EECS 16B

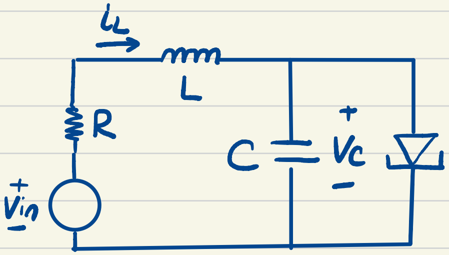
Spring 2022

Lecture 26

4/21/2022

LECTURE 26

- finish tunnel diode circuit example
- complex inner products



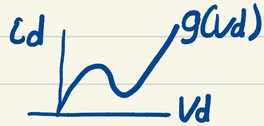
$$\boxed{\begin{aligned} \frac{d}{dt} v_C(t) &= \frac{1}{C}\left(i_L(t) - g(v_C(t))\right) \\ \frac{d}{dt} i_L(t) &= \frac{1}{L}\left(v_{in}(t) - R\, i_L(t) - v_C(t)\right) \end{aligned}}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := \begin{bmatrix} v_C \\ i_L \end{bmatrix}, \quad u := v_{in}, \qquad \frac{d}{dt}\vec{x}(t) = \vec{f}(\vec{x}(t), u(t))$$

$$\text{where } \vec{f}(\vec{x}, u) = \begin{bmatrix} f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u) \end{bmatrix} = \begin{bmatrix} \frac{1}{C}(x_2 - g(x_1)) \\ \frac{1}{L}(u - Rx_2 - x_1) \end{bmatrix}$$

Operating Points :

tunnel diode:



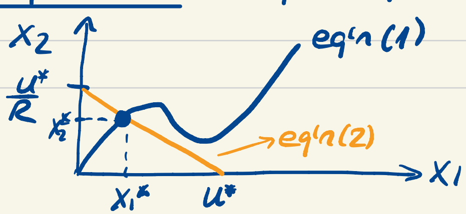
$$f_1(x_1, x_2, u) = 0 \Rightarrow x_2 - g(x_1) = 0 \Rightarrow x_2 = g(x_1) \qquad (1)$$

$$f_2(x_1, x_2, u) = 0 \Rightarrow u - Rx_2 - x_1 = 0 \Rightarrow x_2 = \frac{u - x_1}{R} \qquad (2)$$

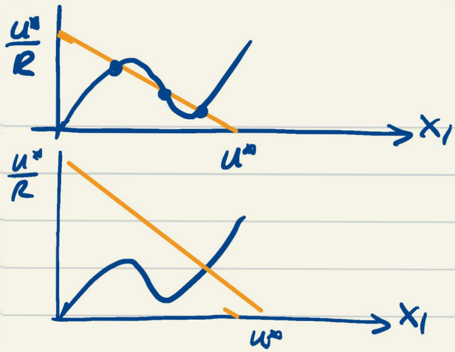
$$\text{Combine (1) \& (2)}: \quad g(x_1) = \frac{u - x_1}{R} \qquad (3)$$

Find x_1^*, u^* satisfying (3). Substitute x_1^* in (1) to obtain x_2^*.

<u>Graphical Interpretation</u>: superimpose graphs (1), (2):



increasing V_{in} (i.e., u^*) raises the orange line



three different equilibria

back to single equilibrium

Circuit interpretation of equilibrium points

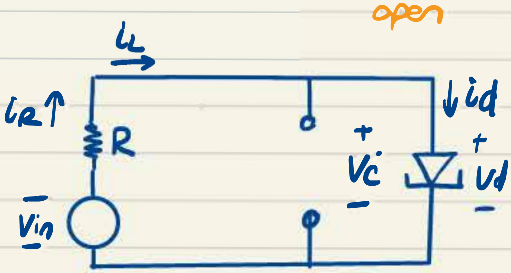
$$\frac{d}{dt}\vec{x}(t) = \vec{f}(\vec{x}(t), u(t))$$

$$f(\vec{x}^*, u^*) = 0 \quad \Rightarrow \quad \frac{d}{dt}\vec{x}(t) = 0 \quad \text{when} \quad \begin{matrix} \vec{x}(0) = \vec{x}^* \\ u(t) = u^* \end{matrix}$$

In this circuit

$$\frac{d}{dt}\vec{x}(t) = \frac{d}{dt}\begin{bmatrix} V_c(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{C} i_c(t) \\ \frac{1}{L} V_L(t) \end{bmatrix}$$

Equilibrium means: $\underbrace{i_c = 0}_{\text{open}}$ and $\underbrace{V_L = 0}_{\text{wire}}$



KCL:

$i_R = i_L = i_d$ (4)

KVL:

$V_{in} - i_R R - V_c = 0$ (5)

$V_d = V_c$ (6)

$$(4) \Rightarrow \underbrace{i_L}_{x_2} = i_d = g(v_d) \overset{(6)}{=} g(\underbrace{V_c}_{x_1}) \quad \rightarrow \text{recovers (1)}$$

$$(4),(5) \Rightarrow \underbrace{V_{in}}_{u} - \underbrace{i_L}_{x_2} R - \underbrace{V_c}_{x_1} = 0 \quad \rightarrow \text{recovers (2)}$$

Linearization:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -\frac{g'(x_1)}{C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}$$

$$A = \begin{bmatrix} -\frac{g'(x_1^*)}{C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}$$

Stability? $tr(A) = -\frac{g'(x_1^*)}{C} - \frac{R}{L}$

$$det(A) = \frac{g'(x_1^*)R}{LC} + \frac{1}{LC}$$

If $g'(x_1^*) > 0$, then $tr(A) < 0$, $det(A) > 0$

so stable from the criterion in the last lecture.

Complex Inner Products

Recall from Lecture 17:

Theorem (Schur Decomposition): For any $A \in \mathbb{R}^{n \times n}$ with real eigenvalues we can find an orthogonal matrix U s.t. $U^T A U$ is upper triangular.

assumed because proof used inner products with evectors — will remove this superfluous assumption with complex inner products

Recall inner product and norm in $\mathbb{R}^n$:

$$\langle \vec{x}, \vec{y} \rangle_{\mathbb{R}^n} = \sum_{i=1}^{n} x_i y_i$$

$$= \vec{y}^T \vec{x} = \vec{x}^T \vec{y}$$

$$\| x \|_{\mathbb{R}^n}^2 = \sum_{i=1}^{n} x_i^2 = \vec{x}^T \vec{x} = \langle \vec{x}, \vec{x} \rangle_{\mathbb{R}^n}$$

Properties that must be satisfied by a norm in any vector space $(\mathbb{V}, \mathbb{F})$

"field" where scalars multiplying vectors come from, e.g. $\mathbb{R}, \mathbb{C}$

space in which vectors live, e.g. $\mathbb{R}^n, \mathbb{C}^n$

i) $\| \vec{x} + \vec{y} \|_{\mathbb{V}} \leq \| \vec{x} \|_{\mathbb{V}} + \| \vec{y} \|_{\mathbb{V}}$ for any $\vec{x}, \vec{y} \in \mathbb{V}$

ii) $\| \alpha \vec{x} \|_{\mathbb{V}} = |\alpha| \, \| \vec{x} \|_{\mathbb{V}}$ for any $\vec{x} \in \mathbb{V}$, $\alpha \in \mathbb{F}$

iii) $\| \vec{x} \|_{\mathbb{V}} \geq 0 \;\; \forall \vec{x} \in \mathbb{V}$ and $\| \vec{x} \|_{\mathbb{V}} = 0 \Leftrightarrow \vec{x} = 0$.

(Convince yourself all 3 hold in $\mathbb{R}^n$ with $\| \vec{x} \| = \sqrt{\sum_i x_i^2}$)

What is a norm that works for $\mathbb{C}^n$?

We can't use $\| \vec{x} \| = \sqrt{\sum_i x_i^2}$.

$\vec{x} = \begin{bmatrix} 1 \\ 2j \end{bmatrix} \in \mathbb{C}^2 \quad x_1^2 + x_2^2 = 1 + (2j)^2 = 1 - 4 = -3 < 0$

Instead, $\boxed{\| \vec{x} \|_{\mathbb{C}^n}^2 = \sum_{i=1}^{n} |x_i|^2}$

$|1|^2 + |2j|^2 = 1 + 4 \Rightarrow \| \vec{x} \|_{\mathbb{C}^2} = \sqrt{5}$

Inner product: $\boxed{\langle \vec{x}, \vec{y} \rangle_{\mathbb{C}^n} = \sum_{i=1}^n x_i \overline{y_i}}$

Note:

- $\langle \vec{x}, \vec{x} \rangle_{\mathbb{C}^n} = \sum_{i=1}^n x_i \overline{x_i} = \sum_{i=1}^n |x_i|^2 = \|\vec{x}\|_{\mathbb{C}^n}^2$
- $\sum_{i=1}^n x_i \overline{y_i} = [\overline{y_1} \cdots \overline{y_n}] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{(\overline{\vec{y}})^T}_{=: \vec{y}^*} \vec{x}$ "conjugate transpose"

$\boxed{\langle \vec{x}, \vec{y} \rangle_{\mathbb{C}^n} = \vec{y}^* \vec{x}}$

- $\langle \vec{y}, \vec{x} \rangle_{\mathbb{C}^n} = \overline{\langle \vec{x}, \vec{y} \rangle_{\mathbb{C}^n}}$

so $\langle \vec{x}, \vec{y} \rangle_{\mathbb{C}^n}$ and $\langle \vec{y}, \vec{x} \rangle_{\mathbb{C}^n}$ not necessarily the same

(unlike $\mathbb{R}^n$ where $\langle \vec{x}, \vec{y} \rangle_{\mathbb{R}^n} = \langle \vec{y}, \vec{x} \rangle_{\mathbb{R}^n}$)

(Use $\langle \vec{y}, \vec{x} \rangle_{\mathbb{C}^n} = \sum_i y_i \overline{x_i}$ and $\langle \vec{x}, \vec{y} \rangle_{\mathbb{C}^n} = \sum x_i \overline{y_i}$

to show $\langle \vec{y}, \vec{x} \rangle_{\mathbb{C}^n} = \overline{\langle \vec{x}, \vec{y} \rangle_{\mathbb{C}^n}}$.)

- $\vec{x}, \vec{y} \in \mathbb{C}^n$ Orthogonal if $\langle \vec{x}, \vec{y} \rangle_{\mathbb{C}^n} = 0$.

  Orthonormal if $\|\vec{x}\|_{\mathbb{C}^n} = 1$, $\|\vec{y}\|_{\mathbb{C}^n} = 1$ also.

Example: $\vec{x} = \begin{bmatrix} 1 \\ j \end{bmatrix}$ $\vec{y} = \begin{bmatrix} j \\ 1 \end{bmatrix}$ $\langle \vec{x}, \vec{y} \rangle_{\mathbb{C}^2} = \vec{y}^* \vec{x}$

$= [\overline{j} \ \overline{1}] \begin{bmatrix} 1 \\ j \end{bmatrix} = [-j \ 1] \begin{bmatrix} 1 \\ j \end{bmatrix} = 0.$

$\|\vec{x}\|^2 = |1|^2 + |j|^2 = 2 = \|\vec{y}\|^2$

- If $Q \in \mathbb{C}^{m \times n}$ has orthonormal columns then

$$Q^* Q = I$$

because

$$\begin{bmatrix} \vec{q}_1^* \\ \vdots \\ \vec{q}_n^* \end{bmatrix} [\vec{q}_1 \cdots \vec{q}_n]$$

$$= \begin{bmatrix} \underbrace{\vec{q}_1^* \vec{q}_1}_{=1} & \cancel{\vec{q}_1^* \vec{q}_2}^{\,0} & \cdots & \cancel{\vec{q}_1^* \vec{q}_n}^{\,0} \\ \cancel{\vec{q}_2^* \vec{q}_1}^{\,0} & \cancel{\vec{q}_2^* \vec{q}_2}^{\,1} & & \\ \vdots & & \ddots & \\ \cancel{\vec{q}_n^* \vec{q}_1}^{\,0} & & & \cancel{\vec{q}_n^* \vec{q}_n}^{\,1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & & 1 \end{bmatrix}.$$

__Defn:__ A square matrix $Q \in \mathbb{C}^{n \times n}$ with orthonormal columns is called a "__unitary__" matrix (generalizing notion of __orthogonal__ matrix to complex matrices).

$$Q^* Q = I \underset{\substack{\text{when} \\ Q \text{ is} \\ \text{square}}}{\Rightarrow} Q^* = Q^{-1}$$

Therefore $Q^* = Q^{-1}$ for unitary matrices.
(similar to $Q^T = Q^{-1}$ for orthogonal matrices)

Schur Decomposition generalized:

> For any $A \in \mathbb{C}^{n \times n}$ we can find a unitary matrix U such that $U^* A U$ is upper triangular.

If $A \in \mathbb{R}^{n \times n}$ but its eigenvalues are complex, previous version not applicable; the generalized version above is.

Gram-Schmidt Orthonormalization generalized to $\mathbb{C}^n$:

Task: given linearly independent $\vec{a}_1, \dots \vec{a}_k \in \mathbb{C}^n$ find orthonormal $\vec{q}_1, \dots \vec{q}_k \in \mathbb{C}^n$ such that $\vec{a}_i$ is a linear combination of $\vec{q}_1$ to $\vec{q}_i$.

$$[\vec{a}_1 \ \cdots \ \vec{a}_k] = \underbrace{[\vec{q}_1 \ \cdots \ \vec{q}_k]}_{\substack{Q \\ \text{with orthonormal} \\ \text{columns}}} \underbrace{\begin{bmatrix} * & * & \cdots & * \\ 0 & * & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & * \end{bmatrix}}_{R:\ \text{upper triangular}}$$

Step 1: $\vec{q}_1 = \frac{1}{\|\vec{a}_1\|_{\mathbb{C}^n}} \vec{a}_1$

Step 2: $\vec{z}_2 = \vec{a}_2 - \langle \vec{a}_2, \vec{q}_1 \rangle_{\mathbb{C}^n} \vec{q}_1$

$\vec{q}_2 = \frac{1}{\|\vec{z}_2\|_{\mathbb{C}^n}} \vec{z}_2$

Step 3: $\vec{z}_3 = \vec{a}_3 - \langle \vec{a}_3, \vec{q}_1 \rangle_{\mathbb{C}^n} \vec{q}_1 - \langle \vec{a}_3, \vec{q}_2 \rangle_{\mathbb{C}^n} \vec{q}_2$

$\vec{q}_3 = \frac{1}{\|\vec{z}_3\|_{\mathbb{C}^n}} \vec{z}_3$

$\vdots$