EECS 16B

Spring 2022

Lecture 24

4/14/2022

LECTURE 24

- nonlinear systems
- linearization

So far discussed linear systems:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t) \qquad \vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i]$$

Today nonlinear:

$$\frac{d}{dt}\vec{x}(t) = \vec{f}(\vec{x}(t), \vec{u}(t)) \qquad \vec{x}[i+1] = \vec{f}(\vec{x}[i], \vec{u}[i])$$

where $\vec{f}: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a vector-valued function

of state $\vec{x} \in \mathbb{R}^n$ and input $\vec{u} \in \mathbb{R}^m$.

Linear systems are a special case: $\vec{f}(\vec{x}, \vec{u}) = A\vec{x} + B\vec{u}$

<u>Example 1:</u>

$$m\ell \frac{d^2\theta(t)}{dt^2} = -k\ell \frac{d\theta(t)}{dt} - mg\sin\theta(t) \qquad -(1)$$

Let $\vec{x}(t) = \begin{bmatrix} \theta(t) \\ \omega(t) \end{bmatrix}$ $\quad \omega(t) = \frac{d\theta}{dt}(t)$ (angular velocity)

$$\frac{d\vec{x}(t)}{dt} = \begin{bmatrix} \frac{d\theta}{dt} \\ \frac{d\omega}{dt} \end{bmatrix} = \begin{bmatrix} \omega(t) \\ \frac{d^2\theta(t)}{dt^2} \end{bmatrix} = \underbrace{\begin{bmatrix} \omega(t) \\ -\frac{k}{m}\omega(t) - \frac{g}{l}\sin\theta(t) \end{bmatrix}}$$

$x_1 = \theta$, $x_2 = \frac{d\theta}{dt} = \omega$

$$= \begin{bmatrix} x_2(t) \\ -\frac{k}{m}x_2(t) - \frac{g}{l}\sin(x_1(t)) \end{bmatrix}$$

$$= \vec{f}(\vec{x}(t))$$

Equilibrium (Operating) Points:

For a continuous-time system with no input

$$\frac{d}{dt}\vec{x}(t) = \vec{f}_c(\vec{x}(t)) \qquad -(2)$$

the solutions of the static eq'n $\boxed{\vec{f}_c(\vec{x}) = 0}$ are called equilibrium points. If $\vec{x}^*$ is an equilibrium i.e. if $\vec{f}_c(\vec{x}^*) = 0$, then $\vec{x}(t) = \vec{x}^*$ is a solution of diff. eq'n above with $\vec{x}(0) = \vec{x}^*$: substitute $\vec{x}(t) = \vec{x}^*$ in (2):

$$\underbrace{\frac{d}{dt}\vec{x}^*}_{=0} = \underbrace{\vec{f}_c(\vec{x}^*)}_{=0}.$$

bc/ $\vec{x}^* = \text{cst.}$

Pendulum example: $\vec{f}(\vec{x}) = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_2 - \frac{g}{l}\sin x_1 \end{bmatrix} = 0$

$$\Rightarrow \begin{cases} x_2 = 0 & \text{--- (3)} \\ -\frac{k}{m}x_2 - \frac{g}{l}\sin x_1 = 0 & \text{---- (4)} \end{cases}$$

Substitute (3) in (4): $\sin x_1 = 0 \Rightarrow x_1 = 0, \pi$

Two equilibrium points: $(x_1, x_2) = (0, 0)$ downward pointing

$(x_1, x_2) = (\pi, 0)$ upward pointing

What about discrete time equilibria?

$$\vec{x}[i+1] = \vec{f_d}(\vec{x}[i]) \quad \text{-- (5)}$$

$\vec{x}^*$ is an equilibrium if : $\boxed{\vec{x}^* = \vec{f_d}(\vec{x}^*)}$.

$\vec{x}[i] = \vec{x}^*$ for all i is a solution of (5) because

$\vec{x}[i] = \vec{x}^* \Rightarrow \vec{x}[i+1] = \vec{f_d}(\vec{x}[i]) = \vec{f_d}(\vec{x}^*) = \vec{x}^*$.

Systems with inputs:

$(\vec{x}^*, \vec{u}^*)$ is an "operating point" of

$$\frac{d}{dt}\vec{x}(t) = \vec{f_c}(\vec{x}(t), \vec{u}(t)) \quad \text{-- (6)}$$

if $\boxed{\vec{f_c}(\vec{x}^*, \vec{u}^*) = 0}$. -- (7)

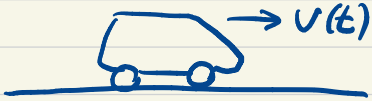
$(\vec{x}^*, \vec{u}^*)$ is an "operating point" of

$$\vec{x}[i+1] = \vec{f_d}(\vec{x}[i], \vec{u}[i])$$

if $\boxed{\vec{x}^* = \vec{f_d}(\vec{x}^*, \vec{u}^*)}$

If we apply the constant input $\vec{u}(t) = \vec{u}^*$ then $\vec{x}(t) = \vec{x}^*$ is a solution of (6) with $\vec{x}(0) = \vec{x}^*$.

Example 2:



$$M \frac{dv(t)}{dt} = -\beta v(t)^2 + \frac{1}{R} u(t)$$

$x = v$ is the (single) state, $f(x,u) = \frac{-\beta}{M} x^2 + \frac{1}{RM} u$.

(x^*, u^*) is an operating point if (from Eq'n 7):

$$f(x^*, u^*) = 0 \Rightarrow u^* = \beta R x^{*2}.$$

If we want speed x^* we must apply torque u^*

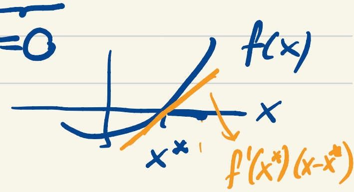
to overcome drag $\beta R x^{*2}$.

Linearization: linear approximation of nonlinear model around an operating point

Easy when $x \in \mathbb{R}$:

1) no input: $\frac{d}{dt} x(t) = f(x(t))$, $f(x^*) = 0$ (8)

Taylor approximation: $f(x) \approx \underbrace{f(x^*)}_{=0} + f'(x^*)(x - x^*)$



Define $\delta x(t) = x(t) - x^*$. Then:

$$\frac{d}{dt}\delta x(t) = \frac{d}{dt}(x(t) - x^*) = \frac{d}{dt}x(t) - \underbrace{\cancel{\frac{d}{dt}x^*}}_{0} = f(x(t)) \overset{(8)}{\approx} f'(x^*)\,\delta x(t)$$

Linearized model:

$$\frac{d}{dt}\delta x(t) = \underbrace{f'(x^*)}_{=:\lambda}\,\delta x(t)$$

2) with input $u \in \mathbb{R}$: $\frac{d}{dt}x(t) = f(x(t), u(t))$

Suppose (x^*, u^*) operating point: $f(x^*, u^*) = 0$

$$f(x,u) \approx \underbrace{f(x^*, u^*)}_{=0} + \underbrace{\frac{\partial f}{\partial x}(x^*, u^*)}_{=:\lambda}\underbrace{(x - x^*)}_{\delta x} + \underbrace{\frac{\partial f}{\partial u}(x^*, u^*)}_{b}\underbrace{(u - u^*)}_{=:\delta u}$$

b/c operating pt.

$$\frac{d}{dt}\delta x(t) = \frac{d}{dt}x(t) - \underbrace{\cancel{\frac{d}{dt}x^*}}_{0} = \lambda\,\delta x(t) + b\,\delta u(t)$$

<u>Example 2</u>: $f(x,u) = \frac{-\beta}{M}x^2 + \frac{1}{RM}u$

$$\frac{\partial f}{\partial x}(x,u) = -\frac{2\beta}{M}x \qquad \frac{\partial f}{\partial u}(x,u) = \frac{1}{RM}$$

$$\lambda = \frac{\partial f}{\partial x}(x^*, u^*) = \frac{-2\beta}{M}x^* \qquad b = \frac{\partial f}{\partial u}(x^*, u^*) = \frac{1}{RM}$$

$$\frac{d}{dt}\delta x(t) = \lambda \delta x(t) + b\delta u(t)$$

where $\delta x(t) = x(t) - x^*$, $\delta u(t) = u(t) - u^*$,

$u^* = \beta R x^{*2}$. Assume we apply $\delta u = 0$

$(u(t) = u^*)$: $\frac{d}{dt}\delta x(t) = \lambda \delta x(t)$

$$\Rightarrow \delta x(t) = e^{\lambda t}\delta x(0)$$

$\lambda = -\frac{2\beta}{M}x^* < 0$ so $\delta x(t) \to 0$, i.e., $x(t) \to x^*$.

If λ not negative enough (slow convergence to x^*) we can apply feedback:

$$\delta u(t) = k\,\delta x(t)$$

Closed-loop system: $\frac{d}{dt}\delta x(t) = (\lambda + bk)\,\delta x(t)$

We can choose k to make $\lambda + bk$ as negative as we wish : $\delta x(t) = \underbrace{e^{(\lambda+bk)t}}_{\to 0 \text{ faster}}\delta x(0)$

$$u(t) = u^* + \delta u(t)$$

$$= \beta R x^{*2} + k\,\delta x(t)$$

$$\boxed{u(t) = \beta R x^{*2} + k(x(t) - x^*)}$$

cruise control

Next assume $\vec{x} \in \mathbb{R}^2$, $u \in \mathbb{R}$.

$\vec{f}(\vec{x},u) \in \mathbb{R}^2$, so we can write $\vec{f}(\vec{x},u)$ as $\begin{bmatrix} f_1(x_1,x_2,u) \\ f_2(x_1,x_2,u) \end{bmatrix}$ where f_1, f_2 are scalar valued functions.

As before,

$$
\begin{aligned}
f_1(x_1,x_2,u) \approx \overbrace{f_1(x_1^*,x_2^*,u^*)}^{=0} &+ \frac{\partial f_1}{\partial x_1}(x_1^*,x_2^*,u^*)\overbrace{(x_1-x_1^*)}^{\delta x_1} \\
&+ \frac{\partial f_1}{\partial x_2}(x_1^*,x_2^*,u^*)\underbrace{(x_2-x_2^*)}_{\delta x_2} \qquad (9) \\
&+ \frac{\partial f_1}{\partial u}(x_1^*,x_2^*,u^*)\underbrace{(u-u^*)}_{\delta u}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
f_2(x_1,x_2,u) \approx \overbrace{f_2(x_1^*,x_2^*,u^*)}^{=0} &+ \frac{\partial f_2}{\partial x_1}(x_1^*,x_2^*,u^*)\overbrace{(x_1-x_1^*)}^{\delta x_1} \\
&+ \frac{\partial f_2}{\partial x_2}(x_1^*,x_2^*,u^*)\underbrace{(x_2-x_2^*)}_{\delta x_2} \qquad (10) \\
&+ \frac{\partial f_2}{\partial u}(x_1^*,x_2^*,u^*)\underbrace{(u-u^*)}_{\delta u}
\end{aligned}
$$

Combining (9)-(10) in matrix/vector form:

$$
\underbrace{\begin{bmatrix} f_1(x_1,x_2,u) \\ f_2(x_1,x_2,u) \end{bmatrix}}_{\vec{f}(\vec{x},u)} \approx \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_1^*,x_2^*,u^*) & \frac{\partial f_1}{\partial x_2}(x_1^*,x_2^*,u^*) \\ \frac{\partial f_2}{\partial x_1}(x_1^*,x_2^*,u^*) & \frac{\partial f_2}{\partial x_2}(x_1^*,x_2^*,u^*) \end{bmatrix}}_{=:A} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial u}(x_1^*,x_2^*,u^*) \\ \frac{\partial f_2}{\partial u}(x_1^*,x_2^*,u^*) \end{bmatrix}}_{=:B} \delta u \qquad (11)
$$

Then, $\frac{d}{dt}\begin{bmatrix}\delta x_1(t)\\ \delta x_2(t)\end{bmatrix} = \frac{d}{dt}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} - \cancelto{0}{\frac{d}{dt}\begin{bmatrix}x_1^*\\ x_2^*\end{bmatrix}} = \vec{f}(\vec{x}(t), u(t))$

and substitution of (11) gives linearized model:

$$\frac{d}{dt}\begin{bmatrix}\delta x_1(t)\\ \delta x_2(t)\end{bmatrix} = A\begin{bmatrix}\delta x_1(t)\\ \delta x_2(t)\end{bmatrix} + B\,\delta u(t),$$

where A, B are as defined in (11).

<u>Easily generalizable to $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$.</u>

$$\vec{f}(\vec{x}, \vec{u}) = \begin{bmatrix} f_1(x_1, \ldots x_n, u_1, \ldots u_m) \\ \vdots \\ f_n(x_1, \ldots x_n, u_1, \ldots u_m) \end{bmatrix}$$

Linearized model at a given operating point $(\vec{x}^*, u^*)$ is:

$$\frac{d}{dt}\vec{\delta x}(t) = A\,\vec{\delta x}(t) + B\,\vec{\delta u}(t)$$

where, $\vec{\delta x}(t) = \vec{x}(t) - \vec{x}^*$, $\vec{\delta u}(t) = \vec{u}(t) - \vec{u}^*$;

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_1^*, \ldots x_n^*, u_1^*, \ldots u_m^*) & \cdots & \frac{\partial f_1}{\partial x_n}(x_1^*, \ldots x_n^*, u_1^*, \ldots u_m^*) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(x_1^*, \ldots x_n^*, u_1^*, \ldots u_m^*) & \cdots & \frac{\partial f_n}{\partial x_n}(x_1^*, \ldots x_n^*, u_1^*, \ldots u_m^*) \end{bmatrix},$$

that is, $A(i,j) = \frac{\partial f_i}{\partial x_j}(x_1^*, \ldots x_n^*, u_1^*, \ldots u_m^*)$, and

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1}(x_1^*, \ldots x_n^*, u_1^*, \ldots u_m^*) & \cdots & \frac{\partial f_1}{\partial u_m}(x_1^*, \ldots x_n^*, u_1^*, \ldots u_m^*) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1}(x_1^*, \ldots x_n^*, u_1^*, \ldots u_m^*) & \cdots & \frac{\partial f_n}{\partial u_m}(x_1^*, \ldots x_n^*, u_1^*, \ldots u_m^*) \end{bmatrix}$$

$$B(i, j) = \frac{\partial f_i}{\partial u_j}(x_1^*, \ldots x_n^*, u_1^*, \ldots u_m^*).$$