EECS 16B

Spring 2022

Lecture 18

3/17/2022

LECTURE 18 :
- recap of upper triangularization
- Spectral Theorem

<u>Last time</u>: any square matrix can be upper triangularized (proven for real matrices with real eigenvalues; true for complex eigenvalues also).

$$U^T A U = T \qquad \begin{array}{l} U: \text{orthogonal} \\ T: \text{upper triangular} \end{array}$$

Note: 1) A and T have the same eigenvalues :

if $(\lambda, \vec{v})$ evalue/evector for T, then $(\lambda, U\vec{v})$ is " " for A

$$A(U\vec{v}) = (UTU^T)(U\vec{v}) = UT\vec{v} = U\lambda\vec{v} = \lambda(U\vec{v})$$

with $\underbrace{U^T U}_{=I}$

multiply from left by U, right by U^T : $A = UTU^T$

thus, $A(U\vec{v}) = \lambda(U\vec{v})$ i.e. λ is evalue for A also.

2) diagonal of entries of T are its evalues (because upper triangular – shown last time)

From (1), (2) : once matrix A is upper triangularized its evalues appear in the diagonal entries of T:

$$U^T A U = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}.$$

4) Induction proof from last lecture can be turned into a recursive algorithm (Algorithm 10 in Note 15):

Define a function:

Given $A \in \mathbb{R}^{(k+1)\times(k+1)}$ with real evalues,

- pick evalue/evector pair $(\lambda_1, \vec{q}_1)$ such that $\|\vec{q}_1\|=1$
- complete $\vec{q}_1$ to an orthonormal basis for $\mathbb{R}^{k+1}$: $\{\vec{q}_1, \vec{q}_2, \dots \vec{q}_{k+1}\}$, e.g., by Gram Schmidt.

  Define $Q = [\vec{q}_1 \ \vec{q}_2 \cdots \vec{q}_{k+1}]$. Then

$$Q^T A Q = \begin{bmatrix} \lambda_1 & * \\ \vec{0} & A_0 \end{bmatrix}.$$

- Return Q and A_0.

Given $n\times n$ matrix A that we want to upper-triangularize:

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(Q, A0) = function_above(A)
U = Q
while size(A0) > 1
    (Q, A0) = function_above(A0)
    U = U * [1 0
             0 Q]
end;
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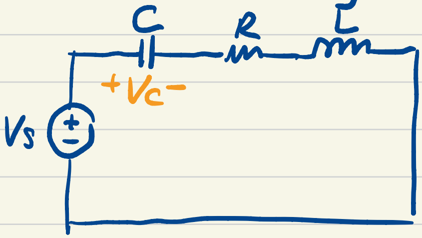
Example: (Critically damped) RLC circuit



HW5:

$x_1(t) = V_c(t)$

$x_2(t) = \frac{d}{dt} V_c(t)$

$$\lambda_{1,2} = \frac{-R}{2L} \mp \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}} \Leftarrow A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}$$

Suppose $\underbrace{\frac{R^2}{L^2} = \frac{4}{LC}}$ $\Rightarrow \lambda_1 = \lambda_2 = \frac{-R}{2L}$ (HW5 assumed $\lambda_1 \neq \lambda_2$)

$$\hookrightarrow \frac{-1}{LC} = \frac{-R^2}{4L^2}$$

$$\Downarrow$$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{R^2}{4L^2} & -\frac{R}{L} \end{bmatrix}$$

Evectors: $\lambda_{1,2} I - A = \begin{bmatrix} -\frac{R}{2L} & \\ & -\frac{R}{2L} \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ \frac{R^2}{4L^2} & \frac{R}{L} \end{bmatrix}$

$$= \begin{bmatrix} -\frac{R}{2L} & -1 \\ \frac{R^2}{4L^2} & \frac{R}{2L} \end{bmatrix}$$

null space?

$$\vec{v} = \begin{bmatrix} 1 \\ -\frac{R}{2L} \end{bmatrix} \cdot \alpha \quad \alpha \neq 0$$

Can't find two lin. ind. evectors.

$\Rightarrow$ not diagonalizable.

Can nevertheless upper triangularize A:

$$U^T A U = \begin{bmatrix} \lambda & * \\ 0 & \lambda \end{bmatrix}, \quad \lambda = \frac{-R}{2L}$$

for some orthogonal U you will find in HW9.

Sol'n of circuit diff. eq'n ($V_s = 0$ for simplicity):

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) \xrightarrow{\vec{y} = U^T\vec{x}} \frac{d}{dt}\vec{y}(t) = U^T A \underbrace{\vec{x}(t)}_{U\vec{y}(t)}$$

$$= \underbrace{U^T A U}_{\begin{bmatrix} \lambda & * \\ 0 & \lambda \end{bmatrix}} \vec{y}(t)$$

$$\frac{d}{dt} y_1(t) = \lambda y_1(t) + * \, y_2(t)$$

$$\frac{d}{dt} y_2(t) = \lambda y_2(t)$$

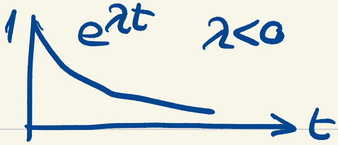
$$y_2(t) = e^{\lambda t} y_2(0)$$

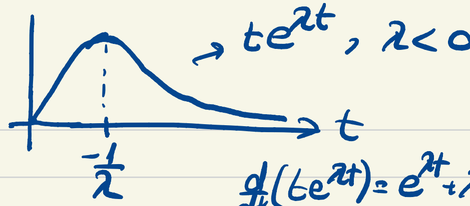
$$\frac{d}{dt} y_1(t) = \lambda y_1(t) + \underbrace{* \, e^{\lambda t} y_2(0)}_{\text{treat as input}}$$

$$y_1(t) = e^{\lambda t} y_1(0) + \underbrace{\int_0^t \underbrace{e^{\lambda(t-\tau)} * \, e^{\lambda \tau} y_2(0)}_{= e^{\lambda t} * \, y_2(0)} \, d\tau}_{= e^{\lambda t} * \, y_2(0) \underbrace{\int_0^t d\tau}_{t}}$$

$$\boxed{\begin{aligned} y_1(t) &= e^{\lambda t} y_1(0) + t e^{\lambda t} * \, y_2(0) \\ y_2(t) &= e^{\lambda t} y_2(0) \end{aligned}}$$

new! signature of repeated eigenvalue





$\frac{d}{dt}(te^{\lambda t}) = e^{\lambda t} + \lambda t e^{\lambda t}$

Similary $t^2 e^{\lambda t}$ would pop up in the solution to:

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} \lambda & * & * \\ 0 & \lambda & * \\ 0 & 0 & \lambda \end{bmatrix} \vec{y}(t)$$

$U^T A U = T \Rightarrow A = U T U^T$ where U: orthogonal
T: upper triangle

called "Schur decomposition"

Spectral Theorem: Motivation:

- For a diagonalizable matrix A we can find V s.t.

$$V^{-1} A V = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

  V: not necessarily orthogonal

- If we instead upper triangularize we find orthogonal U such that

$$U^{-1} A U = U^T A U = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

- For symmetric matrices $(A = A^T)$ we get both:
  Diagonalizable with an orthogonal V:

$$V^{-1} A V = V^T A V = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

Theorem (Spectral Theorem):

Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Then:

(i) The evalues of A are real.

(ii) A diagonalizable.

(iii) Evectors of A are pairwise orthogonal (therefore, if we choose them to have length 1 then they constitute orthonormal matrix:

$$V = [\vec{v}_1 \cdots \vec{v}_n] \text{ orthogonal matrix})$$

Proof:

(i) Let $(\lambda, \vec{v})$ be evalue/evector pair.

$$A\vec{v} = \lambda \vec{v} \qquad \cdots (1)$$

$$\lambda = a + jb \quad (\text{show } b = 0\text{, i.e., } \bar{\lambda} = a - jb = a = \lambda)$$

Take complex conjugates of both sides of (1):

$$\overline{(A\vec{v})} = \overline{(\lambda \vec{v})}$$

$$= \underbrace{\bar{A}}_{=A \text{ because } A \text{ real}}\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}}$$

$$A\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}}$$

Transpose: $\bar{\vec{v}}^T A^T = \bar{\vec{v}}^T \bar{\lambda}$ $= A$ by symmetry

multiply from right by $\vec{v}$:

$$\bar{\vec{v}}^T A \vec{v} = \bar{\lambda}(\bar{\vec{v}}^T \vec{v}) \qquad --(2)$$

Multiply (1) from left by $\bar{\vec{v}}^T$:

$$\bar{\vec{v}}^T A \vec{v} = \lambda(\bar{\vec{v}}^T\vec{v}) \quad --- \quad (3)$$

LHS of (2), (3) are the same so RHS must also be same:

$$\bar{\lambda}\,(\bar{\vec{v}}^T\vec{v}) = \lambda(\underbrace{\bar{\vec{v}}^T\vec{v}})$$

$\Rightarrow$ $\bar{\lambda} = \lambda$

$$[\bar{v}_1 \cdots \bar{v}_n]\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \bar{v}_1 v_1 + \cdots + \bar{v}_n v_n$$

$$= |v_1|^2 + \cdots |v_n|^2$$

$\neq 0$ bc $\vec{v}$ is evector

$\Rightarrow$ λ is real.

Evector $\vec{v}$ also real:

$$(A - \underbrace{\lambda}_{\text{real}} I)\vec{v} = 0$$

$$\underbrace{(A - \lambda I)}_{\text{real}}$$

(ii) Apply Schur decomposition:

$$U^T A U = T$$

$$T^T = U^T A^T (U^T)^T$$

$$= U^T A U = T$$

$T = T^T$ and upper triangular $\Rightarrow$ T diagonal

$$U^T A U = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$