EE 16B

Spring 2022

Lecture 11

2/22/2022

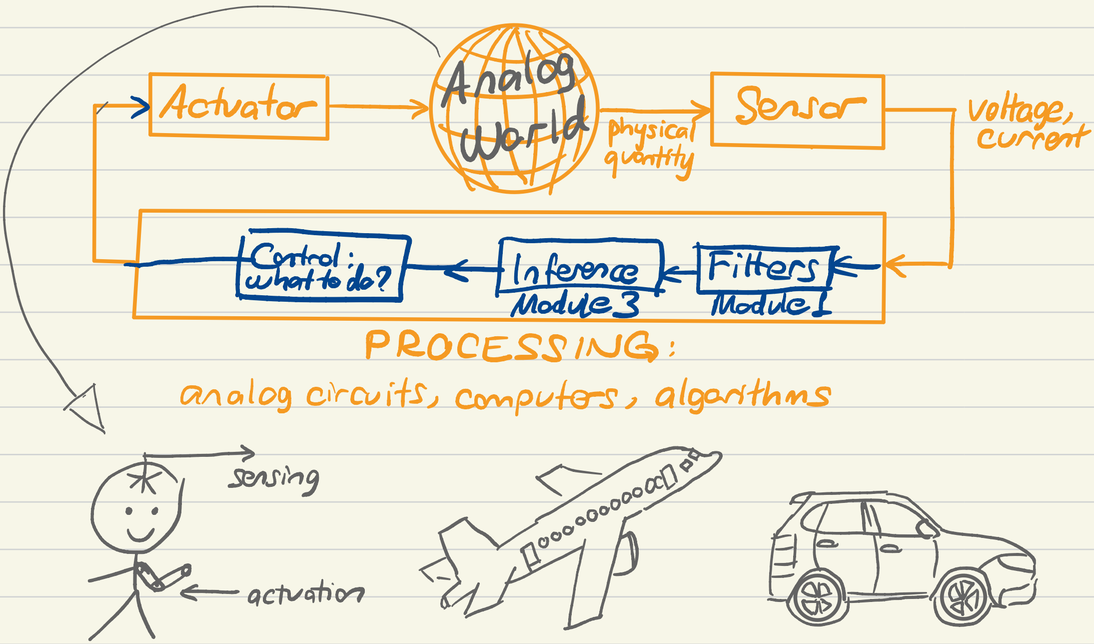
New prof (Murat)

LECTURE 11

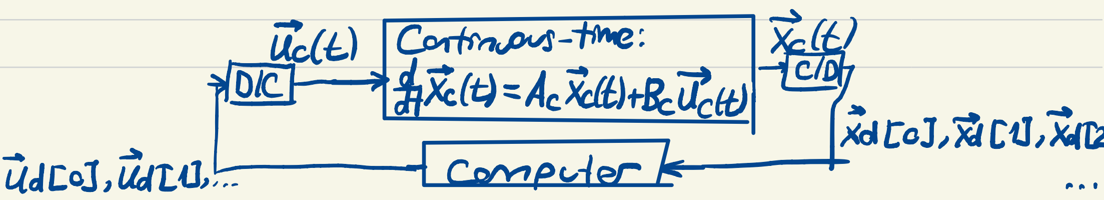
New module: Control and Robotics

Today: - Introduction to Control
- Continuous- to discrete-time
- System identification

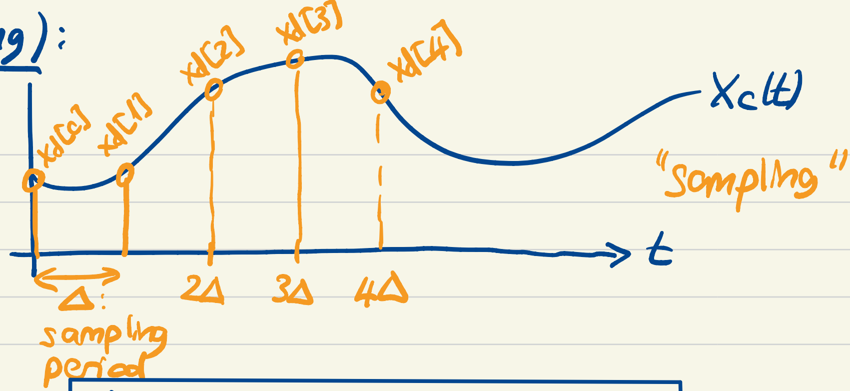
EECS view of the World from Lecture 1:



Control & inference blocks are algorithms executed digitally in a computer, in discrete time. Rest of the system evolves in continuous time. How do we connect the two worlds?

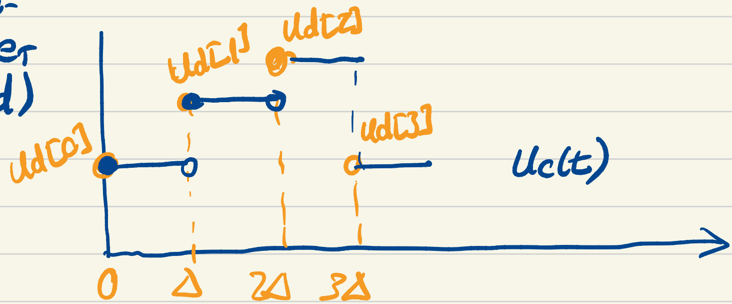


C/D (Sampling):



$$\boxed{\vec{x}_d[i] = \vec{x}_c(i\Delta) \qquad i=0,1,2,\ldots}$$ Sampling

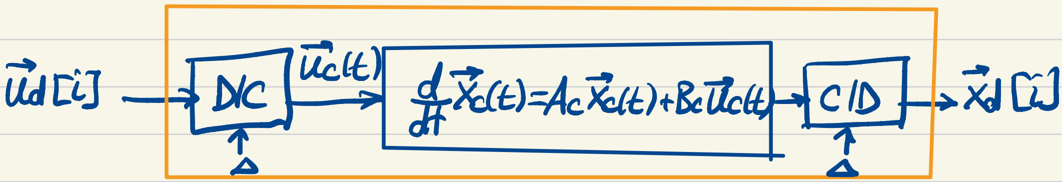
D/C (Zero-order Hold)



$$\boxed{\vec{u}_c(t) = \vec{u}_d[i] \qquad t \in [i\Delta, (i+1)\Delta)}$$

(ZOH)

Discretization



$$\underbrace{\qquad}_{\vec{x}_d[i+1] = A_d \vec{x}_d[i] + B_d \vec{u}_d[i]}$$

– recurrence relation that describes how the system evolves from sample to sample

How to find A_d and B_d given A_c, B_c, Δ.

i.e.

given $\vec{x}_d[i]$ and $\vec{u}_d[i]$ what is $\vec{x}_d[i+1]$?

$\vec{x}_d[i] = \vec{x}_c(i\Delta)$ $\quad$ $\vec{x}_d[i+1] = \vec{x}_c((i+1)\Delta)$ from Sampling eq'n above

$\vec{x}_d[i+1]$ is the sol'n of diff. eq'n

$$\frac{d}{dt}\vec{x}_c(t) = A_c\vec{x}_c(t) + B_c\cancel{\vec{u}_c(t)} \quad = \vec{u}_d[i]$$

at time $\underline{(i+1)\Delta}$ from initial condition $\vec{x}_c(i\Delta) = \vec{x}_d[i]$ at $\underline{t_0 = i\Delta}$.

First, scalar case:

$$\frac{d}{dt}x_c(t) = \lambda x_c(t) + b\,u_d[i] \qquad x_c(t_0) = x_d[i]$$

Discussion 2A/6A:

$$x_c(t) = e^{\lambda(t-t_0)}x_c(t_0) + \underbrace{\left(\int_{t_0}^{t} e^{\lambda(t-\tau)}d\tau\right)}_{\text{see discussion}} b\,u_d[i]$$

Substitute $\underline{t_0 = i\Delta}$, $\underline{t = (i+1)\Delta}$:

$$\underbrace{x_c((i+1)\Delta)}_{x_d[i+1]} = \underbrace{e^{\lambda\Delta}}_{A_d}\underbrace{x_c(i\Delta)}_{x_d[i]} + \underbrace{\left(\frac{e^{\lambda\Delta}-1}{\lambda}\right)}_{\substack{\text{when } \lambda\neq 0, \\ \text{Replace with } \Delta \\ \text{when } \lambda = 0}} b\,u_d[i]$$

$$B_d = \begin{cases} \frac{e^{\lambda\Delta}-1}{\lambda}b & \text{if } \lambda\neq 0 \\ \Delta b & \text{if } \lambda = 0. \end{cases}$$

Summary (scalar case):

$A_c = \lambda$, $B_c = b$ $\Rightarrow$ $A_d = e^{\lambda\Delta}$, $B_d =$ "

Vector Case:

$$\frac{d}{dt}\vec{x}_c(t) = A_c\vec{x}_c(t) + B_c\vec{u}_d[i]$$

$t_0 = i\Delta$

$t = (i+1)\Delta$

Recall Lecture 6: diagonalization

$$\vec{y}_c = V^{-1}\vec{x}_c \Rightarrow \vec{x}_c = V\vec{y}_c$$

$$\frac{d}{dt}\vec{y}_c(t) = V^{-1}\frac{d}{dt}\vec{x}_c(t) = V^{-1}A_c\underbrace{\vec{x}_c(t)}_{V\vec{y}_c(t)} + V^{-1}B_c\vec{u}_d[i]$$

$$\frac{d}{dt}\vec{y}_c(t) = \underbrace{V^{-1}A_cV}_{\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}}\vec{y}_c(t) + \left(V^{-1}B_c\vec{u}_d[i]\right)$$

$$\frac{d}{dt}y_{c,k}(t) = \lambda_k y_{c,k}(t) + \left(V^{-1}B_c\vec{u}_d[i]\right)_k$$

k th entry of vector $\vec{y}_c$

kth entry of vector $V^{-1}B_c\vec{u}_d[i]$

From scalar case above:

$$y_{d,k}[i+1] = e^{\lambda_k\Delta}y_{d,k}[i] + \underbrace{\frac{e^{\lambda_k\Delta}-1}{\lambda_k}}\left(V^{-1}B_c\vec{u}_d[i]\right)_k$$

if $\lambda_k \neq 0$; o/w replace w/ Δ

stack up from $k=1$ to $k=n$

$$\vec{y}_d[i+1] = \begin{bmatrix}e^{\lambda_1\Delta} & & \\ & \ddots & \\ & & e^{\lambda_n\Delta}\end{bmatrix}\vec{y}_d[i] + \begin{bmatrix}\frac{e^{\lambda_1\Delta}-1}{\lambda_1} & & \\ & \ddots & \\ & & \frac{e^{\lambda_n\Delta}-1}{\lambda_n}\end{bmatrix}V^{-1}B_c\vec{u}_d[i]$$

$$\vec{x}_d = V\vec{y}_d \qquad \vec{y}_d = V^{-1}\vec{x}_d$$

multiply both sides by V

$$\vec{x}_d[i+1] = \underbrace{V\begin{bmatrix} e^{\lambda_1\Delta} & & \\ & \ddots & \\ & & e^{\lambda_n\Delta} \end{bmatrix}V^{-1}}_{A_d}\vec{x}_d[i] + \underbrace{V\begin{bmatrix} \frac{e^{\lambda_1\Delta}-1}{\lambda_1} & & \\ & \ddots & \\ & & \frac{e^{\lambda_n\Delta}-1}{\lambda_n} \end{bmatrix}V^{-1}B_c}_{B_d}\vec{u}_d[i]$$

System Identification:

$$\vec{x}_d[i+1] = A_d\,\vec{x}_d[i] + B_d\,\vec{u}_d[i]$$

Can we learn the entries of A_d and B_d by observing the input sequence $\vec{u}_d[0], \vec{u}_d[1], \ldots$ and resulting state sequence $\vec{x}_d[1], \vec{x}_d[2], \ldots$?

Yes, under appropriate conditions. Will use Least Squares.

Least Squares Review:

$$\vec{s} = D\,\vec{p} + \vec{e}$$

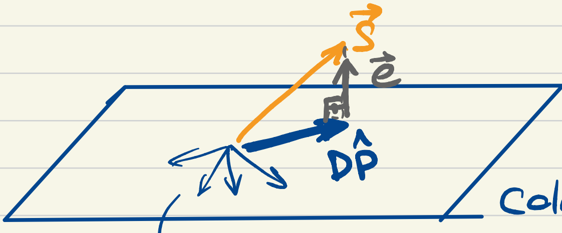
$\vec{s} \in \mathbb{R}^{\ell}$: vector of measurements

$\vec{p} \in \mathbb{R}^{q}$: unknown parameters (typically $q < \ell$)

$D \in \mathbb{R}^{\ell \times q}$: known matrix

Find $\hat{p}$ s.t. $D\hat{p}$ is as close to $\vec{s}$ as possible in the sense that

$\|\vec{e}\| = \|\vec{s} - D\vec{p}\|$ is minimized when $\vec{p} = \hat{p}$.



column space of D

$D\vec{p}$ lives here for any choice of $\vec{p}$

$\|\vec{e}\|$ is minimized when $\vec{e} \perp$ column space of D

$$\underbrace{\begin{matrix} \vec{d}_1^T \vec{e} = 0 \\ \vdots \\ \vec{d}_q^T \vec{e} = 0 \end{matrix}}$$

where $D = [\vec{d}_1 \cdots \vec{d}_q]$

$$D^T = \begin{bmatrix} \vec{d}_1^T \\ \vdots \\ \vec{d}_q^T \end{bmatrix}$$

$$D^T \vec{e} = 0$$

$$\downarrow$$

$$\vec{e} = \vec{s} - D\vec{p} \rightarrow D^T(\vec{s} - D\vec{p}) = 0$$

$$D^T D \vec{p} = D^T \vec{s}$$

If $D^T D$ invertible LS solution is

$$\hat{p} = (D^T D)^{-1} D^T \vec{s}$$

I incorrectly wrote "If D invertible" in Lecture. Corrected after class.