1 Phasors

We consider sinusoidal voltages and currents of a specific form:

$$\begin{array}{r|l} \text{Voltage} & v(t) = V_0 \cos\big(\omega t + \phi_v\big) \\ \text{Current} & i(t) = I_0 \cos\big(\omega t + \phi_i\big), \end{array}$$

where,

a) V_0 is the voltage amplitude and is the highest value of voltage $v(t)$ will attain at any time. Similarly, I_0 is the current amplitude.

b) ω is the frequency of oscillation.

c) ϕ_v and ϕ_i are the phase terms of the voltage and current respectively. These capture a delay, or a shift in time.

We know from Euler's identity that $e^{j\theta} = \cos(\theta) + j\sin(\theta)$. Using this identity, we can obtain an expression for $\cos(\theta)$ in terms of an exponential:

$$\cos(\theta) = \Re\mathfrak{e}(e^{j\theta})$$

Extending this to our voltage signal from above:

$$v(t) = V_0 \cos\big(\omega t + \phi_v\big) = V_0 \Re\mathfrak{e}(e^{j\omega t + j\phi_v}) = V_0 \Re\mathfrak{e}(e^{j\phi_v} e^{j\omega t})$$

Now, since we know that the circuit will not change the frequency of the signal, we can drop the $e^{j\omega t}$, as long as we remember that all signals related to the voltage will be sinusoidal with angular frequency ω. The result is called the phasor form of this signal:

$$\tilde{V} = V_0 e^{j\phi_v}$$

The phasor representation contains the magnitude and phase of the signal, but not the time-varying portion. Phasors let us handle sinusoidal signals much more easily, letting us use circuit analysis techniques that we already know to analyze AC circuits. Note that we can only use this if we know that our signal is a sinusoid.

Within this standard form, the phasor domain representation is as follows. The general equation that relates cosines to phasors is below, where $\tilde{V}$ is the phasor.

$$V_0 \cos\big(\omega t + \phi_v\big) = \Re\mathfrak{e}(\tilde{V} e^{j\omega t})$$

The standard forms for voltage and current phasors are given below:

$$\begin{array}{r|l} \text{Voltage} & \tilde{V} = V_0 e^{j\phi_v} \\ \text{Current} & \tilde{I} = I_0 e^{j\phi_i} \end{array}$$

We define the *impedance* of a circuit component to be $Z = \frac{\tilde{V}}{\tilde{I}}$, where $\tilde{V}$ and $\tilde{I}$ represent the voltage across and the current through the component, respectively.

Phasor Relationship for Resistors

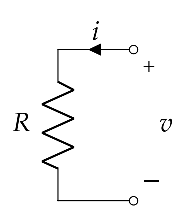


Figure 1: A simple resistor circuit

Consider a simple resistor circuit as in Figure 1, with current being

$$i(t) = I_0 \cos(\omega t + \phi)$$

By Ohm's law,

$$\begin{aligned} v(t) &= i(t)R \\ &= I_0 R \cos(\omega t + \phi) \end{aligned}$$

In phasor domain,

$$\tilde{V} = R\tilde{I}$$

Phasor Relationship for Capacitors

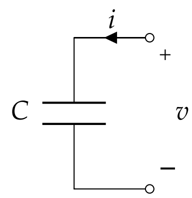


Figure 2: A simple capacitor circuit

Consider a capacitor circuit as in Figure 2, with voltage being

$$v(t) = V_0 \cos(\omega t + \phi)$$

By the capacitor equation,

$$\begin{aligned} i(t) &= C\tfrac{\mathrm{d}v}{\mathrm{d}t}(t) \\ &= -CV_0\omega \sin(\omega t + \phi) \\ &= -CV_0\omega \left(-\cos\left(\omega t + \phi + \frac{\pi}{2}\right)\right) \\ &= CV_0\omega \cos\left(\omega t + \phi + \frac{\pi}{2}\right) \\ &= (\omega C)V_0 \cos\left(\omega t + \phi + \frac{\pi}{2}\right) \end{aligned}$$

In phasor domain,

$$\tilde{I} = \omega C e^{j\frac{\pi}{2}}\tilde{V} = j\omega C\tilde{V}$$

The impedence of a capacitor is an abstraction to model the capacitor as a resistor in the phasor domain. This is denoted Z_C.

$$Z_C = \frac{\tilde{V}}{\tilde{I}} = \frac{1}{j\omega C}$$

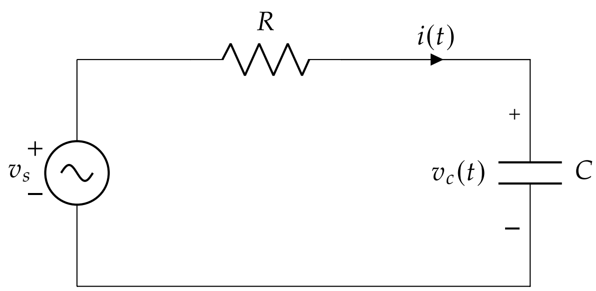
2 Phasor Analysis

Any sinusoidal time-varying function $x(t)$, representing a voltage or a current, can be expressed in the form

$$x(t) = \Re\mathfrak{e}[Xe^{j\omega t}], \tag{1}$$

where X is a time-independent function called the phasor counterpart of $x(t)$. Thus, $x(t)$ is defined in the time domain, while its counterpart X is defined in the phasor domain.

The phasor analysis method consists of five steps. Consider the RC circuit below.



The voltage source is given by

$$v_s(t) = 12\sin\left(\omega t - \frac{\pi}{4}\right), \tag{2}$$

with $\omega = 1 \times 10^3\,\frac{\text{rad}}{\text{s}}$, $R = \sqrt{3}\text{k}\Omega$, and $C = 1\,\mu\text{F}$.

Our goal is to obtain a solution for $i(t)$ with the sinusoidal voltage source $v_s(t)$.

a) **Step 1: Adopt cosine references**

All voltages and currents with known sinusoidal functions should be expressed in the standard cosine format. Convert $v_s(t)$ into a cosine and write down its phasor representation $\tilde{V}_s$.

Answer

$$v_s(t) = 12\cos\left(\omega t - \frac{\pi}{4} - \frac{\pi}{2}\right) = 12\cos\left(\omega t - \frac{3\pi}{4}\right)$$

The phasor is given by

$$\tilde{V}_s = 12e^{-j\frac{3\pi}{4}}$$

Note that

$$v_s(t) = \frac{12e^{-j\frac{3\pi}{4}}e^{j\omega t} + 12e^{j\frac{3\pi}{4}}e^{-j\omega t}}{2} = \frac{\tilde{V}_s e^{j\omega t} + \overline{\tilde{V}}_s e^{-j\omega t}}{2}$$

b) **Step 2: Transform circuits to phasor domain**

The voltage source is represented by its phasor $\tilde{V}_s$. The current $i(t)$ is related to its phasor counterpart $\tilde{I}$ by

$$i(t) = \Re\mathfrak{e}[\tilde{I}e^{j\omega t}].$$

What are the phasor representations of R and C?

Answer

$$Z_R = R$$
$$Z_C = \frac{1}{j\omega C}$$

c) **Step 3: Cast KCL and/or KVL equations in phasor domain**

Use Kirchhoff's laws to write down a loop equation that relates all phasors in Step 2.

Answer

$$\tilde{V}_s = \tilde{I} \cdot Z_R + \tilde{I} \cdot Z_C$$
$$12e^{-j\frac{3\pi}{4}} = \tilde{I}\left(R + \frac{1}{j\omega C}\right)$$

d) **Step 4: Solve for unknown variables**

Solve the equation you derived in Step 3 for $\tilde{I}$ and $\tilde{V}_c$. What are the polar forms of $\tilde{I}$ ($Ae^{j\theta}$, where A is a positive real number) and $\tilde{V}_c$?

Answer

$$\tilde{I} = \frac{12e^{-j\frac{3\pi}{4}}}{R + \frac{1}{j\omega C}} = \frac{j12\omega Ce^{-j\frac{3\pi}{4}}}{1 + j\omega RC}$$
$$\tilde{V}_c = \tilde{I} \cdot Z_C = \frac{j12\omega Ce^{-j\frac{3\pi}{4}}}{1 + j\omega RC} \cdot \frac{1}{j\omega C} = \frac{12e^{-j\frac{3\pi}{4}}}{1 + j\omega RC}$$

To derive the polar form,

$$\tilde{I} = \frac{j12e^{-j\frac{3\pi}{4}} \cdot 10^{-3}}{1 + j\sqrt{3}} = \frac{e^{j\frac{\pi}{2}}12e^{-j\frac{3\pi}{4}} \cdot 10^{-3}}{2e^{j\frac{\pi}{3}}} = 6e^{-j\frac{7\pi}{12}}\,\mathrm{mA}.$$
$$\tilde{V}_c = \frac{12e^{-j\frac{3\pi}{4}}}{1 + j\omega RC} = \frac{12e^{-j\frac{3\pi}{4}}}{1 + j\sqrt{3}} = \frac{12e^{-j\frac{3\pi}{4}}}{2e^{j\frac{\pi}{3}}} = 6e^{-j\frac{13\pi}{12}}\,\mathrm{V}$$

e) **Step 5: Transform solutions back to time domain**

To return to time domain, we apply the fundamental relation between a sinusoidal function and its phasor counterpart. What is $i(t)$ and $v_c(t)$? What is the phase difference between $i(t)$ and $v_c(t)$?

Answer

$$\begin{aligned} i(t) &= \Re[\tilde{I}e^{j\omega t}] \\ &= \Re[6e^{-j\frac{7\pi}{12}}e^{j\omega t}] \\ &= \frac{1}{2}(6e^{-j\frac{7\pi}{12}}e^{j\omega t} + 6e^{j\frac{7\pi}{12}}e^{-j\omega t}) \\ &= 6\cos\left(\omega t - \frac{7\pi}{12}\right)\mathrm{mA} \end{aligned}$$

$$\begin{aligned}
v_c(t) &= \Re\mathfrak{e}[\tilde{V}_c e^{j\omega t}] \\
&= \Re\mathfrak{e}[6e^{-j\frac{13\pi}{12}} e^{j\omega t}] \\
&= \frac{1}{2}(6e^{-j\frac{13\pi}{12}} e^{j\omega t} + 6e^{j\frac{13\pi}{12}} e^{-j\omega t}) \\
&= 6\cos\left(\omega t - \frac{13\pi}{12}\right)\text{V}
\end{aligned}$$

The phase difference between the two, with respect to $i(t)$ is $-\frac{\pi}{2}$.