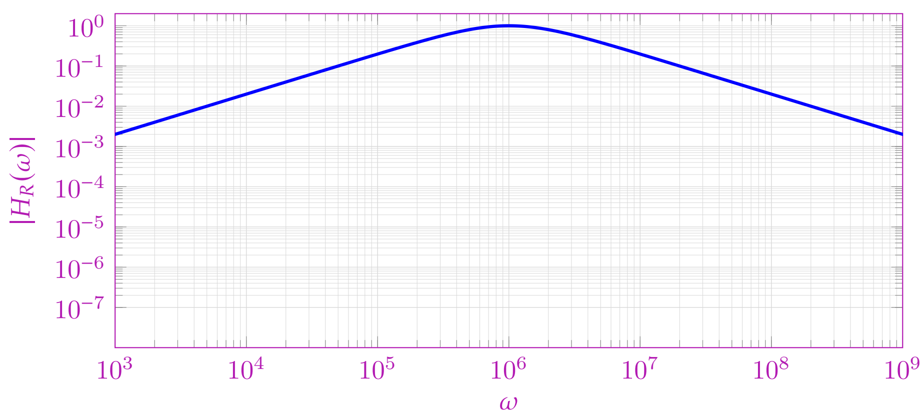


As before, let us see how the transfer function approaches $\omega \to 0$ and $\omega \to \infty$.

$$\lim_{\omega \to 0} H_L(\omega) = \frac{-\omega^2 LC}{1+0} = -\frac{\omega^2}{10^{12}}$$
$$\lim_{\omega \to \infty} H_L(\omega) = \frac{(j\omega)^2 LC}{(j\omega)^2 LC} = 1$$

To estimate the corner frequency, we see that $-\frac{\omega^2}{10^{12}} = 1$ when $\omega = 10^6 \frac{\text{rad}}{\text{s}}$. Thus, we expect the Bode plot to look like a high pass filter that looks like $\frac{\omega^2}{10^{12}}$ (which has a slope of 2 in log-log scale) up until the corner frequency $\omega_c = 10^6 \frac{\text{rad}}{\text{s}}$, at which point the plot looks like a flat line at 1. Note that we have dropped negative signs because we are considering the magnitude plot.

$$
\begin{aligned}
H_R(\omega) &= \frac{j\omega RC}{(j\omega)^2 LC + j\omega RC + 1} \\
&= \frac{j\omega(2\times 10^{-6})}{(j\omega)^2(10^{-12}) + j\omega(2\times 10^{-6}) + 1} \\
|H_R(\omega)| &= \frac{\omega(2\times 10^{-6})}{\sqrt{(1-\omega^2(10^{-12}))^2 + (\omega(2\times 10^{-6}))^2}}
\end{aligned}
$$



Same thing here, look at the limits $\omega \to 0$ and $\omega \to \infty$.

$$
\begin{aligned}
\lim_{\omega\to 0} H_R(\omega) &= \frac{j\omega RC}{1+0} = j\omega(2\times 10^{-6}) \\
\lim_{\omega\to\infty} H_R(\omega) &= \frac{(j\omega)RC}{(j\omega)^2 LC} = \frac{R}{L}\frac{1}{j\omega} = \frac{2\times 10^6}{j\omega}
\end{aligned}
$$

These are equal when $\omega = 10^6 \frac{\text{rad}}{\text{s}}$. This is the center frequency of our bandpass filter. Below this center frequency, the magnitude looks like $\omega(2\times 10^{-6})$ (slope 1 in log-log) and above the center frequency, the magnitude looks like $\frac{2\times 10^6}{\omega}$ (slope -1).