EECS 16B

Spring 2022

Lecture 25

4/19/2022

LECTURE 25 - Linearization continued

Given nonlinear system $\frac{d}{dt}\vec{x}(t) = \vec{f}(\vec{x}(t), \vec{u}(t))$, where

$$x(t) \in \mathbb{R}^n,\; u(t) \in \mathbb{R}^m,\; \vec{f}(\vec{x},\vec{u}) = \begin{bmatrix} f_1(x_1, \dots x_n, u_1, \dots u_m) \\ \vdots \\ f_n(x_1, \dots x_n, u_1, \dots u_m) \end{bmatrix},\; \vec{f}(\vec{x}^*, \vec{u}^*) = 0.$$

Linearized model at operating point $(\vec{x}^*, u^*)$ is :

$$\boxed{\frac{d}{dt}\vec{\delta x}(t) = A\,\vec{\delta x}(t) + B\,\vec{\delta u}(t)}$$

where , $\vec{\delta x}(t) = \vec{x}(t) - \vec{x}^*$, $\vec{\delta u}(t) = \vec{u}(t) - \vec{u}^*$,

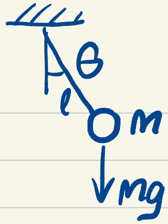
$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_1^*, \dots x_n^*, u_1^*, \dots u_m^*) & \cdots & \frac{\partial f_1}{\partial x_n}(x_1^*, \dots x_n^*, u_1^*, \dots u_m^*) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(x_1^*, \dots x_n^*, u_1^*, \dots u_m^*) & \cdots & \frac{\partial f_n}{\partial x_n}(x_1^*, \dots x_n^*, u_1^*, \dots u_m^*) \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1}(x_1^*, \dots x_n^*, u_1^*, \dots u_m^*) & \cdots & \frac{\partial f_1}{\partial u_m}(x_1^*, \dots x_n^*, u_1^*, \dots u_m^*) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1}(x_1^*, \dots x_n^*, u_1^*, \dots u_m^*) & \cdots & \frac{\partial f_n}{\partial u_m}(x_1^*, \dots x_n^*, u_1^*, \dots u_m^*) \end{bmatrix}.$$

(i,j) entry of A and B given by :

$$\begin{cases} A(i,j) = \frac{\partial f_i}{\partial x_j}(x_1^*, \dots x_n^*, u_1^*, \dots u_m^*) \\ B(i,j) = \frac{\partial f_i}{\partial u_j}(x_1^*, \dots x_n^*, u_1^*, \dots u_m^*). \end{cases}$$

Example 1:



$x_1(t) := \theta(t)$

$x_2(t) := \frac{d\theta}{dt}(t)$

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} x_2(t) \\ -\frac{k}{m}x_2(t) - \frac{g}{l}\sin(x_1(t)) \end{bmatrix}}_{\begin{matrix} f_1(x_1(t), x_2(t)) \\ f_2(x_1(t), x_2(t)) \end{matrix}}$$

$$f_1(x_1, x_2) = x_2$$

$$f_2(x_1, x_2) = -\frac{k}{m}x_2 - \frac{g}{l}\sin x_1$$

Downward pointing equilibrium: $(x_1, x_2) = (0, 0)$
Upward " " : $(x_1, x_2) = (\pi, 0)$

$$J_{\vec{x}}\vec{f}(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_1,x_2) & \frac{\partial f_1}{\partial x_2}(x_1,x_2) \\ \frac{\partial f_2}{\partial x_1}(x_1,x_2) & \frac{\partial f_2}{\partial x_2}(x_1,x_2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{k}{m} \end{bmatrix}$$

$$A = J_{\vec{x}}\vec{f}(\vec{x}^*) \qquad A_{down} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix} = J_{\vec{x}}\vec{f}(0,0)$$

$$A_{up} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{m} \end{bmatrix} = J_{\vec{x}}\vec{f}(\pi,0)$$

Stability criteria (continuous-time) for $A \in \mathbb{R}^{2\times 2}$:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \lambda I - A = \begin{bmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{bmatrix}$$

$$\det(\lambda I - A) = (\lambda - a_{11})(\lambda - a_{22}) - a_{12}a_{21} = \lambda^2 \underbrace{-(a_{11}+a_{22})}_{-tr(A)}\lambda + \underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{\det(A)}$$

Recall trace of A: $tr(A) = a_{11} + a_{12}$

$$\lambda_1 = \frac{tr(A) - \sqrt{tr(A)^2 - 4det(A)}}{2} \qquad \lambda_2 = \frac{tr(A) + \sqrt{tr(A)^2 - 4det(A)}}{2}$$

(i) $det(A) \leq 0 \Rightarrow$ unstable

because $\sqrt{tr(A)^2 - 4det(A)} \geq \sqrt{tr(A)^2} = |tr(A)|$

therefore $\lambda_2 \geq \frac{tr(A) + |tr(A)|}{2} \geq 0$

(ii) $tr(A) \geq 0 \Rightarrow$ unstable

- If $tr(A)^2 - 4det(A) < 0$ then $\sqrt{tr(A)^2 - 4det(A)}$ imaginary and $Re(\lambda_1) = Re(\lambda_2) = \frac{tr(A)}{2} \geq 0$.
- If $tr(A)^2 - 4det(A) \geq 0$ then $\sqrt{tr(A)^2 - 4det(A)} \geq 0$ and $\lambda_2 \geq \frac{tr(A)}{2} \geq 0$.

(iii) $det(A) > 0$ and $tr(A) < 0 \Rightarrow$ stable

- If $tr(A)^2 - 4det(A) < 0$ then $\sqrt{tr(A)^2 - 4det(A)}$ imaginary and $Re(\lambda_1) = Re(\lambda_2) = \frac{tr(A)}{2} < 0$.
- If $tr(A)^2 - 4det(A) \geq 0$, then $\sqrt{tr(A)^2 - 4det(A)} \in [0, |tr(A)|)$ where the upper bound follows from $det(A) > 0$. Thus, $\lambda_1 = \frac{tr(A) - \sqrt{\cdot}}{2} \leq \frac{tr(A)}{2} < 0$, $\lambda_2 = \frac{tr(A) + \sqrt{\cdot}}{2} < \frac{tr(A) + |tr(A)|}{2} \leq 0$.

Combining (i)-(iii): stable if and only if $det(A) > 0$ and $tr(A) < 0$.

$$A_{down} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{M} \end{bmatrix} \qquad A_{up} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{M} \end{bmatrix}$$

$tr(A_{down}) = -\frac{k}{M} < 0$ ✓ $\qquad tr(A_{up}) = -\frac{k}{M} < 0$ ✓

$det(A_{down}) = \frac{g}{l} > 0$ ✓ $\qquad det(A_{down}) = -\frac{g}{l} < 0$ ✗

stable $\qquad$ unstable

Look at evalues of A_{up} for more insight:

$$\det(\lambda I - A_{up}) = \det\left(\begin{bmatrix} \lambda & -1 \\ -g/\ell & \lambda + \frac{k}{m} \end{bmatrix}\right) = \lambda^2 + \frac{k}{m}\lambda - \frac{g}{\ell}$$

$$\lambda_{1,2} = \frac{-\frac{k}{m} \mp \sqrt{\left(\frac{k}{m}\right)^2 + 4\frac{g}{\ell}}}{2}$$

$$\lambda_2 = \frac{-\frac{k}{m} + \sqrt{\left(\frac{k}{m}\right)^2 + 4\frac{g}{\ell}}}{2} > 0 \quad \text{since} \quad \sqrt{\left(\frac{k}{m}\right)^2 + 4\frac{g}{\ell}} > \sqrt{\left(\frac{k}{m}\right)^2} = \frac{k}{m}$$

Note: λ_2 becomes more positive (therefore instability more severe) when ℓ is smaller. Try balancing a shorter stick in your hand!

Linearization in Discrete Time:

$$\vec{x}[i+1] = \vec{f}(\vec{x}[i], \vec{u}[i]) \qquad (1)$$

$$\vec{x}^* = \vec{f}(\vec{x}^*, \vec{u}^*)$$

$$\vec{f}(\vec{x}, \vec{u}) \approx \underbrace{\vec{f}(\vec{x}^*, \vec{u}^*)}_{\vec{x}^*} + A\underbrace{(\vec{x} - \vec{x}^*)}_{\delta\vec{x}} + B\underbrace{(\vec{u} - \vec{u}^*)}_{\delta\vec{u}} \qquad (2)$$

where A and B are obtained as in page 1.

Linearization: replace $\vec{f}$ in (1) with approximation in (2):

$$\vec{x}[i+1] = \vec{x}^* + A\,\delta\vec{x}[i] + B\,\delta\vec{u}[i]$$

$$\underbrace{\vec{x}[i+1] - \vec{x}^*}_{= \delta\vec{x}[i+1]} = A\,\delta\vec{x}[i] + B\,\delta\vec{u}[i]$$

Example 2: Population growth model

$$x[i+1] = r\,\underbrace{x[i]}_{\text{population of a species in generation } i}$$

Assume $r>1 \Rightarrow$ exponential, unbounded growth.
Unrealistic because resources run out when x gets large.
A more relastic model where cst. r is replaced with x-dependent growth rate $r(1-\frac{x}{N})$:

$$x[i+1] = \underbrace{r\left(1-\frac{x[i]}{N}\right)x[i]}_{f(x[i])} \quad \text{nonlinear}$$

$$f(x) = r\left(1-\frac{x}{N}\right)x = rx - \frac{r}{N}x^2 \,, \quad f'(x) = r - \frac{2r}{N}x \quad (3)$$

Equilibrium (discrete time):

$$x = f(x)$$
$$x = r\left(1-\frac{x}{N}\right)x$$
$$x\left(1-r+\frac{r}{N}x\right) = 0$$

Two roots: $x=0$ (extinct) $\quad x = \frac{N}{r}(r-1) > 0$ since $r>1$

Linearized model:

i) around $x^*=0$: $\delta x[i+1] = \underbrace{f'(0)}_{=r \text{ from } (3)}\delta x[i]$

$r>1 \Rightarrow$ unstable

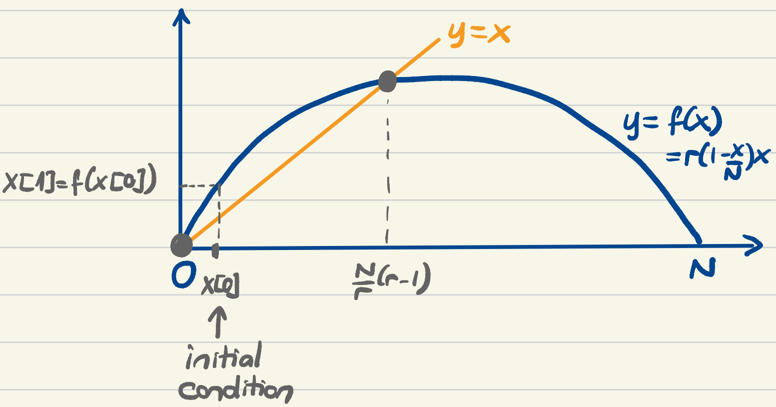
ii) around $x^* = \frac{N}{r}(r-1)$:

$$f'\left(\frac{N}{r}(r-1)\right) = r - \frac{2\cancel{r}}{\cancel{N}}\left(\frac{\cancel{N}}{\cancel{r}}(r-1)\right) = r - 2r + 2$$
$$= 2 - r$$

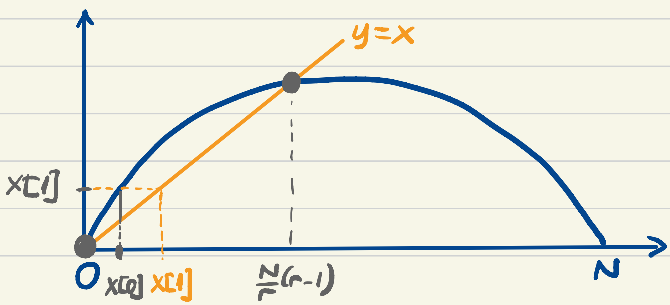
$$\delta x[i+1] = \underbrace{(2-r)}_{\in(-1,1)\ \text{for stability}} \delta x[i]$$

i.e. $r \in (1,3)$ " "

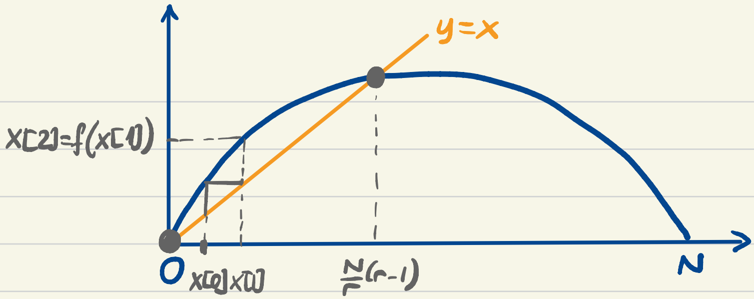
"Cobweb diagrams": graphical method to visualize how solutions of scalar discrete-time equations evolve (beyond scope for this course but kinda fun):

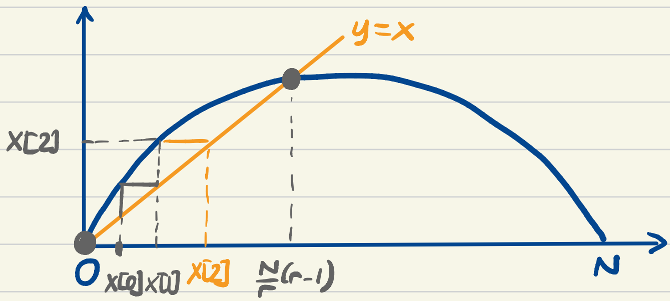


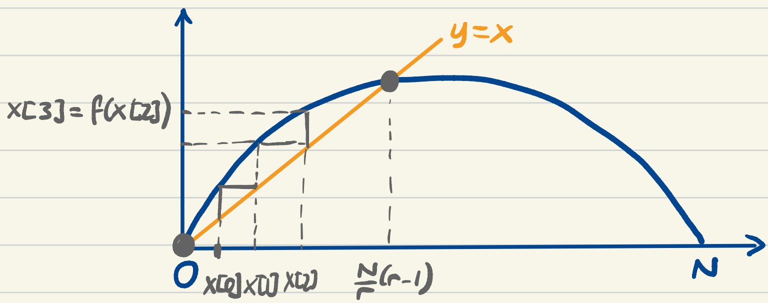
intersections of $y=f(x)$ and $y=x$ are equilibrium points b/c $f(x)=x$ at those points



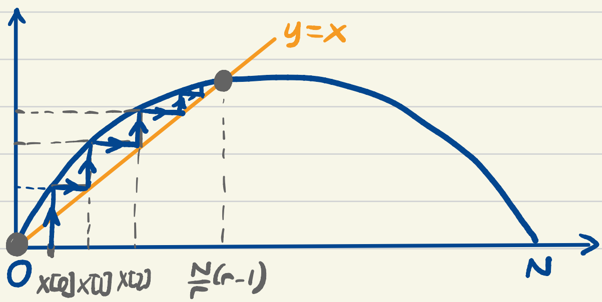
used $y=x$ axis to bring value $x[1]$ from vertical to horizontal axis







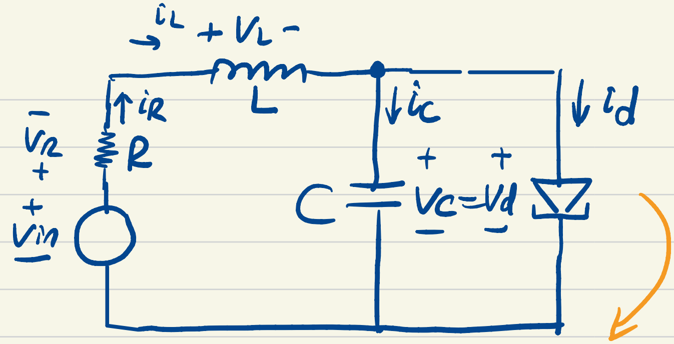
Continuing in this manner, you can see that the solution from x[0] evolves as shown below, indicating convergence to $x^* = \frac{N}{r}(r-1)$.



Examples above didn't have inputs. Ex. 3 below has V_{in} as input.

Example 3:

tunnel diode circuit



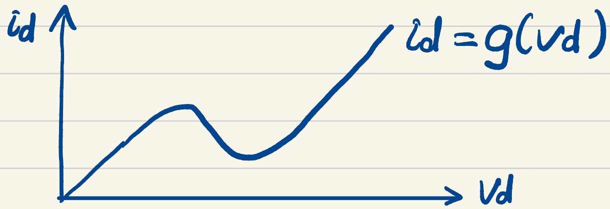
KCL:

(3) $i_R = i_L = i_C + i_d$

KVL:

(4) $V_{in} = V_R + V_L + V_C$

(5) $V_d = V_C$



tunnel diode voltage-current characteristic

$$\vec{x} = \begin{bmatrix} V_C \\ i_L \end{bmatrix}$$

$$C\frac{dV_C}{dt} = i_C(t) \overset{(3)}{=} i_L(t) - i_d(t) = i_L(t) - g(V_d(t))$$

$$\overset{(5)}{=} i_L(t) - g(V_C(t))$$

$$L\frac{di_L}{dt} = V_L(t) \overset{(4)}{=} V_{in}(t) - V_R(t) - V_C(t)$$

$$\overset{\text{resistor eqn}}{=} V_{in}(t) - R\, i_R(t) - V_C(t)$$

$$\overset{(3)}{=} V_{in}(t) - R\, i_L(t) - V_C(t)$$

Variables in orange above are neither state nor input variables. We use the substitutions indicated above each equation to make sure the right hand side depends only on $\vec{x} = \begin{bmatrix} V_C \\ i_L \end{bmatrix}$ and $u = V_{in}$, so we get a model of the form

$$\frac{d}{dt}\vec{x}(t) = \vec{f}(\vec{x}(t), u(t)).$$

$$\boxed{\begin{aligned} \frac{d}{dt}V_C(t) &= \frac{1}{C}i_L(t) - \frac{1}{C}g(V_C(t)) \\ \frac{d}{dt}i_L(t) &= \frac{1}{L}V_{in}(t) - \frac{R}{L}i_L(t) - \frac{1}{L}V_C(t) \end{aligned}}$$

Can't write in matrix/vector form because of nonlinearity $g(\cdot)$.