1 RC Circuits

In this problem, we will be using differential equations to find the voltage across a capacitor over time in an RC circuit. We set up our problem by first defining three functions over time: $I(t)$ is the current at time t, $V(t)$ is the voltage across the circuit at time t, and $V_o(t)$ is the voltage across the capacitor at time t.

Recall from 16A that the voltage across a resistor is defined as $V_R = RI_R$ where I_R is the current across the resistor. Also, recall that the voltage across a capacitor is defined as $V_o = \frac{Q}{C}$ where Q is the charge across the capacitor.

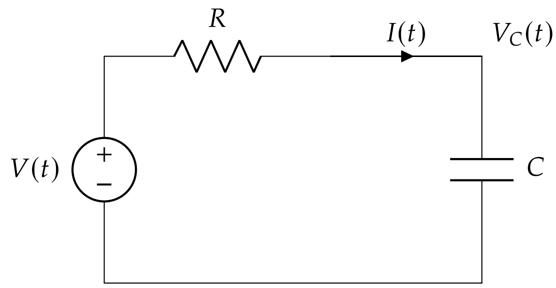


Figure 1: Example Circuit

a) First, find an equation that relates the current across the capacitor $I(t)$ with the voltage across the capacitor $V_C(t)$.

Answer

Differentiating $V_C(t) = \frac{Q(t)}{C}$ in terms of t, we get

$$\frac{dV_C(t)}{dt} = \frac{dQ(t)}{dt}\frac{1}{C}$$

By definition, the change in charge is the current across the capacitor, so

$$\frac{d}{dt}V_C(t) = I(t)\frac{1}{C}$$

b) Write a system of equations that relates the functions $I(t)$, $V_C(t)$, and $V(t)$.

Answer

Kirchhoff's law states that the voltage across a closed loop is 0.

$$RI(t) + V_C(t) - V(t) = 0$$

$$RI(t) + V_C(t) = V(t) \tag{1}$$

Alternatively, we can use Ohm's law to say

$$I(t) = \frac{V(t) - V_C(t)}{R}$$

$$RI(t) + V_C(t) = V(t) \tag{2}$$

c) Rewrite the previous equation in part (b) in the form of a differential equation involving only $V_C(t)$ and $V(t)$.

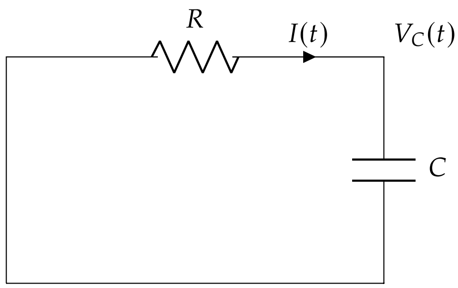


Figure 2: Circuit for part (d)

Answer

From part (a), we have

$$I(t) = \frac{dV_C(t)}{dt}C$$

Substituting this into Equation 2 gives us

$$RC\frac{dV_C(t)}{dt} + V_C(t) = V(t)$$

d) Let's suppose that at $t = 0$, the capacitor is charged to a voltage V_{DD} ($V_C(0) = V_{DD}$). Let's also assume that $V(t) = 0$ for all $t \geq 0$. Solve the differential equation for $V_C(t)$ for $t \geq 0$.

Answer

Because $V(t) = 0$, our differential equation simplifies to

$$RC\frac{dV_C(t)}{dt} + V_C(t) = 0$$

Doing some algebraic manipulations gives us

$$\frac{dV_C(t)}{dt} = -\frac{1}{RC}V_C(t)$$

This equation tells us that we are looking for some function $V_C(t)$ such that when we take its derivative, we get the same function $V_C(t)$ multiplied by a scalar $-\frac{1}{RC}$. Because the derivative is equal to a scalar times itself, we think that the solution $V_C(t)$ will probably be of the form Ae^{bt}, where A and b are both constants. In this case we see that $b = -\frac{1}{RC}$, and we find that

$$V_C(t) = Ae^{-\frac{1}{RC}t}$$

We still need to solve for the constant A in front of the exponential, and we use $V_C(0) = K$ to help us find A. Setting $t = 0$ in the equation gives us

$$\begin{aligned} V_C(0) &= Ae^{-\frac{1}{RC}0} \\ &= Ae^0 \\ &= A \\ &= V_{DD} \end{aligned}$$

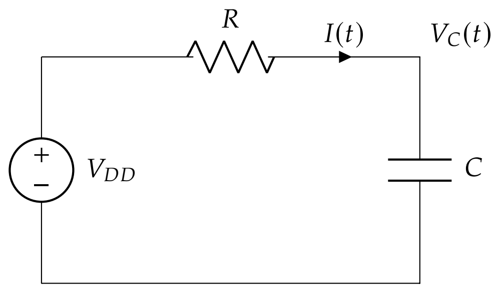


Figure 3: Circuit for part (e)

Thus, we see that $A = V_{DD}$, and our solution is

$$V_C(t) = V_{DD}e^{-\frac{1}{RC}t}$$

e) Now, let's suppose that we start with an uncharged capacitor $V_C(0) = 0$. We apply some constant voltage $V(t) = V_{DD}$ across the circuit. Solve the differential equation for $V_C(t)$ for $t \geq 0$.

Answer

Substituting $V(t) = V_{DD}$ into our solution from part (c):

$$RC\frac{dV_C(t)}{dt} + V_o(t) = V_{DD}$$

Rearranging the differential equation we get

$$\frac{d}{dt}V_C = -\frac{1}{RC}V_C(t) + \frac{V_{DD}}{RC}$$

This differential equation looks similar to the one in the previous part but we notice that there is an extra $\frac{V_{DD}}{RC}$ term. If we were to again guess the solution $V_C(t) = Ae^{-\frac{1}{RC}t}$ we see that:

$$-\frac{1}{RC}Ae^{-\frac{1}{RC}t} = -\frac{1}{RC}Ae^{-\frac{1}{RC}t} + \frac{V_{DD}}{RC}$$
$$0 = \frac{V_{DD}}{RC}$$

We end up with an equation that is impossible to solve meaning there must have been a problem with our guess. Therefore, we will improve our guess by trying the solution $V_C(t) = Ae^{-\frac{1}{RC}t} + B$.

Taking its derivative and plugging it into the differential equation, we get

$$\frac{dV_C(t)}{dt} = -\frac{1}{RC}V_C(t) + \frac{V_{DD}}{RC}$$
$$-\frac{1}{RC}Ae^{-\frac{1}{RC}t} = -\frac{1}{RC}(Ae^{-\frac{1}{RC}t} + B) + \frac{V_{DD}}{RC}$$
$$0 = -\frac{B}{RC} + \frac{V_{DD}}{RC}$$

This tells us that $B = V_{DD}$. Now to solve for A, we plug in the initial condition $V_C(0) = 0$

$$V_C(0) = Ae^{-\frac{1}{RC}\cdot 0} + V_{DD} = A + V_{DD} = 0$$

It follows that $A = -V_{DD}$ so our final solution to the differential equation will be

$$V_C(t) = -V_{DD}e^{-\frac{1}{RC}t} + V_{DD} = V_{DD}(1 - e^{-\frac{1}{RC}t})$$

Alternate Method using Substitution of Variables:

We want to arrange this equation to be in a form that we know how to solve:

$$\frac{d}{dt}V_C = \frac{V_{DD} - V_C(t)}{RC}$$

This is not quite the form we have seen before, as the term on the right is not equal to the term being differentiated. Let's instead define a new variable $\tilde{V}_C(t) = V_C(t) - V_{DD}$. Note that $\frac{d\tilde{V}_C(t)}{dt} = \frac{dV_C(t)}{dt}$. We can substitute these into our differential equation and obtain

$$RC\frac{dV_C(t)}{dt} + V_C(t) - V_{DD} = 0$$

$$RC\frac{d\tilde{V}_C(t)}{dt} + \tilde{V}_C(t) = 0$$

In this equation, we have now removed V_{DD} from the left hand because of how we defined $\tilde{V}_C(t)$. We can now solve the differential equation using the same method as in the previous part to get

$$\tilde{V}_C(t) = Ae^{-\frac{t}{RC}}$$

Substituting $V_C(t) = V_{DD} + \tilde{V}_C(t)$ back into this equation gives us

$$V_C(t) = V_{DD} + Ae^{-\frac{t}{RC}}$$

Using in the initial condition $V_C(0) = 0$, we get:

$$0 = V_{DD} + Ae^{-\frac{0}{RC}} = V_{DD} + A \implies A = -V_{DD}$$

Therefore,

$$\begin{aligned} V_C(t) &= V_{DD} - V_{DD}e^{-\frac{t}{RC}} \\ &= V_{DD}(1 - e^{-\frac{t}{RC}}) \end{aligned}$$

2 Graphing RC Responses

Consider the following RC Circuit with a single resistor R, capacitor C, and voltage source $V(t)$.

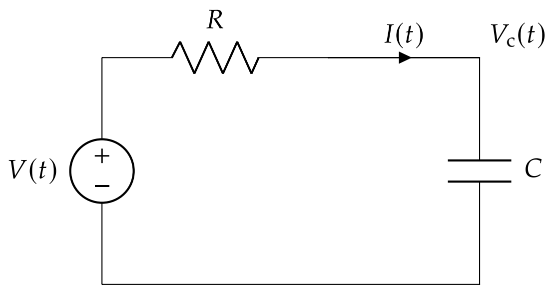


Figure 4: Example Circuit

a) Let's suppose that at $t = 0$, the capacitor is charged to a voltage V_{DD} ($V_c(0) = V_{DD}$) and that $V(t) = 0$ for all $t \geq 0$. Plot the response $V_c(t)$.

Answer

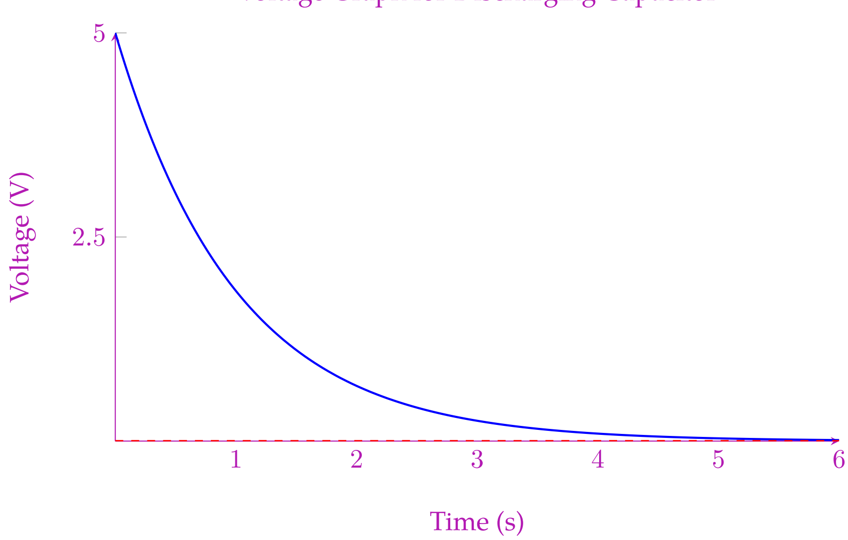
We can represent the RC Circuit with the following first order differential equation

$$\frac{dV_c(t)}{dt} = -\frac{1}{RC}V_c(t)$$

Since the initial condition is $V_c(0) = V_{DD}$, the solution to this differential equation will be $V_c(t) = V_{DD}e^{-\frac{t}{RC}}$.

To plot this response by using a graphing tool or by plotting points and connecting the dots. We've plotted the response when $V_{DD} = 5$ and $RC = 1$.

Voltage Graph for Discharging Capacitor



b) Now let's suppose that at $t = 0$, the capacitor is uncharged ($V_c(0) = 0$) and that $V(t) = V_{DD}$ for all $t \geq 0$. Plot the response $V_c(t)$.

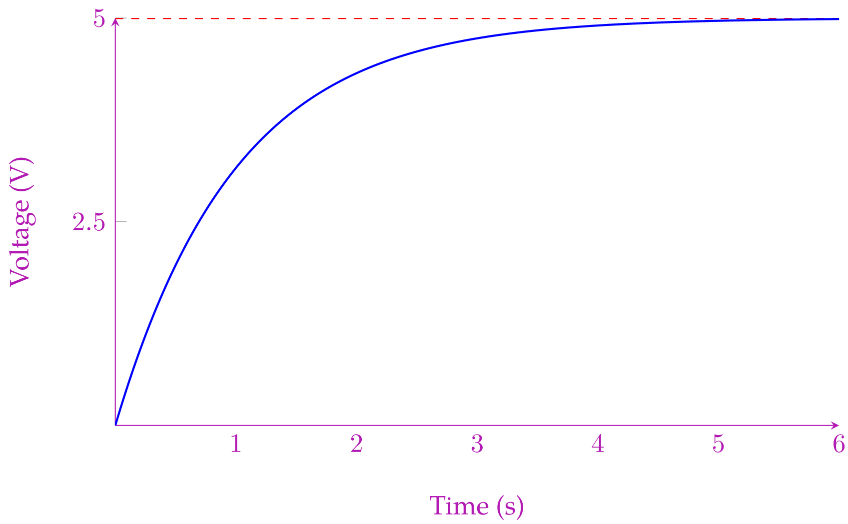
Answer

We can represent the RC Circuit with the following first order differential equation

$$\frac{dV_c(t)}{dt} = -\frac{1}{RC}V_c(t) + \frac{V_{DD}}{RC}$$

The solution to this differential equation is $V_c(t) = V_{DD}(1 - e^{-\frac{t}{RC}})$ and we can plot its response again by using a graphing tool. We've plotted the response when $V_{DD} = 5$ and $RC = 1$.

Voltage Graph for Charging Capacitor



To better understand our responses, we now define a **time constant** which is a measure of how long it takes for the capacitor to charge or discharge. Mathematically, we define τ as the time at which $V_C(\tau)$ is $\frac{1}{e} = 36.8\%$ away from its steady state value.

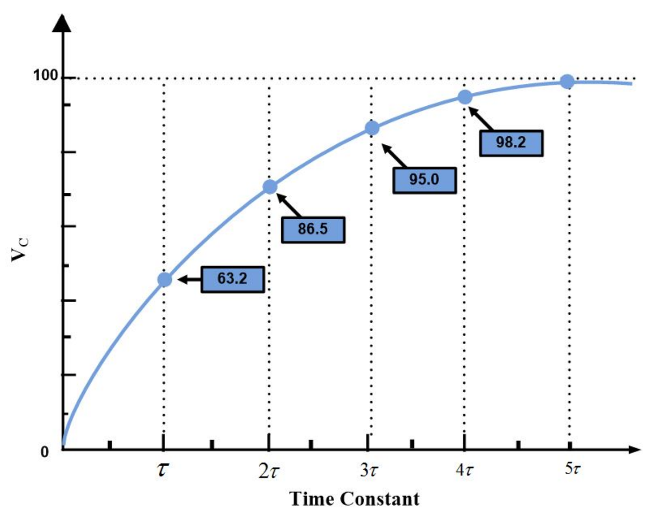


Figure 5: Different values of capacitor voltage at different times, relative to τ.

c) Suppose that $V_{DD} = 5\,\text{V}$, $R = 100\,\Omega$, and $C = 10\,\mu\text{F}$. What is the time constant τ for this circuit?

Answer

The time constant for an RC circuit with a single resistor and capacitor will be $\tau = RC$. To show this, we look at the discharging case in part (a), let $V_c(\tau) = \frac{V_{DD}}{e}$ and solve for τ.

$$
\begin{aligned}
V_c(\tau) = V_{DD}e^{-\frac{\tau}{RC}} &= \frac{V_{DD}}{e} \\
e^{-\frac{\tau}{RC}} &= \frac{1}{e} \\
ln(e^{-\frac{\tau}{RC}}) &= ln(\frac{1}{e}) \\
-\frac{\tau}{RC} &= -1 \\
\tau &= RC
\end{aligned}
$$

Alternatively we could've solved for τ by considered the charging case from part (b) and solved for $V_c(\tau) = V_{DD}(1 - \frac{1}{e})$.

d) Going back to part (b), on what order of magnitude of time (nanoseconds, milliseconds, 10's of seconds, etc.) does this circuit settle (V_c is $> 95\%$ of its value as $t \longrightarrow \infty$)?

Answer

The time constant τ of an RC circuit is just $\tau = RC$. For our circuit:

$$\tau = RC = 100\,\Omega \cdot 10\,\mu\text{F} = 0.001\,\text{s}$$

After 3 time constants, the voltage will be 95% of its steady state value

$$3\tau = 0.003\,\text{s}$$

The circuit will settle on the order of milliseconds. Alternatively, this value can be found by using algebra:

$$
\begin{aligned}
0.95V_{DD} &= V_{DD}(1 - e^{-\frac{t}{RC}}) \\
-0.05 &= -e^{-\frac{t}{RC}} \\
0.05 &= e^{-\frac{t}{RC}} \\
ln(0.05) &= -\frac{t}{RC} \\
-3 &= -\frac{t}{0.001} \\
t &= 0.003\,\text{seconds}
\end{aligned}
$$

e) Give 2 ways to reduce the settling time of the circuit if we are allowed to change one component in the circuit.

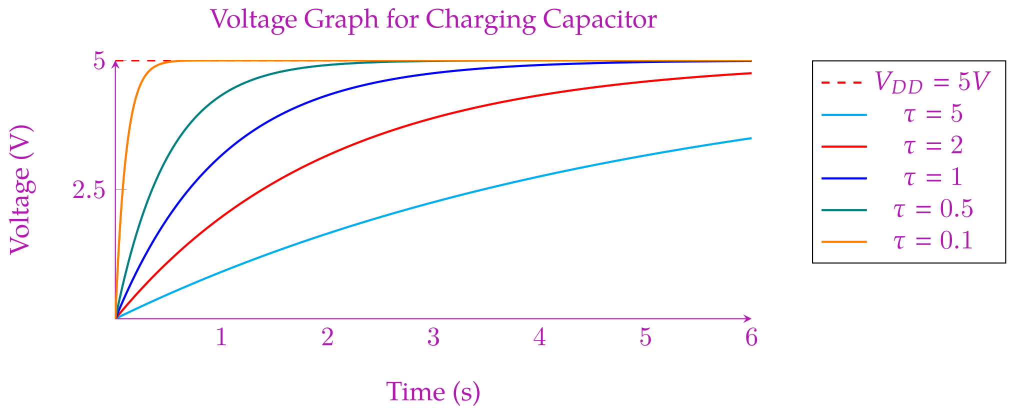
Answer

To reduce settling time we reduce τ. We can achieve this by

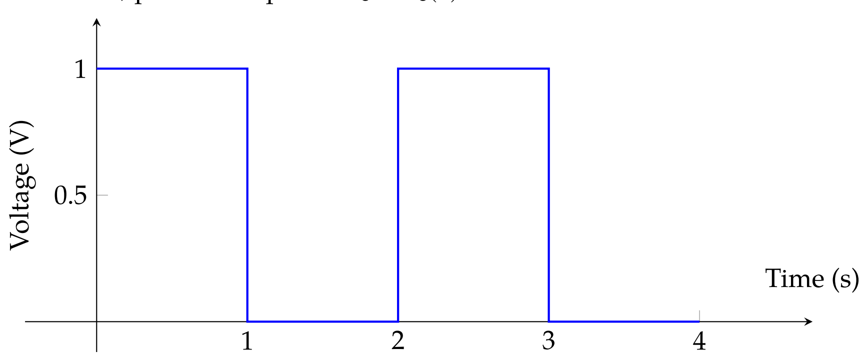
a) Lowering the value of R or
b) Lowering the value of C.

Notice how the value of V_{DD} does not change the settling time.

We've plotted the responses for a couple of τ values below. As τ approaches 0, the response V_c will approach an ideal square wave.

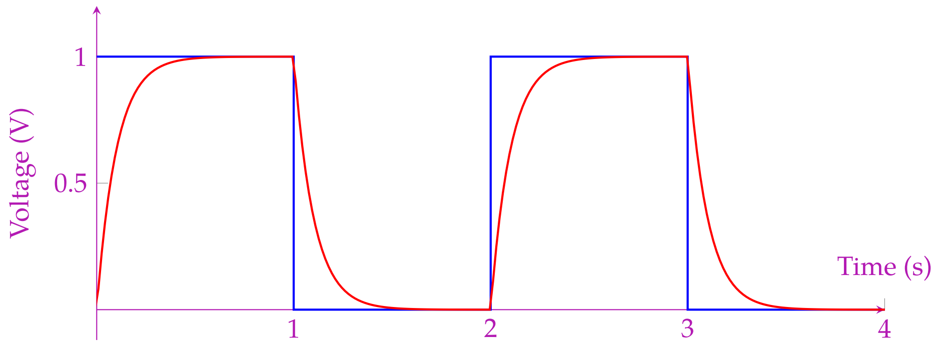


f) Suppose we have a source $V(t)$ that alternates between 0 and $V_{DD} = 1\,\text{V}$. Given $RC = 0.1\,\text{s}$, plot the response V_c if $V_c(0) = 0$.



Answer

The input switches between high and low every second while $\tau = RC = 0.1\,\text{s}$. This means the capacitor will charge and discharge for 10τ so we can approximate it as fully charged and discharged after 1 second.



g) Now suppose we have the same source $V(t)$ but $RC = 1\,\mathrm{s}$, plot the response V_c if $V_c(0) = 0$.

Answer

The input has stayed the same while $\tau = RC = 1\,\mathrm{s}$. This means the capacitor will only charge and discharge for one time constant or up to around 63%.