EECS 16B

Spring 2022

Lecture 15

3/8/2022

LECTURE 15

- complete feedback stabilization
- controllability

Definition: We say that a system is (bounded-input, bounded state) stable if state x is bounded for any initial condition and bounded disturbance pair. Unstable otherwise, i.e., if x grows unbounded for <u>some</u> initial condition, disturbance.

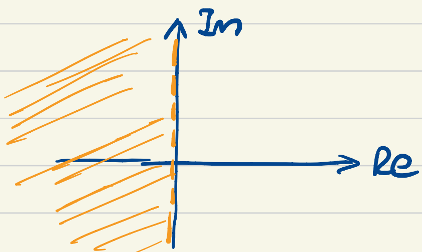
Stability Criteria:

Continuous-time

$$\frac{d}{dt}\vec{x}(t) = A_c\vec{x}(t)+\vec{w}(t)$$

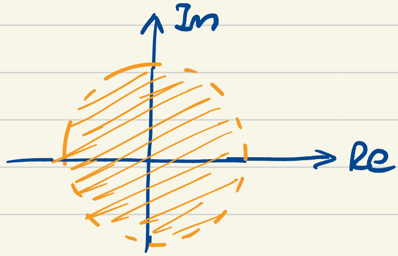
Discrete-time

$$\vec{x}[i+1] = A_d\vec{x}[i]+\vec{w}[i]$$



$\mathrm{Re}\,\lambda_k < 0$

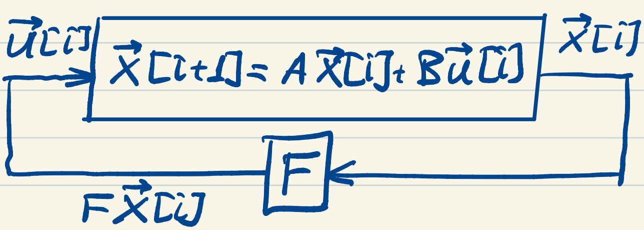
for each eigenvalue of A_c



$|\lambda_k| < 1$

for each eigenvalue of A_d

Stabilization by Feedback:



$$\Rightarrow \vec{x}[i+1] = \underbrace{(A+BF)}_{=:A_{CL}}\vec{x}[i]$$

We can assign the eigenvalues of A_{CL} arbitrarily with choice of F if we can bring (A,B) to ...

Controller Canonical Form (for scalar u):

$$TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & \vdots \\ & & & \ddots & \\ & & & & 1 \\ * & * & \cdots & & * \end{bmatrix} \qquad TB = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

__Claim:__ An invertible $n \times n$ matrix T bringing TAT^{-1} and TB to the form above exists if

$$[A^{n-1}B \quad A^{n-2}B \quad \cdots \quad AB \quad B]$$

is nonsingular, i.e. the columns $A^{n-1}B, \ldots, AB, B$ are linearly independent.

Proof: Let q^T denote the top row of

$$[A^{n-1}B \quad A^{n-2}B \quad \cdots \quad AB \quad B]^{-1}.$$

$$\begin{bmatrix} q^T \\ \text{— second row —} \\ \vdots \\ \text{— nth row —} \end{bmatrix} [A^{n-1}B \quad A^{n-2}B \quad \cdots \quad AB \quad B] = \begin{bmatrix} 1 & 0 & \cdots & \\ 0 & 1 & \cdots & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

$$\left.\begin{aligned} q^T A^{n-1} B &= 1 \\ q^T A^{n-2} B &= 0 \\ &\vdots \\ q^T A B &= 0 \\ q^T B &= 0 \end{aligned}\right\} \quad \cdots (1)$$

Take $T = \begin{bmatrix} q^T \\ q^TA \\ \vdots \\ q^TA^{n-1} \end{bmatrix}$

$$TB = \begin{bmatrix} q^TB \\ q^TAB \\ \vdots \\ q^TA^{n-1}B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$TA = \begin{bmatrix} q^TA \\ q^TA^2 \\ \vdots \\ q^TA^n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & \\ 0 & 0 & 1 & 0 & \cdots \\ & & & \ddots & \\ 0 & \cdots & & & 1 \\ * & \cdots & & & * \end{bmatrix} \underbrace{\begin{bmatrix} q^T \\ q^TA \\ \vdots \\ q^TA^{n-1} \end{bmatrix}}_{T}$$

TAT^{-1} has this form

How do we know this T is invertible? Here is how: note

$$\begin{bmatrix} q^T \\ q^TA \\ \vdots \\ q^TA^{n-1} \end{bmatrix} \begin{bmatrix} A^{n-1}B & \cdots & AB & B \end{bmatrix}$$

$$= \begin{bmatrix} q^TA^{n-1}B & q^TA^{n-2}B & \cdots & q^TAB & q^TB \\ q^TA^nB & q^TA^{n-1}B & \cdots & & q^TAB \\ q^TA^{n+1}B & & & \ddots & \\ \vdots & & & & q^TA^{n-1}B \end{bmatrix}$$

$$\overset{\text{by (1)}}{=} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \ddots & 0 \\ * & \cdots & & 1 \end{bmatrix} \quad \text{-- (2)}$$

If rows of T were linearly dependent we could find $\vec{w} \neq 0$ such that $\vec{w}^T T = 0$. But then (2) implies $\vec{w}^T \begin{bmatrix} 1 & & 0 \\ & \ddots & \vdots \\ * & & 1 \end{bmatrix} = 0$ $\Rightarrow \vec{w} = 0$

Contradiction!

Note: The condition of the Claim is a certificate for being able to assign evalues of $A+BF$ by choice of F. We don't need to bring the system to the canonical form to design F.

Example 1: $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $n=2$

Are B and AB linearly independent? YES. $AB = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$A + BF = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix}}_{\begin{bmatrix} 0 & 0 \\ f_1 & f_2 \end{bmatrix}} = \begin{bmatrix} 1 & 1 \\ f_1 & 2+f_2 \end{bmatrix} = A_{CL}$$

Note: evalues of A are $\{1,2\}$.

Evalues of A_{CL}: $\det(\lambda I - A_{CL}) = \det\left(\begin{bmatrix} \lambda-1 & -1 \\ -f_1 & \lambda-2-f_2 \end{bmatrix}\right)$

$$= \lambda^2 - (3+f_2)\lambda + 2 + f_2 - f_1$$

Let's say we want closed-loop evalues at $\{0,0\}$.

That means characteristic polynomial $= \lambda^2$

Match coefficients:

$$3 + f_2 = 0, \quad 2 + f_2 - f_1 = 0$$

$$f_1 = -1, \quad f_2 = -3$$

Note: upper triangular matrix

$$A = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

has evalues $\lambda_1, \ldots \lambda_n$ (diagonal entries). Why?

Look at $\lambda_n I - A = \begin{bmatrix} & & \\ - & - & - \\ 0 & \cdots & 0 \end{bmatrix}$ singular b/c last row $= \vec{0}^T$

$\lambda_{n-1} I - A = \begin{bmatrix} 0 & \cdots & 0 & * \\ 0 & \cdots & 0 & \lambda_{n-1}-\lambda_n \end{bmatrix}$ } linearly dependent rows $\Rightarrow$ singular

⋮

think about $\lambda_n - z I - A$ and other λ_i's.

Example 2 (Ex. 3 in last lecture):

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ linearly dependent on B

$$A + BF = \begin{bmatrix} 1+f_1 & 1+f_2 \\ 0 & 2 \end{bmatrix}$$

2 is an eigenvalue independent of f_1, f_2
can't overcome instability

Controllability:

Recall solution of discrete-time system:

$$\vec{x}[i+1] = A\vec{x}[i] + B\,u[i]$$ (assume single input)

$i=0$: $$\vec{x}[1] = \underbrace{A\vec{x}[0] + Bu[0]}$$

$i=1$: $$\vec{x}[2] = A\vec{x}[1] + Bu[1]$$

$$= \underbrace{A^2\vec{x}[0] + AB\,u[0] + Bu[1]}$$

$$\vec{x}[3] = A\vec{x}[2] + Bu[2]$$

$$= A^3\vec{x}[0] + A^2B\,u[0] + AB\,u[1] + Bu[2]$$

⋮

$$\vec{x}[l] = A^l\vec{x}[0] + A^{l-1}Bu[0] + \cdots + ABu[l-2] + Bu[l-1]$$

$$\vec{x}[\ell] - A^{\ell}\vec{x}[0] = [A^{\ell-1}B, A^{\ell-2}B, \ldots, AB, B] \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[\ell-2] \\ u[\ell-1] \end{bmatrix}$$

Can we find input sequence $u[0], \ldots u[\ell-1]$ that brings state $\vec{x}$ from $\vec{x}[0]$ to a target $\vec{x}_{target}$ at time ℓ?

Yes, if $\vec{x}_{target} - A^{\ell}\vec{x}[0]$ lies in the column space of

$$C_\ell := [A^{\ell-1}B, \ldots, AB, B].$$

"Controllability": ability to reach any target state $\vec{x}_{target}$ from any $\vec{x}[0]$.

Def'n: A system is called controllable if, given any target state $\vec{x}_{target}$ and initial condition $\vec{x}[0]$, we can find a time ℓ and input sequence $u[0], \ldots u[\ell-1]$ such that $\vec{x}[\ell] = \vec{x}_{target}$.

How do we check that? If C_ℓ has n linearly independent columns for some ℓ, then column space is $\mathbb{R}^n$, n: state dimension. This means we can make $C_\ell \begin{bmatrix} u[0] \\ \vdots \\ u[\ell-1] \end{bmatrix}$ anything we want in $\mathbb{R}^n$ by choosing this vector. In particular, we can assign $C_\ell \begin{bmatrix} u[0] \\ \vdots \\ \vdots \end{bmatrix} = \vec{x}_{target} - A^\ell \vec{x}[0]$. Then,

$$\vec{x}[\ell] = A^\ell \vec{x}[0] + C_\ell \begin{bmatrix} u[0] \\ \vdots \\ \vdots \end{bmatrix} = \cancel{A^\ell \vec{x}[0]} + \vec{x}_{target} - \cancel{A^\ell \vec{x}[0]}$$

Example 1: $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $AB = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$C_1 = B$ dim $= 1$

$C_2 = [AB \;\; B] = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ dim $= 2 = n$

✓ controllable

Example 2: $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$C_1 = B$ dim $= 1$

$C_2 = [AB \;\; B] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ dim $= 1$

$C_3 = [A^2B \;\; AB \;\; B] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

dim $= 1$

dim can't increase further
stuck at $1 < n = 2$. ✓ uncontrollable

What we observe about dimension getting stuck is a consequence of the following result...

Lemma: If $A^\ell B$ is linearly dependent on
$$\{A^{\ell-1}B, \dots AB, B\}$$
then $A^{\ell+1}B$ is also linearly dependent on
$$\{A^{\ell-1}B, \dots AB, B\}.$$

Proof: $A^\ell B = \alpha_{\ell-1}A^{\ell-1}B + \dots + \alpha_1 AB + \underline{\alpha_0 B}$

for some $\alpha_{\ell-1}, \dots \alpha_1, \alpha_0$ by linear dependence.

Then,
$$\begin{aligned} A^{\ell+1}B &= A \cdot A^\ell B \\ &= A(\alpha_{\ell-1}A^{\ell-1}B + \dots \alpha_1 AB + \alpha_0 B) \\ &= \alpha_{\ell-1}A^\ell B + \alpha_{\ell-2}A^{\ell-1}B + \dots \alpha_1 A^2 B + \alpha_0 AB \\ &= \star A^{\ell-1}B + \star A^{\ell-2}B + \dots + \underset{\substack{\| \\ \alpha_{\ell-1}\alpha_0}}{\star} B \end{aligned}$$

$\square$

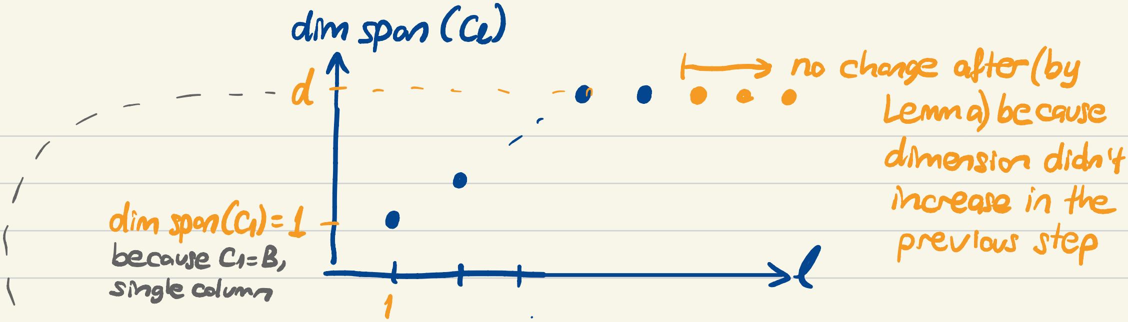
Note: $C_{\ell+1} = [A^\ell B \ \underbrace{A^{\ell-1}B \dots AB \ B}_{C_\ell}]$

Lemma implies:

if $\dim\operatorname{span}(C_{\ell+1}) = \dim\operatorname{span}(C_\ell) = d$

then $\dim\operatorname{span}(C_{\ell+2}) = \dim\operatorname{span}(C_{\ell+3}) = \dots = d$.

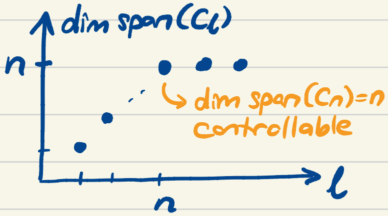
Once the dimension stops growing it stops for good.

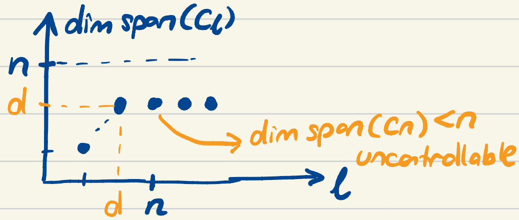


Let d denote the dimension at which we get stuck.
If $d = n$: CONTROLLABLE
If $d < n$: UNCONTROLLABLE (will never reach n).

We could write a code that increments ℓ by one as long dim span (C_ℓ) keeps growing and terminates once growth stops (it has to stop because dimension can't exceed n). Then we can apply the test above with the dimension d reached to check controllability...

OR we can be smarter: dimension grows by one at each step as long as it grows (we are adding a single column when we increment ℓ by 1). If we are able to reach n, we will reach it at $\ell = n$. Otherwise, growth will have stopped before n, so dim span(C_n) will be $d < n$. Thus, all we have to do is check if dim span$(C_n) = n$. Controllable if so, uncontrollable if not. One-shot test!





$$\boxed{\begin{array}{l}\text{Controllability condition: } \dim \operatorname{span}(C_n) = n \\ \text{i.e., } C_n = [A^{n-1}B \; \cdots \; AB \;\; B] \text{ has } n \text{ linearly indep. columns.}\end{array}}$$