1 Complex Algebra

a) Express the following values in polar forms: -1, j, $-j$, $\sqrt{j}$, and $\sqrt{-j}$. Recall $j^2 = -1$.

Answer

A complex number can be represented in the following forms:

$$z = a + jb = rcos(\theta) + jrsin(\theta) = re^{j\theta}, \tag{1}$$

where, $r = \sqrt{a^2 + b^2}$, $\theta = tan^{-1}\left(\frac{b}{a}\right)$ and a, b are real numbers.

$$\begin{aligned}
-1 &= j^2 = e^{j\pi} = e^{-j\pi} \\
j &= e^{j\frac{\pi}{2}} = \sqrt{-1} \\
-j &= -e^{j\frac{\pi}{2}} = e^{-j\frac{\pi}{2}} \\
\sqrt{j} &= (e^{j\frac{\pi}{2}})^{\frac{1}{2}} = e^{j\frac{\pi}{4}} = \frac{1+j}{\sqrt{2}} \\
\sqrt{-j} &= (e^{-j\frac{\pi}{2}})^{\frac{1}{2}} = e^{-j\frac{\pi}{4}} = \frac{1-j}{\sqrt{2}}
\end{aligned}$$

b) Represent $\sin\theta$ and $\cos\theta$ using complex exponentials. (*Hint:* Use Euler's identity $e^{j\theta} = \cos\theta + j\sin\theta$.)

Answer

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}, \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}.$$

c) For complex number $z = x + jy$ show that $|z| = \sqrt{z\bar{z}}$, where $\bar{z}$ is the complex conjugate of z.

Answer

We can follow the definition of complex conjugate and magnitude:

$$\sqrt{z\bar{z}} = \sqrt{(x + jy)(x - jy)} = \sqrt{x^2 + y^2} = |z|$$

For the next four parts, let $A = 1 - j\sqrt{3}$ and $B = \sqrt{3} + j$.

d) Express A and B in polar form.

Answer

Following the definitions in part a:

$$|A| = 2, \quad |B| = 2, \quad \theta_A = -\frac{\pi}{3}, \quad \theta_B = \frac{\pi}{6}.$$

Hence,

$$A = 2e^{-j\frac{\pi}{3}} \quad B = 2e^{j\frac{\pi}{6}}.$$

e) Find AB, $A\bar{B}$, $\frac{A}{B}$, $A + \bar{A}$, $A - \bar{A}$, $\overline{AB}$, $\bar{A}\bar{B}$, and $\sqrt{B}$.

Answer

$$AB = 4 \cdot e^{-j\frac{\pi}{6}} = 2\sqrt{3} - 2j$$
$$A\bar{B} = 4 \cdot e^{-j\frac{\pi}{2}} = -4j$$
$$\frac{A}{B} = e^{-j\frac{\pi}{2}} = -j$$
$$A + \bar{A} = 2$$

Note, $A + \bar{A}$ is a purely real number.

$$A - \bar{A} = -2j\sqrt{3}$$

Note, $A - \bar{A}$ is a purely imaginary number.

$$\overline{AB} = 2\sqrt{3} + 2j$$
$$\bar{A}\bar{B} = (1 + j\sqrt{3})(\sqrt{3} - j) = \sqrt{3} + \sqrt{3} + j(3 - 1) = 2\sqrt{3} + 2j$$

Note, $\overline{AB} = \bar{A}\bar{B}$.

$$\sqrt{B} = \sqrt{2} \cdot e^{j\frac{\pi}{12}}$$

f) Show the number A in complex plane, marking the distance from origin and angle with real axis.

Answer

Location of A in the complex plane is shown in the following figure.

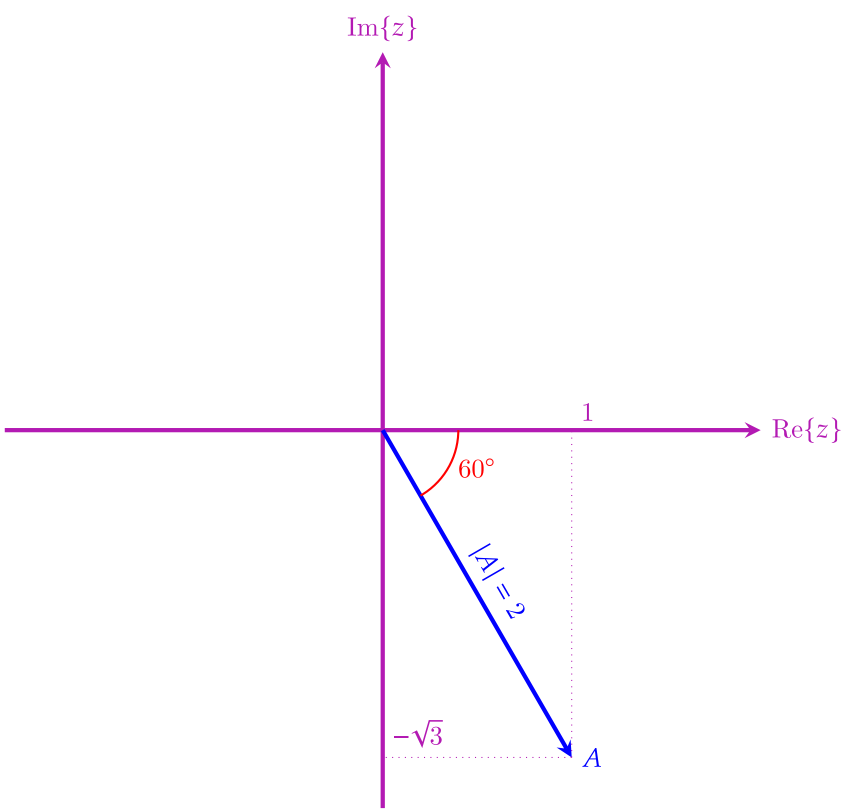

Figure 1: Location of A in complex plane

g) Show that multiplying A with j is equivalent to rotating the magnitude of the complex number by $\pi/2$ or 90 degrees in the complex plane.

Answer

Multiplying A by j:

$$jA = e^{j\pi/2} \times 2e^{-j\pi/3} = 2e^{j\pi/6} = \sqrt(3) + j \tag{2}$$

The rotation is demonstrated in the following complex plane plot.

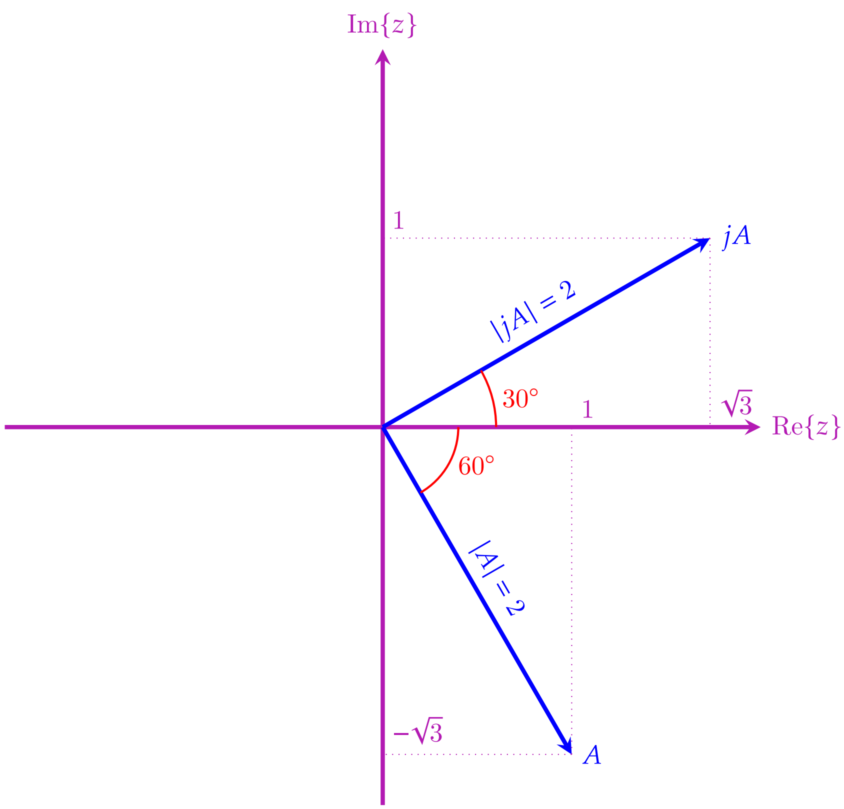

Figure 2: Rotation of A because of multiplication by j

h) What are the roots of $z^2 = 1$? What about $z^3 = 1$? How many roots does $z^n = 1$ have? What is the general form for the solutions of $z^n = 1$?

Answer

For the roots of $z^2 = 1$, z^2 is a standard quadratic, so we have $z = 1, -1$ by solving the quadratic equation. To solve $z^3 = 1$, we substitute $1 = e^{j0}, e^{j2\pi}, e^{j4\pi}, \ldots$, which are the complex representations of 1. For each complex representation of 1:

$$z^3 = e^{j0}, e^{j2\pi}, e^{j4\pi}$$

Then take the cube root for each case:

$$z = e^{\frac{j0}{3}}, e^{\frac{j2\pi}{3}}, e^{\frac{j4\pi}{3}}$$

Notice that for $e^{j6\pi}$ and greater, the solutions repeat the above three solutions.

The general solution for $z^n = 1$ is

$$z = e^{j2\pi\frac{k}{n}}$$

for $0 \leq k < n$.

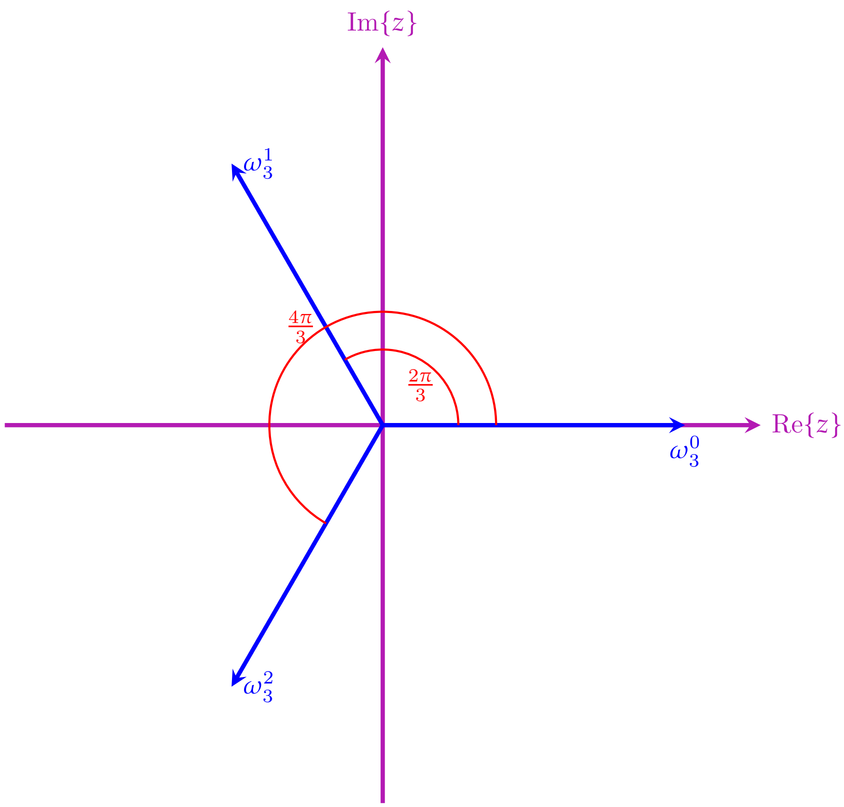

Figure 3: Roots of unity for case of $n = 3$