EECS 16B

Spring 2022

Lecture 14

3/3/2022

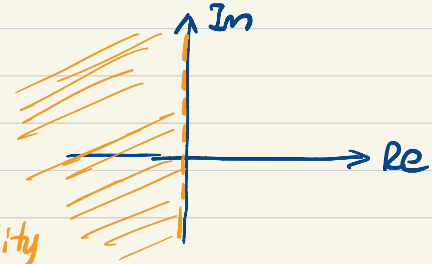
LECTURE 14

- stabilization by feedback
- controller canonical form

Summary of stability conditions:

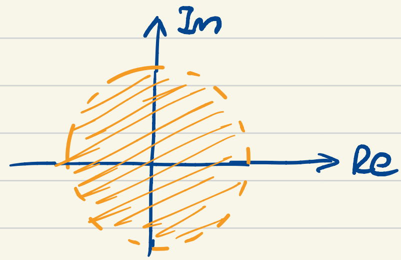
continuous-time

$$\frac{d}{dt}\vec{x}(t) = A_c\vec{x}(t) + \vec{w}(t)$$



discrete-time

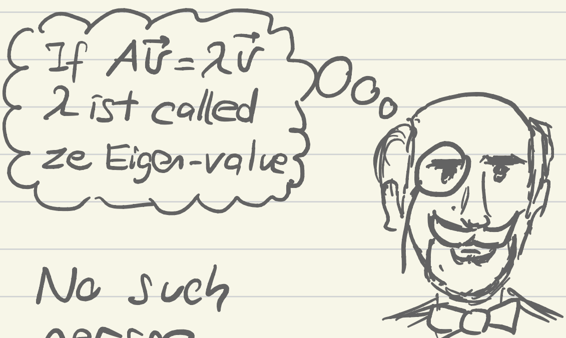
$$\vec{x}[i+1] = A_d\vec{x}[i] + \vec{w}[i]$$



Stability conditions:

continuous-time: $\mathrm{Re}\,\lambda_k < 0$ for each eigenvalue of A_c, $k = 1, \ldots, n$

discrete-time: $|\lambda_k| < 1$ for each eigenvalue of A_d, $k = 1, \ldots, n$



No such person existed. "Eigen" means "own."

Professor Wolfgang Friedrich Eigen (1865-1931)

Stabilization by Feedback:

$$\vec{x}[i+1] = A\vec{x}[i] + B\underbrace{\vec{u}[i]}_{\text{control input}} + \underbrace{\vec{w}[i]}_{\text{disturbance input}}$$

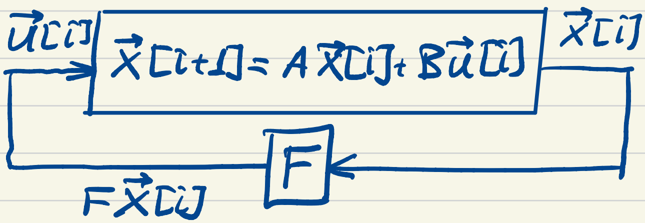
What if A has an evalue with $|\lambda| > 1$?
Can we achieve stability by designing $\vec{u}$?

Try the feedback: $\underbrace{\vec{u}[i]}_{\in\mathbb{R}^m} = \underbrace{F}_{m\times n}\underbrace{\vec{x}[i]}_{\in\mathbb{R}^n}$

If $m=1$ (single input), $F \in \mathbb{R}^{1\times n}$ (row vector):

$$F = [f_1 \ f_2 \ \cdots \ f_n]$$

$$u[i] = F\vec{x}[i] = f_1 x_1[i] + f_2 x_2[i] + \cdots + f_n x_n[i]$$



Substitute $\vec{u}[i] = F\vec{x}[i]$ in $\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i]$:

$$\vec{x}[i+1] = \underbrace{(A+BF)}_{A_{CL}}\vec{x}[i] + \vec{w}[i]$$

"Closed-loop system"

Can we design F such that evalues of $A_{CL} = A + BF$ are inside the unit circle?

Let's try some examples:

<u>Example 1</u>: (scalar) $x[i+1] = 2\,x[i] + u[i]$

↳ >1 : unstable without feedback

$$u[i] = f\,x[i]$$

closed-loop:

$$x[i+1] = (2+f)\,x[i]$$

For stability, $|2+f| < 1$ i.e., $2+f \in (-1, 1)$

$$\Downarrow$$

$$f \in (-3, -1)$$

<u>Example 2</u>: $\vec{x}[i+1] = \underbrace{\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}}_{A} \vec{x}[i] + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u[i]$

$$A_{CL} = A + BF = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix}}_{\begin{bmatrix} 0 & 0 \\ f_1 & f_2 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 & 1 \\ 3+f_1 & 2+f_2 \end{bmatrix}$$

Evalues of A:

$$\det(\lambda I - A)$$

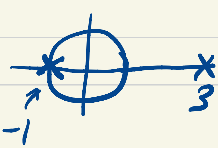
$$= \det\left(\begin{bmatrix} \lambda & -1 \\ -3 & \lambda-2 \end{bmatrix}\right)$$

$$= \lambda(\lambda-2) - 3$$

$$= \lambda^2 - 2\lambda - 3$$

$$= (\lambda-3)(\lambda+1)$$

unstable



$$\det(\lambda I - (A+BF))$$

$$= \det\left(\begin{bmatrix} \lambda & -1 \\ -3-f_1 & \lambda-2-f_2 \end{bmatrix}\right)$$

$$= \lambda^2 - (2+f_2)\lambda - (3+f_1) \quad \text{-- (1)}$$

Suppose we want A_{CL} to have evalues at λ_1, λ_2. $\Rightarrow$ characteristic polynomial of A_{CL} should be $(\lambda-\lambda_1)(\lambda-\lambda_2) = \lambda^2 - (\lambda_1+\lambda_2)\lambda + \lambda_1\lambda_2$ --(2)

Choose f_1, f_2 such that coefficients of (1) and (2) match:

$-3-f_1 = \lambda_1\lambda_2$ $\qquad$ $2+f_2 = \lambda_1+\lambda_2$

$f_1 = -\lambda_1\lambda_2 - 3$ $\qquad$ $f_2 = \lambda_1+\lambda_2-2$

Does this always work? Not for any A, B:

<u>Example 3</u>: $\vec{x}[i+1] = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}_{A} \vec{x}[i] + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u[i]$

Evalues of A: 1, 2 unstable (evalues = diagonal entries because A is upper-triangular)

$$A+BF = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+f_1 & 1+f_2 \\ 0 & 2 \end{bmatrix}$$

evalues: $1+f_1$, 2 (again, because upper-triangular)

Can't be changed by F.
System remains unstable no matter how F is selected.

We will see what makes Example 2 work and Example 3 fail...

Controller Canonical Form (single input, $m=1$)

A special structure of A and B in which we can arbitrarily assign evalues of

$$A_{CL} = A + BF$$

with the choice of F:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ & & & & 0 \\ 0 & \cdots & & 0 & 1 \\ a_1 & a_2 & \cdots & & a_n \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Example 2 had this form: $n=2$, $a_1=3$, $a_2=2$

Nice properties of this form:

1) Characteristic polynomial of A is transparent:

$$\det(\lambda I - A) = \lambda^n - a_n \lambda^{n-1} - a_{n-1}\lambda^{n-2} \cdots - a_2\lambda - a_1$$

Example 2: $\lambda^2 - 2\lambda - 3$

For $n=3$: $\lambda^3 - a_3\lambda^2 - a_2\lambda - a_1$ is the characteristic polynomial of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 & a_2 & a_3 \end{bmatrix}.$$ (Check.)

2) $A+BF$ has the same structure as A:

$$\begin{bmatrix} 0 & 1 & 0 & \cdots \\ & \ddots & \ddots & \\ & & & 1 \\ a_1 & \cdots & \cdots & a_n \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [f_1 \cdots f_n]}_{\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \\ f_1 & \cdots & f_n \end{bmatrix}}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots \\ & \ddots & \ddots & \\ & & & 1 \\ a_1+f_1 & \cdots & \cdots & a_n+f_n \end{bmatrix}$$

Same structure as A, but $a_k \to a_k + f_k$, $k=1,\dots n$.

From Properties 1 and 2,

$$\det(\lambda I - A_{CL}) = \lambda^n - (a_n+f_n)\lambda^{n-1} \cdots - (a_2+f_2)\lambda - (a_1+f_1).$$

Suppose we want $A_{CL} = A+BF$ to have evalues $\lambda_1, \dots \lambda_n$. Then, the characteristic polynomial of A_{CL} should be:

$$(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$$

$$= \lambda^n - (\lambda_1 + \cdots \lambda_n)\lambda^{n-1} \cdots + (-1)^n \lambda_1 \cdots \lambda_n$$

$$a_1 + f_1 = -(-1)^n \lambda_1 \cdots \lambda_n = (-1)^{n+1} \lambda_1 \cdots \lambda_n \Rightarrow f_1 = (-1)^{n-1} \lambda_1 \cdots \lambda_n - a_1$$

$$\vdots \qquad\qquad \vdots$$

$$a_n + f_n = \lambda_1 + \cdots + \lambda_n \qquad \Rightarrow f_n = \lambda_1 + \cdots \lambda_n - a_n$$

Can we bring A, B to canonical form by a change of variables?

$$\vec{y} = T\vec{x} \quad , \quad T: n\times n, \text{ invertible}$$
(to be determined)

$$\vec{x} = T^{-1}\vec{y}.$$

$$\vec{y}[i+1] = T\vec{x}[i+1] = T(A\vec{x}[i] + Bu[i])$$
$$= TA\vec{x}[i] + TBu[i]$$
$$\vec{y}[i+1] = \underbrace{TAT^{-1}}_{\text{new } A}\vec{y}[i] + \underbrace{TB}_{\text{new } B}u[i]$$

We want new A, B in canonical form:

$$TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & \\ & & \ddots & & \\ & & & \ddots & 1 \\ * & * & \cdots & \cdots & * \end{bmatrix} \qquad TB = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Can we find such T?

I use these lines to separate columns.

<u>Claim</u>: Yes, if $[A^{n-1}B \mid \cdots \mid AB \mid B]$ ($n\times n$ matrix) is invertible. (Will show next time)

Example 3: $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $[AB \mid B] = \left[\begin{array}{c|c} 1 & 1 \\ 0 & 0 \end{array}\right]$

not invertible

When $[A^{n-1}B | \cdots | AB | B]$ is invertible, feedback design is easy in $\vec{y}$ coordinates:

$$\vec{y}[i+1] = \underbrace{\begin{bmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ & & & 1 \\ * & - & - & * \end{bmatrix}}_{A_y} \vec{y}[i] + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{B_y} u[i]$$

$$u[i] = F_y \vec{y}[i]$$

Can assign evalues of $A_y + B_y F_y$ by choice of F_y, because (A_y, B_y) in canonical form.

In the original coordinates:

$$u[i] = F_y \vec{y}[i] = \underbrace{F_y T}_{F} \vec{x}[i]$$

$$\vec{x}[i+1] = (\underbrace{A + BF}_{A_{CL}}) \vec{x}[i]$$

<u>Note:</u> Evalues of $A_{CL} = A + BF$ are the same as those of $A_y + B_y F_y$ (which we designed by choice of F_y). To see this, suppose $(\lambda, \vec{v})$ evalue, evector pair for $A+BF$, i.e. $(A+BF)\vec{v} = \lambda \vec{v}$. --- (3)

Multiply (3) from left by T:

since λ is scalar

$$T(A+BF)\vec{v} = T\lambda\vec{v} = \lambda T\vec{v}$$

thus, $$(TA+TBF)\vec{v} = \lambda T\vec{v} \qquad \text{--- (4)}$$

Substitute $TA = A_y T$, $TB = B_y$, $F = F_y T$ in (4):

since $TAT^{-1} = A_y$ as observed above

$$(A_y T + B_y F_y T)\vec{v} = \lambda T\vec{v}$$

$$(A_y + B_y F_y) T\vec{v} = \lambda T\vec{v}.$$

This means λ is an evalue of $A_y + B_y F_y$ as well and $\vec{v}_y := T\vec{v}$ is the corresponding evector.

<u>To summarize,</u> $A+BF$ and $A_y + B_y F_y$ have the same evalues. Therefore, we can design F_y in the canonical $\vec{y}$ coordinates to assign evalues of $A_y + B_y F_y$ and those are also evalues of $A_{cL} = A+BF$ in original $\vec{x}$ coordinates.

<u>Next time:</u> Proof of Claim and understanding the condition therein.