EE 16B
Spring 2022
Lecture 13
3/11/2022

LECTURE 13

– stability:
- discrete-time
- continuous-time

<u>Definition</u>: We say that a system is (bounded-input, bounded state) stable if state x is bounded for any initial condition and bounded disturbance pair. Unstable otherwise: i.e. unstable if x grows unbounded for <u>some</u> initial condition and bounded input pair.

When is $x[i+1] = \lambda x[i] + e[i]$ stable?

- $|\lambda| > 1$: UNSTABLE
  any nonzero initial condition and zero input enough for unboundedness: $x[l] = \lambda^l x[0]$
- $|\lambda| = 1$: UNSTABLE
  can find bounded inputs that lead to unbdd. states. Example: $\lambda = 1$, $e[i] = 1$ for all i:

$$x[1] = x[0] + 1$$
$$x[2] = x[1] + 1 = x[0] + 2$$
$$\vdots$$
$$x[l] = x[0] + l$$

<u>Note</u>: there may be other bounded inputs with which the states remain bounded (e.g., $e[i] = (-1)^i$ in the example above), but one bounded input that makes the state unbounded is enough for instability per definition above.

Example above is for $\lambda = 1$ only, but for any λ with $|\lambda| = 1$ (including complex λ on unit circle), a bounded input of the form $e[i] = \lambda^i$ will drive x unbounded.

- $|\lambda| < 1$: STABLE (to be shown today)

<u>Claim:</u> If $|\lambda|<1$ then for any $x[0]$ and any bounded input e, the solutions of

$$x[i+1] = \lambda x[i] + e[i]$$

remain bounded.

<u>Proof:</u>

$$x[1] = \lambda x[0] + e[0]$$

$$x[2] = \lambda x[1] + e[1] = \lambda^2 x[0] + \lambda e[0] + e[1]$$

$$x[3] = \lambda x[2] + e[2] = \lambda^3 x[0] + \lambda^2 e[0] + \lambda e[1] + e[2]$$

$$\vdots$$

$$x[l] = \underbrace{\lambda^l x[0]}_{\text{bounded and} \to 0 \text{ as } l\to\infty \text{ bc/ } |\lambda|<1} + \sum_{k=0}^{l-1} \lambda^k e[l-1-k] \qquad (1)$$

Show the summation term is also bounded when e is bounded, that is, there is a number M such that

$$|e[i]| \leq M \quad \text{for all } i.$$

$$\left|\sum_{k=0}^{l-1} \lambda^k e[l-1-k]\right| \leq \sum_{k=0}^{l-1} \underbrace{|\lambda^k e[l-1-k]|}_{= |\lambda|^k \underbrace{|e[l-1-k]|}_{\leq M}}$$

$$\leq \sum_{k=0}^{l-1} M |\lambda|^k$$

Recall:

$$= M \underbrace{\sum_{k=0}^{l-1} |\lambda|^k}$$

bounded and converges to $\frac{1}{1-|\lambda|}$ monotonically as $l \to \infty$ when $|\lambda|<1$

$$1+x+x^2+x^3+\dots = \frac{1}{1-x}$$

if $|x|<1$ (geometric series)

$$\sum_{k=0}^{l-1} |\lambda|^k \leq \frac{1}{1-|\lambda|}$$

$$\left|\sum_{k=0}^{l-1} \lambda^k e[l-1-k]\right| \leq \frac{M}{1-|\lambda|},$$ therefore bounded.

Both terms on the right hand side of (1) are bounded, therefore $x[l]$ remains bounded. $\square$

end of proof

Vector Case:

Eq'n (2)

$$\vec{x}[i+1] = A\,\vec{x}[i] + \vec{e}[i] \qquad \vec{x} \in \mathbb{R}^n,\ \vec{e} \in \mathbb{R}^n$$
$$A \in \mathbb{R}^{n \times n}$$

Solution (by recursion as in scalar case above):

$$\vec{x}[l] = \underbrace{A^l}_{= \underbrace{A \cdot A \cdots A}_{l \text{ times}}} \vec{x}[0] + \sum_{k=0}^{l-1} A^k \vec{e}[l-1-k]$$

When does $\vec{x}[l]$ remain bounded? No immediate answer b/c A is matrix.

Split into scalar eq'ns:

$\vec{y} := V^{-1}\vec{x}$ where $V = [\underbrace{\vec{v}_1 \cdots \vec{v}_n}]$ ... (Eq'n 3)

e values of A, assumed linearly independent so V^{-1} exists (i.e., A is diagonalizable)

$\vec{x} = V\vec{y}$ (4)

$$
\begin{aligned}
\vec{y}[i+1] &= V^{-1}\vec{x}[i+1] \\
&= V^{-1}(A\vec{x}[i] + \vec{e}[i]) && \text{by (2)} \\
&= V^{-1}A\vec{x}[i] + V^{-1}\vec{e}[i] \\
&= \underbrace{V^{-1}AV}_{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}\vec{y}[i] + V^{-1}\vec{e}[i] && \text{by (4)}
\end{aligned}
$$

because $A\vec{v}_i = \lambda_i \vec{v}_i$

$$\underbrace{A[\vec{v}_1 \cdots \vec{v}_n]}_{V} = [\lambda_1\vec{v}_1 \cdots \lambda_n\vec{v}_n] = \underbrace{[\vec{v}_1 \cdots \vec{v}_n]}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$AV = V\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$V^{-1}AV = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$y_k[i+1] = \lambda_k y_k[i] + (V^{-1}\vec{e}[i])_k$

→ scalar eq'n

$|\lambda_k| < 1 \Rightarrow y_k$ bounded when e is bounded

$k = 1, \ldots n$

Therefore if $|\lambda_k| < 1$ for all k, then $\vec{y}$ is bounded, so $\vec{x} = V\vec{y}$ also bounded.

What if A is not diagonalizable?

We can still bring it to an upper-triangular form (will see how next week):

$$\vec{y}[i+1] = \begin{bmatrix} \lambda_1 & \star & \cdots & \star \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ & & \lambda_{n-1} & \star \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \vec{y}[i] + V^{-1}\vec{e}[i].$$

V: appropriate matrix we'll see next week

$\star$: some number ("stuff")

- $y_n[i+1] = \lambda_n y_n[i] + (V^{-1}\vec{e}[i])_n$

  $|\lambda_n| < 1 \Rightarrow y_n$ bounded (by scalar result above)

- $y_{n-1}[i+1] = \lambda_{n-1} y_{n-1}[i] + \underbrace{\underbrace{\star\, y_n[i]}_{\text{bounded by first bullet above}} + \underbrace{(V^{-1}\vec{e}[i])_{n-1}}_{\text{also bounded}}}_{\text{treat this sum as a bounded input}}$

  $|\lambda_{n-1}| < 1 \Rightarrow y_{n-1}$ is bounded

- $y_{n-2}[i+1] = \lambda_{n-2} y_{n-2}[i] + \underbrace{\star\, y_{n-1}[i] + \star\, y_n[i] + (V^{-1}\vec{e}[i])_{n-2}}_{\text{bounded by previous two bullets}}$

  $|\lambda_{n-2}| < 1 \Rightarrow y_{n-2}$ bounded

⋮

Conclusion: Discrete-time system (2) is stable if each eigenvalue λ_k, $k=1,\dots,n$, of A satisfies

$$|\lambda_k| < 1,$$

i.e., all eigenvalues strictly inside the unit circle in the complex plane.

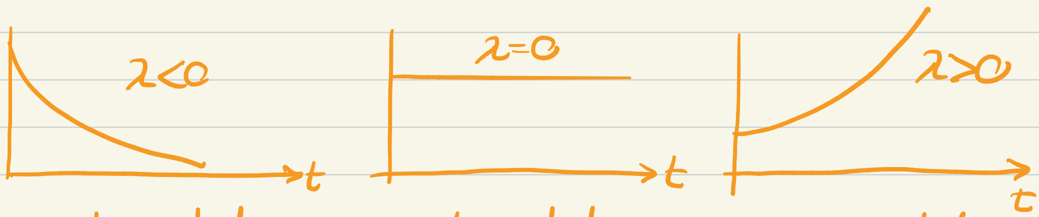
Stability of continuous-time sustems

Same stability definition that's at the very top (agnostic to continuous- or discrete-time), but different conditions for stability.

Scalar case first: $\frac{d}{dt}x(t) = \lambda x(t) + w(t)$ ← disturbance

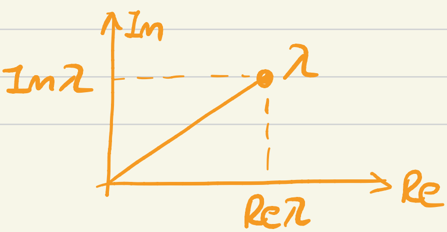
$$x(t) = e^{\lambda t}x(0) + \int_0^t e^{(t-\tau)} w(\tau)\,d\tau$$

When is $e^{\lambda t}$ bounded?



bounded | bounded | unbounded

What about complex λ?

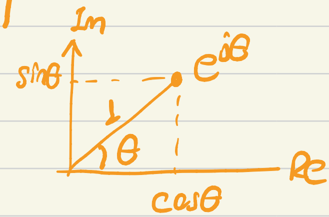


$$\lambda = \mathrm{Re}\lambda + j\,\mathrm{Im}\lambda$$

$$e^{\lambda t} = e^{(\mathrm{Re}\lambda + j\,\mathrm{Im}\lambda)t}$$

$$= e^{(\mathrm{Re}\lambda)t} e^{j(\mathrm{Im}\lambda)t}$$

$$\Rightarrow |e^{\lambda t}| = |e^{(\mathrm{Re}\lambda)t}| \cdot |e^{j(\mathrm{Im}\lambda)t}|$$



Recall $e^{j\theta} = \cos\theta + j\sin\theta$

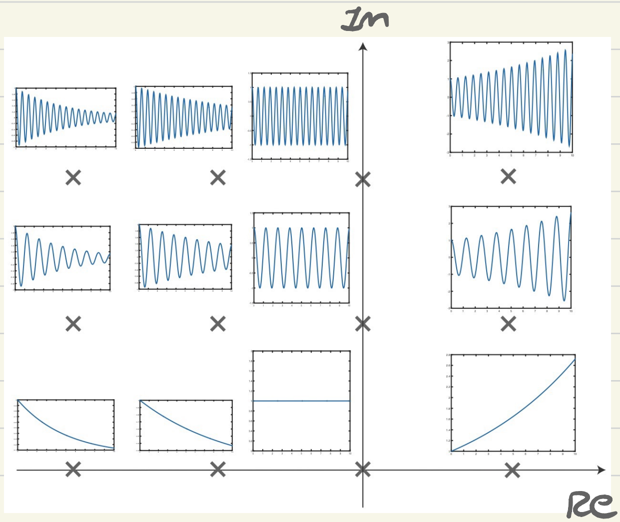
and $|e^{j\theta}| = \sqrt{(\cos\theta)^2 + (\sin\theta)^2}$

$= 1$ for any θ.

Therefore $|e^{j(\mathrm{Im}\lambda)t}| = 1$ (view $(\mathrm{Im}\lambda)t$ as θ above).

$$\Rightarrow |e^{\lambda t}| = |e^{(\mathrm{Re}\lambda)t}| \underbrace{|e^{j(\mathrm{Im}\lambda)t}|}_{=1} = |e^{(\mathrm{Re}\lambda)t}|.$$

Bounded if $\mathrm{Re}\lambda \leq 0$; unbounded if $\mathrm{Re}\lambda > 0$.



$e^{\lambda t}$ for various values of λ in the complex plane (only real part of $e^{\lambda t}$ shown when complex)

$\longleftarrow$ bounded when $\mathrm{Re}\lambda \leq 0$

$\longrightarrow$ unbounded when $\mathrm{Re}\lambda > 0$

Note the oscillations when λ is complex. This is because $e^{\lambda t} = e^{(\mathrm{Re}\lambda)t} e^{j(\mathrm{Im}\lambda)t} = e^{(\mathrm{Re}\lambda)t}(\cos((\mathrm{Im}\lambda)t) + j\sin(\mathrm{Im}\lambda)t)$.

envelope: decaying when $\mathrm{Re}\,\lambda<0$, constant when $\mathrm{Re}\,\lambda=0$, growing when $\mathrm{Re}\,\lambda>0$

oscillations when $\mathrm{Im}\,\lambda \neq 0$. Only the real part, $\cos((\mathrm{Im}\,\lambda)t)$ shown in the plots above

Back to stability of

$$\frac{d}{dt}x(t) = \lambda x(t) + w(t)$$

- $\mathrm{Re}\,\lambda > 0$: UNSTABLE

  nonzero $x(0)$ and $w(t)=0$ enough for unbounded state:

  $$x(t) = x(0)e^{\lambda t}$$

- $\mathrm{Re}\,\lambda = 0$: UNSTABLE

  $$x(t) = \underbrace{e^{\lambda t}x(0)}_{\text{bounded}} + \underbrace{\int_0^t e^{\lambda(t-\tau)}w(\tau)d\tau}_{\text{can be driven unbounded by a suitable choice of } w}$$

  Example: $\lambda=0 \Rightarrow x(t) = x(0) + \int_0^t w(\tau)d\tau$

  $w(\tau)=1 \Rightarrow x(t) = x(0)+t$ unbounded

- $\mathrm{Re}\,\lambda < 0$: STABLE

  proof similar to discrete-time with $|\lambda|<1$

  $$x(t) = \underbrace{e^{\lambda t}x(0)}_{\text{bounded and } \to 0} + \underbrace{\int_0^t e^{\lambda(t-\tau)}w(\tau)d\tau}_{\text{can derive a bound on this when } w \text{ bounded, i.e., } |w(\tau)|\leq M \text{ for some } M.}$$

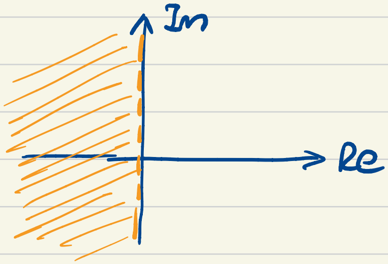
__Vector Case:__ Similar arguments to discrete-time (with diagonalization or upper triangularization) lead to the conclusion that

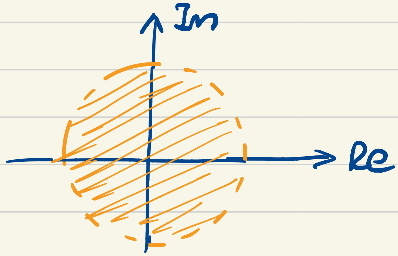
$$\frac{d}{dt}\vec{x}(t) = A_c\vec{x}(t) + \vec{w}(t)$$

is stable if $\mathrm{Re}\,\lambda_k < 0$ for each eigenvalue λ_k of A_c, $k=1,\dots n$

__Summary:__

continuous-time	discrete-time
$\frac{d}{dt}\vec{x}(t) = A_c\vec{x}(t) + \vec{w}(t)$	$\vec{x}[i+1] = A_d\vec{x}[i] + \vec{w}[i]$





stability condition:

continuous-time	discrete-time
$\mathrm{Re}\,\lambda_k < 0$ for each eigenvalue of A_c, $k=1,\dots,n$	$\lvert\lambda_k\rvert < 1$ for each eigenvalue of A_d, $k=1,\dots n$

i.e. eigenvalues must be in the shaded regions above.