EE 16B

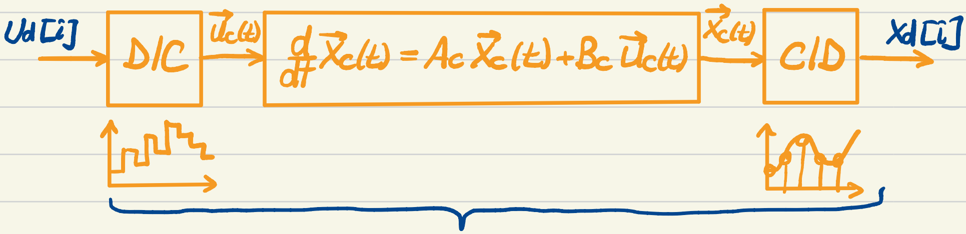
Spring 2022

Lecture 12

2/24/2022

LECTURE 12: – System identification
– Stability

Last lecture:



$$\vec{x}_d[i+1] = A_d X_d[i] + B_d U_d[i]$$

$\vec{x}$: vector of state variables (V_c, I_L in RLC circuits; position, velocity in mechanical systems; concentrations of reactants in a chemical reaction system; flows and pressures in an engine model; etc.)

$\vec{u}$: control input (voltage and current sources in a circuit; forces and torques applied to a mechanical system; etc)

Today: discrete-time only – drop subscript "d":

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i]$$

<u>System Identification</u> (using Least Squares) $D\vec{p} + \vec{e} = \vec{s}$

$\|\vec{e}\|$ is minimized by $\hat{p} = (D^TD)^{-1}D^T\vec{s}$ if D^TD invertible

Scalar case:

error, disturbance

$$x[i+1] = \lambda x[i] + b u[i] + e[i]$$

$$\begin{cases} x[1] = \lambda x[0] + b u[0] + e[0] \\ x[2] = \lambda x[1] + b u[1] + e[1] \end{cases}$$

$$\left\{ \begin{array}{l} \vdots \\ x[l] = \lambda x[l-1] + bu[l-1] + e[l-1] \end{array} \right.$$

$$\underbrace{\begin{bmatrix} x[0] & u[0] \\ x[1] & u[1] \\ \vdots & \vdots \\ x[l-1] & u[l-1] \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \lambda \\ b \end{bmatrix}}_{\vec{p}} + \underbrace{\begin{bmatrix} e[0] \\ e[1] \\ \vdots \\ e[l-1] \end{bmatrix}}_{\vec{e}} = \underbrace{\begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[l] \end{bmatrix}}_{\vec{s}}$$

if D^TD invertible, LS solution: $\hat{p} = (D^TD)^{-1}D^T\vec{s}$

$$\begin{bmatrix} \hat{\lambda} \\ \hat{b} \end{bmatrix} = (D^TD)^{-1} D^T \begin{bmatrix} x[1] \\ \vdots \\ x[l] \end{bmatrix}$$

Vector case: $\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{e}[i]$

$\vec{e} \in \mathbb{R}^n$
$\vec{x} \in \mathbb{R}^n$
$\vec{u} \in \mathbb{R}^m$
$A \in \mathbb{R}^{n \times n}$
$B \in \mathbb{R}^{n \times m}$

$$\left\{ \begin{array}{l} \vec{x}[1] = A\vec{x}[0] + B\vec{u}[0] + \vec{e}[0] \\ \vdots \\ \vec{x}[l] = A\vec{x}[l-1] + B\vec{u}[l-1] + \vec{e}[l-1] \end{array} \right.$$

transpose

$$\vec{x}[0]^TA^T + \vec{u}[0]^TB^T + \vec{e}[0]^T = \vec{x}[1]^T$$
$$\vdots$$
$$\vec{x}[l-1]^TA^T + \vec{u}[l-1]^TB^T + \vec{e}[l-1]^T = \vec{x}[l]^T$$

$$\underbrace{\begin{bmatrix} \vec{x}[0]^T & \vec{u}[0]^T \\ \vec{x}[1]^T & \vec{u}[1]^T \\ \vdots & \\ x[l-1]^T & \vec{u}[l-1]^T \end{bmatrix}}_{=:D} \underbrace{\begin{bmatrix} A^T \\ B^T \end{bmatrix}}_{(n+m)\times n} + \underbrace{\begin{bmatrix} \vec{e}[0]^T \\ \vec{e}[1]^T \\ \vdots \\ e[l-1]^T \end{bmatrix}}_{\begin{bmatrix} e_1[0] & \cdots & e_n[0] \\ \vdots & & \vdots \\ e_1[l-1] & \cdots & e_n[l-1] \end{bmatrix}} = \underbrace{\begin{bmatrix} \vec{x}[1]^T \\ \vec{x}[2]^T \\ \vdots \\ \vec{x}[l]^T \end{bmatrix}}_{\begin{bmatrix} x_1[1] & \cdots & x_n[1] \\ \vdots & & \vdots \\ x_1[l] & \cdots & x_n[l] \end{bmatrix}}$$

"defined as"

$$\begin{bmatrix} A^T \\ B^T \end{bmatrix} = \begin{bmatrix} \vec{p}_1 & \cdots & \vec{p}_n \end{bmatrix} \qquad \begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_n \end{bmatrix} \qquad \begin{bmatrix} \vec{x}_1 & \cdots & \vec{x}_n \end{bmatrix}$$

columns of $\begin{bmatrix} A^T \\ B^T \end{bmatrix}$

$$D \begin{bmatrix} \vec{p}_1 & \cdots & \vec{p}_n \end{bmatrix} + \begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_n \end{bmatrix} = \begin{bmatrix} \vec{x}_1 & \cdots & \vec{x}_n \end{bmatrix}$$

where $\vec{e}_1 = \begin{bmatrix} e_1[0] \\ \vdots \\ e_1[l-1] \end{bmatrix} \cdots \vec{e}_n = \begin{bmatrix} e_n[0] \\ \vdots \\ e_n[l-1] \end{bmatrix}$

$\vec{x}_1 = \begin{bmatrix} x_1[1] \\ \vdots \\ x_1[l] \end{bmatrix} \cdots \vec{x}_n = \begin{bmatrix} x_n[1] \\ \vdots \\ x_n[l] \end{bmatrix}$

$$\begin{bmatrix} D\vec{p}_1 & \cdots & D\vec{p}_n \end{bmatrix} + \begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_n \end{bmatrix} = \begin{bmatrix} \vec{x}_1 & \cdots & \vec{x}_n \end{bmatrix}$$

$$D\vec{p}_i + \vec{e}_i = \vec{x}_i \qquad i = 1, \dots n$$

If D^TD is invertible, LS sol'n is

$$\hat{p}_1 = (D^TD)^{-1} D^T \vec{x}_1$$
$$\vdots$$
$$\hat{p}_n = (D^TD)^{-1} D^T \vec{x}_n$$

$$\begin{bmatrix} \hat{A}^T \\ \hat{B}^T \end{bmatrix} = \begin{bmatrix} \hat{p}_1 & \cdots & \hat{p}_n \end{bmatrix} = (D^TD)^{-1} D^T \underbrace{\begin{bmatrix} \vec{x}_1 & \cdots & \vec{x}_n \end{bmatrix}}$$

$$= \begin{bmatrix} \vec{x}[1]^T \\ \vdots \\ \vec{x}[l]^T \end{bmatrix}$$ by def'n of $\vec{x}_1 \dots \vec{x}_n$ at the top of the page

Summary:

$$\begin{bmatrix} \hat{A}^T \\ \hat{B}^T \end{bmatrix} = (D^TD)^{-1} D^T \begin{bmatrix} \vec{x}[1]^T \\ \vdots \\ \vec{x}[l]^T \end{bmatrix}$$

Choose time window $[0 \cdots l]$, apply input sequence

$\vec{u}[0] \dots \vec{u}[\ell-1]$, observe resulting states $\vec{x}[0], \vec{x}[1] \cdots \vec{x}[\ell]$.

Form $D = \begin{bmatrix} \vec{x}[0]^T & \vec{u}[0]^T \\ \vdots & \vdots \end{bmatrix}$ and use formula above for $\hat{A}, \hat{B}$.

Stability:

Go back to scalar model, remove control input u:

$$x[i+1] = \lambda x[i] + e[i]$$

Does the sequence $x[0], x[1], \dots$ remain bounded?

Take $\lambda = 2$ and ignore disturbance e:

$$x[i+1] = 2x[i]$$

$$x[1] = 2x[0], \quad x[2] = 2x[1] = 4x[0], \quad x[3] = 2x[2] = 8x[0]$$

$$x[\ell] = 2^\ell x[0]$$

blows up unless $x[0] = 0$. Even with $x[0] = 10^{-9}$

$$\ell = 40: \quad x[40] = \underbrace{2^{40}}_{\approx 10^{12}} \cdot 10^{-9} \approx 1000$$

Take $\lambda = \frac{1}{2}$:

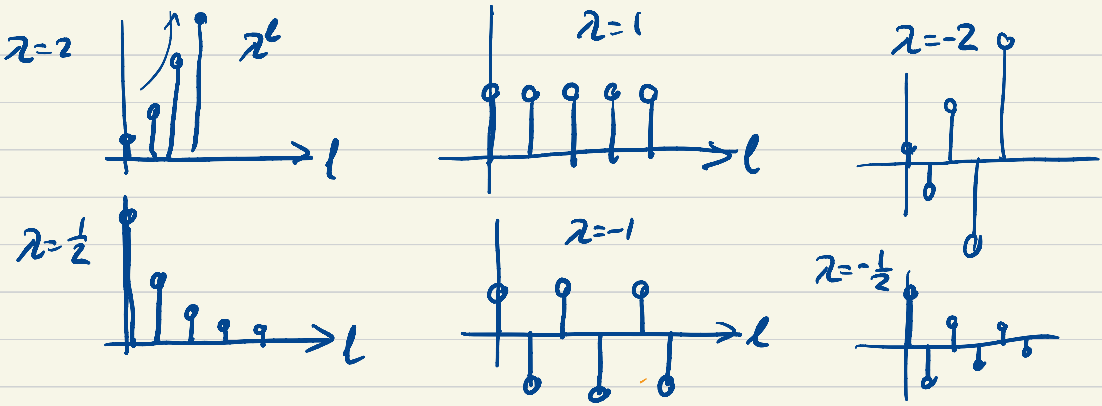
$$x[\ell] = \frac{1}{2^\ell} x[0]$$

bounded and $x[\ell] \to 0$ as $\ell \to \infty$

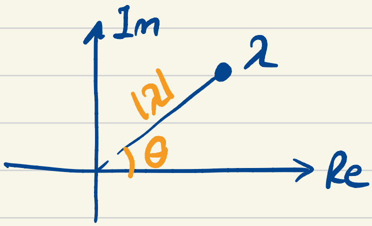
For general λ, solution of

$$x[i+1] = \lambda x[i]$$

is $X[\ell] = \lambda^{\ell} X[0]$. Bounded if $|\lambda| \leq 1$.



What about complex λ? When does λ^{ℓ} remain bounded?

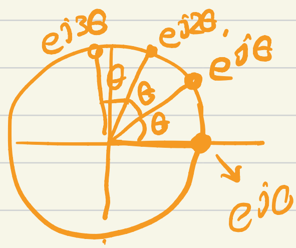


$$\lambda = |\lambda| e^{j\theta}$$

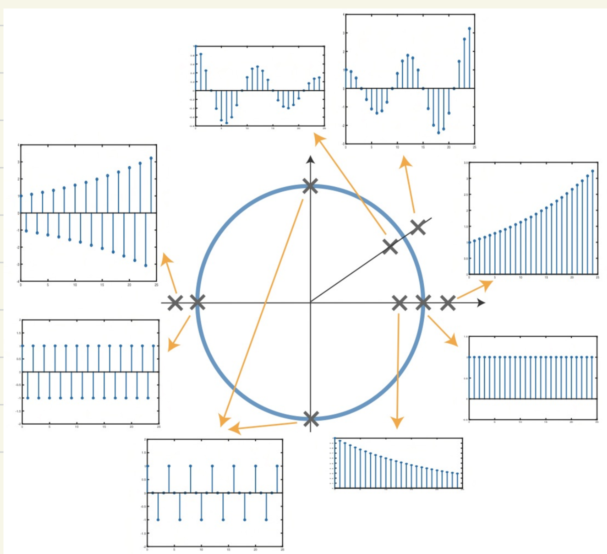
$$\lambda^{\ell} = |\lambda|^{\ell} \underbrace{e^{j\theta\ell}}$$

$$|e^{j\theta\ell}| = 1$$

$$\Rightarrow |\lambda^{\ell}| = |\lambda|^{\ell}$$



Therefore, $|\lambda| \leq 1$ required for boundedness of λ^{ℓ}.



λ^{ℓ}, $\ell = 0, 1, 2, 3, \ldots$ for various choices of λ in the complex plane, each marked with a cross. Only the real part of λ^{ℓ} is shown when λ is complex.

$|\lambda|<1$: λ^{l} bounded and $\to 0$ as $l \to \infty$

$|\lambda|>1$: λ^{l} unbounded as $l \to \infty$

$|\lambda|=1$: really safe?

Not safe when we factor in disturbance.

$$x[i+1] = \lambda x[i] + e[i]$$

Take $\lambda = 1$:

$$x[i+1] = x[i] + e[i]$$

$$x[1] = x[0] + e[0]$$

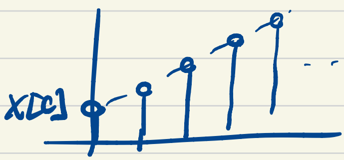
$$x[2] = x[1] + e[1]$$
$$= x[0] + e[0] + e[1]$$

$$x[3] = x[2] + e[2]$$
$$= x[0] + e[0] + e[1] + e[2]$$

$$\vdots$$

$$x[l] = x[0] + e[0] + e[1] + \dots + e[l-1]$$

Even a small, constant will make $x[l]$ grow unbounded



<u>Definition</u>: We say that a system is (bounded-input, bounded state) stable if state x is bounded for any initial condition and any bounded disturbance. Unstable otherwise: i.e. unstable if x grows unbounded for <u>some</u> initial condition and <u>some</u> bounded input.

When is the system

$$x[i+1] = \lambda x[i] + e[i]$$

stable according to def'n above?

$|\lambda| > 1$: unstable (zero input and nonzero initial already lead to unbounded x: $\lambda^i x[0] \to \infty$)

$|\lambda| = 1$: unstable (see example above with $\lambda = 1$)

$|\lambda| < 1$: stable (will show next time)