Homework 7

This homework is due on Friday, March 11, 2022, at 11:59PM. Self-grades and HW re-submissions are due on the following Friday, March 18, 2022, at 11:59PM.

1. **Reading Lecture Notes**

   Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: Note 11, and Note 12.

   (a) **How would you use feedback control to choose the closed-loop eigenvalues of a closed-loop discrete-time system?**

   Solution: We let $\vec{u}[i] = F\vec{x}[i]$ so $\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] = A\vec{x}[i] + BF\vec{x}[i] = (A+BF)\vec{x}[i]$. Then, we calculate the determinant of $\lambda I - (A+BF)$ to get the characteristic polynomial as a symbolic function of the entries of F. Meanwhile, we calculate the target characteristic polynomial by taking our desired eigenvalues and computing $\prod_i(\lambda - \lambda_i)$ as a polynomial. By matching the coefficients of λ^k, we get a system of equations where the unknowns are the entries of F. Solving that system of equations gives us the entries of the F matrix that makes our closed-loop dynamics $A_{cl} = A + BF$ have the desired eigenvalues $\{\lambda_i\}$.

   The same thing can be done with continuous-time systems as well. Now, $\vec{u}(t) = F\vec{x}(t)$ so that $\frac{d}{dt}\vec{x} = A\vec{x} + B\vec{u} = A\vec{x} + BF\vec{x} = (A+BF)\vec{x}$.

   In both cases, the closed-loop dynamics becomes $A_{cl} = A + BF$ and its eigenvalues can be changed by setting F.

   (b) **What is the matrix test for controllability of a general linear discrete-time system $\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]$ with a scalar input $u[i]$?**

   Solution: We construct the controllability matrix $\mathcal{C} = \begin{bmatrix} \vec{b} & A\vec{b} & A^2\vec{b} & \dots & A^{n-1}\vec{b} \end{bmatrix}$. If $\text{rank}(\mathcal{C}) = n$, meaning it is full rank, then the system is controllable.

   Just as a reminder, this means that given any initial state, we can construct a sequence of inputs that lead us to any goal state in n timesteps. Why? Because

$$\vec{x}[n] = A^n\vec{x}[0] + \sum_{k=0}^{n-1} A^{n-1-k}\vec{b}u[k] = A^n\vec{x}[0] + \mathcal{C}\begin{bmatrix} u[n-1] \\ u[n-2] \\ \vdots \\ u[1] \\ u[0] \end{bmatrix}.$$

   If $\text{rank}(\mathcal{C}) = n$, this matrix is invertible and the previous equation can always be solved for the vector of $u[i]$ given an initial condition and a desired state $\vec{x}[n]$.

   (c) **If $\vec{b}$ above were an eigenvector of A, why would this imply that the system is not controllable if the dimension of $\vec{x}$ is larger than 1?**

   Solution: Because the matrix

$$\mathcal{C} = \begin{bmatrix} \vec{b} & A\vec{b} & A^2\vec{b} & \dots & A^{n-1}\vec{b} \end{bmatrix} \tag{1}$$

$$= \begin{bmatrix} \vec{b} & \lambda\vec{b} & \lambda^2\vec{b} & \dots & \lambda^{n-1}\vec{b} \end{bmatrix} \tag{2}$$

   since $A\vec{b} = \lambda\vec{b}$ if $\vec{b}$ is an eigenvector of A. Since all the columns of $\mathcal{C}$ are multiples of each other, the rank is just 1 which is less than $n > 1$. So the system is not controllable. We can only control the system in the direction of $\vec{b}$.

2. Stability Criterion

Consider the complex plane below, which is broken into non-overlapping regions A through H. The circle drawn on the figure is the unit circle $|\lambda| = 1$.

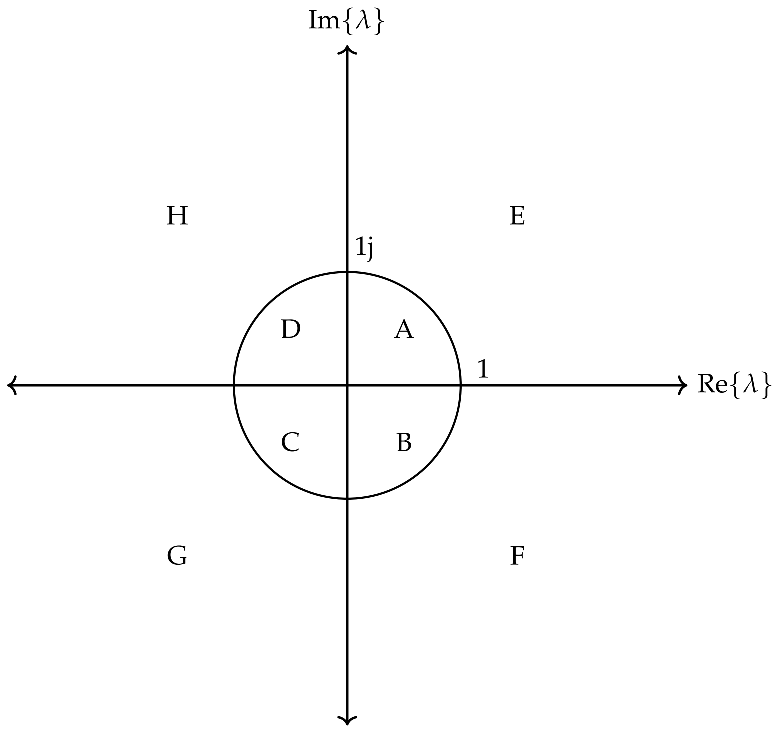


Figure 1: Complex plane divided into regions.

Consider the continuous-time system $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + v(t)$
and the discrete-time system $y[i+1] = \lambda y[i] + w[i]$. Here $v(t)$ and $w[i]$ are both disturbances to their respective systems.

In which regions can the eigenvalue λ be for the system to be *stable*? Fill out the table below to indicate *stable* regions. Assume that the eigenvalue λ does not fall directly on the boundary between two regions.

	A	B	C	D	E	F	G	H
Continuous Time System $x(t)$	◯	◯	◯	◯	◯	◯	◯	◯
Discrete Time System $y[i]$	◯	◯	◯	◯	◯	◯	◯	◯

Solution: For the continuous time system to be stable, we need the real part of λ to be less than zero. Hence, C, D, G, H satisfy this condition.

On the other hand, for the discrete time system to be stable, we need the norm of λ to be less than one. Hence, A, B, C, D satisfy this condition.

3. Bounded-Input Bounded-Output (BIBO) Stability

BIBO stability is a system property where bounded inputs lead to bounded outputs. It's important because we want to certify that, provided our system inputs are bounded, the outputs will not "blow up". In this problem, we gain a better understanding of BIBO stability by considering some simple continuous and discrete systems, and showing whether they are BIBO stable or not.

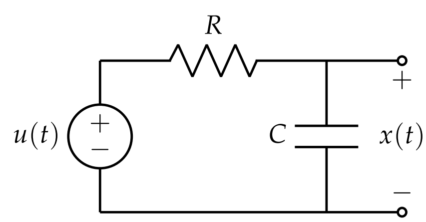
Recall that for the following simple scalar differential equation, we have the corresponding solution:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = ax(t) + bu(t) \qquad x(t) = \mathrm{e}^{at}x(0) + \int_0^t \mathrm{e}^{a(t-\tau)}bu(\tau)\,\mathrm{d}\tau\,. \tag{3}$$

And for the following discrete system, we have the corresponding solution:

$$x[i+1] = ax[i] + bu[i] \qquad x[i] = a^i x[0] + \sum_{k=0}^{i-1} a^k bu[i-1-k] \tag{4}$$

(a) Consider the circuit below with $R = 1\Omega$, $C = 0.5\text{F}$. Let $x(t)$ be the voltage over the capacitor.



This circuit can be modeled by the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -2x(t) + 2u(t) \tag{5}$$

Intuitively, we know that the voltage on the capacitor can never exceed the (bounded) voltage from the voltage source, so this system is BIBO stable. **Show that this system is BIBO stable, meaning that $x(t)$ remains bounded for all time if the input $u(t)$ is bounded. Equivalently, show that if we assume $|u(t)| < \epsilon$, $\forall t \geq 0$ and $|x(0)| < \epsilon$, then $|x(t)| < M$, $\forall t \geq 0$ for some positive constant M.** Thinking about this helps you understand what bounded-input-bounded-output stability means in a physical circuit.

(HINT: eq. (3) may be useful. You may want to write the expression for $x(t)$ in terms of $u(t)$ and $x(0)$ and then take the norms of both sides to show a bound on $|x(t)|$. Remember that norm in 1D is absolute value. Some helpful formulas are $|ab| = |a||b|$, the triangle inequality $|a+b| \leq |a| + |b|$, and the integral version of the triangle inequality $\left|\int_a^b f(\tau)\,\mathrm{d}\tau\right| \leq \int_a^b |f(\tau)|\,\mathrm{d}\tau$, which just extends the standard triangle inequality to an infinite sum of terms.)

Solution:

Using eq. (3), we get the solution to the scalar differential equation as

$$x(t) = \mathrm{e}^{-2t}x(0) + \int_0^t \mathrm{e}^{-2(t-\tau)}2u(\tau)\,\mathrm{d}\tau\,. \tag{6}$$

Then we can try to bound $x(t)$ for $t \geq 0$. We first use the triangle inequality ($|a+b| \leq |a| + |b|$) to get

$$|x(t)| = \left|\mathrm{e}^{-2t}x(0) + \int_0^t \mathrm{e}^{-2(t-\tau)}2u(\tau)\,\mathrm{d}\tau\right| \tag{7}$$

$$|x(t)| \leq \left|\mathrm{e}^{-2t}x(0)\right| + \left|\int_0^t \mathrm{e}^{-2(t-\tau)}2u(\tau)\,\mathrm{d}\tau\right| \tag{8}$$

We then use the property that the integral of absolute value will always be greater than the absolute value of the integral (equation (8) to (9)), and that an exponential is always positive (equation (9) to (10)):

$$|x(t)| \leq \left|e^{-2t}x(0)\right| + \int_0^t \left|e^{-2(t-\tau)}2u(\tau)\right| d\tau \tag{9}$$

$$= e^{-2t}|x(0)| + \int_0^t e^{-2(t-\tau)}2|u(\tau)| d\tau \tag{10}$$

Finally, plugging in our bounds for $|u(\tau)|$ and $|x(0)|$ and doing the integral:

$$|x(t)| \leq e^{-2t}\epsilon + \int_0^t e^{-2(t-\tau)}2\epsilon\, d\tau \tag{11}$$

$$= e^{-2t}\epsilon + 2\epsilon e^{-2t}\int_0^t e^{2\tau}\, d\tau \tag{12}$$

$$= e^{-2t}\epsilon + 2\epsilon e^{-2t}\frac{1}{2}\left(e^{2t}-1\right) \tag{13}$$

$$= e^{-2t}\epsilon + \epsilon\left(1 - e^{-2t}\right) \tag{14}$$

$$= \epsilon,\ \forall t \geq 0 \tag{15}$$

So we see that our state's magnitude is bounded for all time. Note that the negative exponent of the exponential is what makes this system stay bounded.

(b) Assume $x(0) = 0$. **Show that the system eq. (3) is BIBO unstable when $a = j2\pi$ by constructing a bounded input that leads to an unbounded $x(t)$.**

It can be shown that the system eq. (3) is unstable for any purely imaginary a by a similar construction of a bounded input.

Solution: Recall the solution of $x(t)$ with the initial condition at zero

$$x(t) = \int_0^t e^{a(t-\tau)}bu(\tau)\, d\tau\,. \tag{16}$$

Remember, the style of argumentation here is the "counterexample" style. The question asks you to show that *some* bounded input exists that will make the state grow without bound.

Because we know we can get an integral to diverge if we are just integrating a nonzero constant, we decide to try the bounded input $u(t) = \epsilon e^{j2\pi t}$, whose magnitude is equal to ϵ for all t.

Plugging this input and a value in, we see

$$x(t) = \int_0^t e^{j2\pi(t-\tau)}b\epsilon e^{j2\pi\tau}\, d\tau = \int_0^t e^{j2\pi t}b\epsilon\, d\tau\,. \tag{17}$$

Factoring out the terms that do not depend on τ, we are left with

$$x(t) = b\epsilon e^{j2\pi t}\int_0^t d\tau\,. \tag{18}$$

Solving this integral, we get

$$x(t) = b\epsilon t e^{j2\pi t}. \tag{19}$$

Now taking the magnitude of $x(t)$ using the fact that $|e^{j\omega t}| = 1$ for all ω, we get $|x(t)| = \epsilon|b|t$ which clearly diverges as $t \to \infty$.

(c) Consider the discrete-time system and its solution in eq. (4). **Show that if $|a| > 1$, then even if $x[0] = 0$, a bounded input can result in an unbounded output, i.e. the system is BIBO unstable.** *(HINT: The formula for the sum of a geometric sequence may be helpful.)*

Solution: Consider when $x[0] = 0$ and $u[i] = 1\ \forall i$. This gives

$$x[i] = a^i x[0] + \sum_{k=0}^{i-1} a^k b u[i-1-k] \tag{20}$$

$$= \sum_{k=0}^{i-1} a^k b \tag{21}$$

$$= b\frac{a^i - 1}{a-1} \quad \text{(as this is the sum of a geometric series)} \tag{22}$$

When $|a| > 1$, then a^i has magnitude that grows without bound, and thus $|x[i]|$ does as well. We also know this from the convergence criteria for geometric series; when the common ratio $a > 1$, the series does not converge to a finite number as $i \to \infty$.

(d) Consider the discrete-time system

$$x[i+1] = -3x[i] + u[i]. \tag{23}$$

Is this system stable or unstable? Give an initial condition $x(0)$ and a sequence of non-zero inputs for which the state $x[i]$ will always stay bounded. *(HINT: See if you can find any input pattern that results in an oscillatory behavior.))*

Solution:

The system is unstable since the eigenvalue -3 has magnitude ≥ 1. To see this more explicitly, any non-zero $x[0]$ and (bounded) $u[i] = 0\ \forall i \in \mathbb{N}$ will lead to unbounded x.

Consider $x[0] = 0$ and the input $u[i] = 1, 3, 1, 3, 1, 3, \ldots$.

t	0	1	2	3	$\cdots$
$x[i]$	0	1	0	1	$\cdots$
$u[i]$	1	3	1	3	$\cdots$
$-3x[i] + u[i]$	1	0	1	0	$\cdots$

In this case, we get $x[i] = 0$ when t is even, and $x[i] = 1$ when i is odd. In fact, there are an infinite number of input sequences that would result in bounded outputs.

4. Eigenvalue Placement through State Feedback

Consider the following discrete-time linear system:

$$\vec{x}[i+1] = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i]. \tag{24}$$

In standard language, we have $A = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in the form: $\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]$.

(a) **Is this system controllable?**

Solution: We calculate the controllability matrix

$$\mathcal{C} = \begin{bmatrix} \vec{b} & A\vec{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{25}$$

Observe that the $\mathcal{C}$ matrix has linearly independent columns and hence our system is controllable.

(b) **Is this discrete-time linear system stable in open loop (without feedback control)?**

Solution: We have to calculate the eigenvalues of matrix A. Thus,

$$0 = \det(\lambda I - A) \tag{26}$$

$$= \det \begin{bmatrix} \lambda + 2 & -2 \\ 2 & \lambda - 3 \end{bmatrix} \tag{27}$$

$$= \lambda^2 - \lambda - 2 \tag{28}$$

$$\implies \lambda_1 = 2, \qquad \lambda_2 = -1 \tag{29}$$

Since at least one eigenvalue has a magnitude that is greater than or equal to 1, the discrete-time system is unstable. In this case, both of the eigenvalues are unstable.

(c) Suppose we use state feedback of the form $u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i] = F\vec{x}[i]$.

Find the appropriate state feedback constants, f_1, f_2 so that the state space representation of the resulting closed-loop system has eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.

Solution: The closed loop system using state feedback has the form

$$\vec{x}[i+1] = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i] \tag{30}$$

$$= \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i] \tag{31}$$

$$= \left(\begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} f_1 & f_2 \\ f_1 & f_2 \end{bmatrix} \right) \vec{x}[i] \tag{32}$$

Thus, the closed loop system has the form

$$\vec{x}[i+1] = \underbrace{\begin{bmatrix} -2 + f_1 & 2 + f_2 \\ -2 + f_1 & 3 + f_2 \end{bmatrix}}_{A_{\text{cl}}} \vec{x}[i] \tag{33}$$

Finding the characteristic polynomial of the above system, we have

$$\det\left(\lambda I - \begin{bmatrix} -2 + f_1 & 2 + f_2 \\ -2 + f_1 & 3 + f_2 \end{bmatrix} \right) = (\lambda + 2 - f_1)(\lambda - 3 - f_2) - (-2 - f_2)(2 - f_1) \tag{34}$$

$$= \lambda^2 - f_1\lambda - f_2\lambda - \lambda + f_1 f_2 - 6 - 2f_2 + 3f_1 \tag{35}$$

$$-(-4 + f_1 f_2 + 2f_1 - 2f_2) \tag{36}$$
$$= \lambda^2 - (1 + f_1 + f_2)\lambda + f_1 - 2 \tag{37}$$

However, we want to place the eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$. That means we want

$$\lambda^2 - (1 + f_1 + f_2)\lambda + f_1 - 2 = \left(\lambda + \frac{1}{2}\right)\left(\lambda - \frac{1}{2}\right) \tag{38}$$

or equivalently:

$$\lambda^2 - (1 + f_1 + f_2)\lambda + f_1 - 2 = \lambda^2 - \frac{1}{4} \tag{39}$$

Equating the coefficients of the different powers of λ on both sides of the equation, we get,

$$1 + f_1 + f_2 = 0 \tag{40}$$
$$f_1 - 2 = -\frac{1}{4} \tag{41}$$

Solving the above system of equations gives us $f_1 = \frac{7}{4}, f_2 = -\frac{11}{4}$.

(d) We are now ready to go through some numerical examples to see how state feedback works. Consider the first discrete-time linear system. Enter the matrix A and vector $\vec{b}$ from (a) for the system $\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i] + \vec{w}[i]$ into the Jupyter notebook "`eigenvalue_placement.ipynb`" and use the randomly generated $\vec{w}[i]$ as the disturbance introduced into the state equation. Observe how the norm of $\vec{x}[i]$ evolves over time for the given A. **What do you see happening to the norm of the state?**

Solution: See Jupyter notebook "`eigenvalue_placement_sol.ipynb`" for solution. The norm of $\vec{x}(t)$ increases with time for the given A. This is because the matrix A has eigenvalues with magnitude greater than one as we discussed in (b) and thus the state keeps growing at each time step.

(e) Add the feedback computed in part (c) to the system in the notebook and **explain how the norm of the state changes.**

Solution: The eigenvalues of the closed loop system are at $\frac{1}{2}$ and -$\frac{1}{2}$. Thus, the norm of the state variable is now bounded with time. Check the solution in the Jupyter notebook.

(f) Now suppose we've got a different system described by the controlled scalar difference equation $z[i+1] = z[i] + 2z[i-1] + u[i]$. To convert this second-order discrete time system to a two-dimensional first-order discrete time system, we will let $\vec{y}[i] = \begin{bmatrix} z[i-1] \\ z[i] \end{bmatrix}$.

Write down the system representation for $\vec{y}$ in the following matrix form:

$$\vec{y}[i+1] = A_y \vec{y}[i] + \vec{b_y} u[i]. \tag{42}$$

Specify the values of the matrix A_y and the vector $\vec{b_y}$.

Solution: From the problem, we have $z[i+1] = 2z[i-1] + z[i] + u[i]$, which will become the second row of our system. We can then write the equation in matrix form as

$$\vec{y}[i+1] = \begin{bmatrix} z[i] \\ z[i+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} z[i-1] \\ z[i] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[i] \tag{43}$$

where $A_y = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \vec{b_y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(g) It turns out that the original $\vec{x}[i]$ system can be converted to the $\vec{y}[i]$ system using a change of basis P. Let this coordinate change be written as $\vec{y}[i] = P\vec{x}[i]$. **First express A_y and $\vec{b_y}$ symbolically**

in terms of A, $\vec{b}$, and P. Then, confirm numerically that $P = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ is the correct change of basis matrix between the two systems.

Solution: As we know from before, $\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]$. Then,

$$\begin{aligned} \vec{y}[i+1] &= P\vec{x}[i+1] && (44) \\ &= P(A\vec{x}[i] + \vec{b}u[i]) && (45) \\ &= PA\vec{x}[i] + P\vec{b}u[i] && (46) \\ &= PAP^{-1}\vec{y}[i] + P\vec{b}u[i] && (47) \end{aligned}$$

Thus,

$$\begin{aligned} A_y &= PAP^{-1} && (48) \\ \vec{b_y} &= P\vec{b}. && (49) \end{aligned}$$

We confirm that,

$$\begin{aligned} PAP^{-1} &= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} && (50) \\ &= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} && (51) \\ &= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = A_y && (52) \end{aligned}$$

(Note: the above is not a typo. The inverse of this particular P matrix is really itself.)
We also confirm that

$$P\vec{b} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{b_y} \quad (53)$$

(h) For the $\vec{y}$ system from part (f), **design a feedback gain matrix $\begin{bmatrix} \overline{f}_1 & \overline{f}_2 \end{bmatrix}$ to place the closed-loop eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$. Additionally, confirm that this matrix is just a change of basis of the gain matrix from part (c), i.e. $\begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} \overline{f}_1 & \overline{f}_2 \end{bmatrix} P$.**

Note that this means you can solve for the closed-loop gains of your system in any basis, and then transform it to the basis you care about.

Solution: Solving for the new feedback matrix: The closed loop system using state feedback has the form

$$\begin{aligned} \vec{y}[i+1] &= \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \vec{y}[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[i] && (54) \\ &= \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \vec{y}[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(\begin{bmatrix} \overline{f}_1 & \overline{f}_2 \end{bmatrix} \vec{y}[i] \right) && (55) \\ &= \left(\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} \overline{f}_1 & \overline{f}_2 \end{bmatrix} \right) \vec{y}[i] && (56) \\ &= \underbrace{\begin{bmatrix} 0 & 1 \\ 2 + \overline{f}_1 & 1 + \overline{f}_2 \end{bmatrix}}_{A_{\text{cl}}} \vec{y}[i] && (57) \end{aligned}$$

Thus, finding the eigenvalues of the above system we have

$$\det(\lambda I - \begin{bmatrix} 0 & 1 \\ 2 + \overline{f}_1 & 1 + \overline{f}_2 \end{bmatrix}) = 0 \Rightarrow \lambda^2 - (1 + \overline{f}_2)\lambda - (2 + \overline{f}_1) = 0 \quad (58)$$

However, we want to place the eigenvalue at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$. Thus, this means that

$$\lambda^2 - (1+\overline{f}_2)\lambda - \overline{f}_1 - 2 = \left(\lambda + \frac{1}{2}\right)\left(\lambda - \frac{1}{2}\right) \tag{59}$$

$$= \lambda^2 - \frac{1}{4}. \tag{60}$$

Equating the co-efficients of λ on both sides, we get

$$1 + \overline{f}_2 = 0 \tag{61}$$

$$-\overline{f}_1 - 2 = -\frac{1}{4} \tag{62}$$

The above system of equations gives us $\overline{f}_1 = \frac{-7}{4}, \overline{f}_2 = -1$.
Matrix multiplication by the basis P confirms that

$$\begin{bmatrix}\overline{f}_1 & \overline{f}_2\end{bmatrix} P = \begin{bmatrix}-\frac{7}{4} & -1\end{bmatrix}\begin{bmatrix}-1 & 1 \\ 0 & 1\end{bmatrix} = \begin{bmatrix}\frac{7}{4} & -\frac{11}{4}\end{bmatrix} = \begin{bmatrix}f_1 & f_2\end{bmatrix} \tag{63}$$

5. Open-Loop and Closed-Loop Control

In last week's lab-related System ID problem, we built SIXT33N's motor control circuitry and developed a linear model for the velocity of each wheel. We are one step away from our goal: to have SIXT33N drive in a straight line! We will see how to use the model we developed in the System ID problem to control SIXT33N's trajectory to be a straight line.

More specifically, in this problem, we will explore how to use open-loop and closed-loop control to drive the trajectory of your car in a straight line.

Part 1: Open-Loop Control

An open-loop controller is one in which the input is predetermined using your system model and the goal, and not adjusted at all during operation. To design an open-loop controller for your car, you would set the PWM duty-cycle value of the left and right wheels (inputs $u_L[i]$ and $u_R[i]$) such that the predicted velocity of both wheels is your target wheel velocity (v_t). You can calculate these inputs from the target velocity v_t and the $\theta_L, \theta_R, \beta_L, \beta_R$ values you learned from data. In the System ID problem and lab, we have modeled the velocity of the left and right wheels as

$$v_L[i] = d_L[i+1] - d_L[i] = \theta_L u_L[i] - \beta_L; \tag{64}$$
$$v_R[i] = d_R[i+1] - d_R[i] = \theta_R u_R[i] - \beta_R \tag{65}$$

where $d_{L,R}[i]$ represent the distance traveled by each wheel.

(a) Find the open-loop control that would give us $v_L[i] = v_R[i] = v_t$. That is, **solve the model (Equations (64) and (65)) for the inputs $u_L[i]$ and $u_R[i]$ that make the velocities $v_L[i] = v_R[i] = v_t$.**

Solution: Starting from Equations (64) and (65) and substituting in the target velocity v_t, we get the following equations.

$$v_t = \theta_L u_L[i] - \beta_L \tag{66}$$
$$v_t = \theta_R u_R[i] - \beta_R \tag{67}$$
$$v_t + \beta_L = \theta_L u_L[i] \tag{68}$$
$$v_t + \beta_R = \theta_R u_R[i] \tag{69}$$
$$\frac{v_t + \beta_L}{\theta_L} = u_L[i] \tag{70}$$
$$\frac{v_t + \beta_R}{\theta_R} = u_R[i] \tag{71}$$

In practice, the $\theta_L, \theta_R, \beta_L, \beta_R$ parameters are learned from noisy data, and so can be wrong. This means that we will calculate the velocities for the two wheels incorrectly. When the velocities of the two wheels disagree, the car will go in a circle instead of a straight line. Thus, to make the car go in a straight line, we need the distances traveled by both wheels to be the same at each timestep.

This prompts us to simplify our model. Instead of having two state variables $\vec{v}_L$ and $\vec{v}_R$, we can just have a state variable determining how far we are from the desired behavior of going in a line – a state which we will want to drive to 0.

This prompts us to define our state variable δ to be the *difference* in the distance traveled by the left wheel and the right wheel at a given timestep:

$$\delta[i] := d_L[i] - d_R[i] \tag{72}$$

We want to find a scalar discrete-time model for $\delta[i]$ of the form

$$\delta[i+1] = \lambda_{\text{OL}}\delta[i] + f(u_L[i], u_R[i]). \tag{73}$$

Here λ_{OL} is a scalar and $f(u_L[i], u_R[i])$ is the control input to the system (as a function of $u_L[i]$ and $u_R[i]$).

(b) Suppose we apply the open-loop control inputs $u_L[i], u_R[i]$ to the original system. **Using Equations (64) and (65), write $\delta[i+1]$ in terms of $\delta[i]$, in the form of Equation (73). What is the eigenvalue λ_{OL} of the model in Equation (73)? Would the model in Equation (73) be stable with open-loop control if it also had a disturbance term?**

(HINT: For open-loop control, we set the velocities to $v_L[i] = v_R[i] = v_t$. What happens when we substitute that into Equations (64) and (65) and then apply the definition of $\delta[i]$ and $\delta[i+1]$?)

Solution: Proceeding by the hint,

$$\begin{aligned}
\delta[i+1] &= d_L[i+1] - d_R[i+1] && (74)\\
&= (v_L[i] + d_L[i]) - (v_R[i] + d_R[i]) && (75)\\
&= v_t + d_L[i] - (v_t + d_R[i]) && (76)\\
&= d_L[i] - d_R[i] && (77)\\
&= \delta[i] && (78)
\end{aligned}$$

From the derivation above, $\lambda_{\text{OL}} = 1$ and $f(u_L[i], u_R[i]) = 0$. To check stability, we already know our eigenvalue does not meet the stability criteria: $|\lambda_{\text{OL}}| = 1$, so we have an unstable system if we add disturbances (whereas if we don't then the system is *marginally stable*).

Part 2: Closed-Loop Control

Now, in order to make the car drive straight, we must implement closed-loop control – that is, control inputs that depend on the current state and are calculated dynamically – and use feedback in real time.

(c) **If we want the car to drive straight starting from some timestep $i_{\text{start}} > 0$, i.e., $v_L[i] = v_R[i]$ for $i \geq i_{\text{start}}$, what condition does this impose on $\delta[i]$ for $i \geq i_{\text{start}}$?**

Solution: Let $i \geq i_{\text{start}}$. Then

$$\begin{aligned}
0 &= v_L[i] - v_R[i] && (79)\\
&= d_L[i+1] - d_L[i] - (d_R[i+1] - d_R[i]) && (80)\\
&= (d_L[i+1] - d_R[i+1]) - (d_L[i] - d_R[i]) && (81)\\
&= \delta[i+1] - \delta[i]. && (82)
\end{aligned}$$

Thus

$$\delta[i+1] = \delta[i], \qquad i \geq i_{\text{start}} \tag{83}$$

which implies that

$$\delta[i] = \delta[i_{\text{start}}], \qquad i \geq i_{\text{start}}. \tag{84}$$

In other words, we have that for every timestep beyond i_{start}, the difference in distances the wheels have traveled does not change.

(d) **How is the condition you found in the previous part different from the condition:**

$$\delta[i] = 0, \qquad i \geq i_{\text{start}}? \tag{85}$$

Assume that $i_{\text{start}} > 0$, and that $d_L[0] = 0, d_R[0] = 0$.

This is a subtlety that is worth noting and often requires one to adjust things in real systems.

Solution: At time $i = 0$, the car has not moved yet, so $\delta[0] = d_L[0] - d_R[0] = 0$. If at some later time i_{start} we have $\delta[i_{\text{start}}] = 0$ and $\delta[i] = 0$ for later times as well, we remain moving in the same direction we started with. When $\delta[i] \neq 0$, this means the wheels have moved different distances, and therefore has moved along a curved path and changed the direction the car is pointing.

While not required, Fig. 2 illustrates the two different cases where $\delta[i] = 0$ for all times $i \geq 0$ (left) and when $\delta \neq 0$ initially but we have $\delta[i_{\text{start}}] = 0$ for some $i = i_{\text{start}}$ and $\delta[i] = 0$ for $i \geq i_{\text{start}}$ (right).

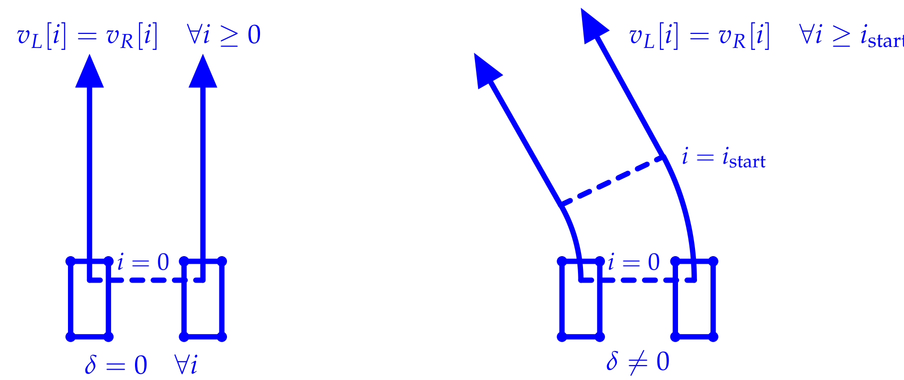


Figure 2

(e) From here, assume that we have reset the distance travelled counters at the beginning of this maneuver so that $\delta[0] = 0$. We will now implement a feedback controller by selecting two dimensionless positive coefficients, f_L and f_R, such that the closed loop system is stable with eigenvalue λ_{CL}. To implement closed-loop feedback control, we want to adjust $v_L[i]$ and $v_R[i]$ at each timestep by an amount that's proportional to $\delta[i]$. Not only do we want our wheel velocities to be some target velocity v_t, we also wish to drive $\delta[i]$ towards zero. This is in order to have the car drive straight along the initial direction it was pointed in when it started moving. If $\delta[i]$ is positive, the left wheel has traveled more distance than the right wheel, so relatively speaking, we can slow down the left wheel and speed up the right wheel to cancel this difference (i.e., drive it to zero) in the next few timesteps. The action of such a control is captured by the following velocities.

$$v_L[i] = v_t - f_L \delta[i]; \tag{86}$$
$$v_R[i] = v_t + f_R \delta[i]. \tag{87}$$

Give expressions for $u_L[i]$ and $u_R[i]$ as a function of v_t, $\delta[i]$, f_L,f_R, and our system parameters $\theta_L.\theta_R, \beta_L, \beta_R$, to achieve the velocities above.

Solution: As in the open loop case, we substitute the velocity expressions above into the equations that relate $v[i]$ and $u[i]$.

For the left wheel we have:

$$v_t - f_L \delta[i] = \theta_L u_L[i] - \beta_L \tag{88}$$
$$v_t - f_L \delta[i] + \beta_L = \theta_L u_L[i] \tag{89}$$
$$\frac{v_t - f_L \delta[i] + \beta_L}{\theta_L} = u_L[i] \tag{90}$$

For the right wheel we have:

$$v_t + f_R \delta[i] = \theta_R u_R[i] - \beta_R \tag{91}$$
$$v_t + f_R \delta[i] + \beta_R = \theta_R u_R[i] \tag{92}$$
$$\frac{v_t + f_R \delta[i] + \beta_R}{\theta_R} = u_R[i] \tag{93}$$

(f) Using the control inputs $u_L[i]$ and $u_R[i]$ found in part (e), **write the closed-loop system equation for $\delta[i+1]$ as a function of $\delta[i]$. What is the closed-loop eigenvalue λ_{CL} for this system in terms of λ_{OL}, f_L, and f_R?**

Solution: We can take the system equation explicitly in terms of $u_L[i]$ and $u_R[i]$ from the solution of part (c) in **??**, and substitute into this equation our control expressions from the previous part.

$$\delta[i+1] = \delta[i] + \theta_L u_L[i] - \theta_R u_R[i] - \beta_L + \beta_R \tag{94}$$

$$= \delta[i] + \theta_L\left(\frac{v_t - f_L\delta[i] + \beta_L}{\theta_L}\right) - \theta_R\left(\frac{v_t + f_R\delta[i] + \beta_R}{\theta_R}\right) - \beta_L + \beta_R \tag{95}$$

$$= \delta[i] + v_t - f_L\delta[i] - (v_t + f_R\delta[i]) \tag{96}$$

$$= \delta[i] - f_L\delta[i] - f_R\delta[i] \tag{97}$$

$$= (1 - f_L - f_R)\delta[i] \tag{98}$$

We see that our λ_{CL} will end up being $1 - f_L - f_R$, which is equal to $\lambda_{\mathrm{OL}} - f_L - f_R$.

(g) **What is the condition on f_L and f_R for the closed-loop system to be stable in the previous part?**

Solution:

$$|\lambda_{\mathrm{CL}}| < 1 \tag{99}$$

$$\implies |1 - f_L - f_R| < 1 \tag{100}$$

$$\implies -1 < 1 - f_L - f_R < 1 \tag{101}$$

$$\implies 0 < f_L + f_R < 2 \tag{102}$$

Stability in this case means that δ is bounded and will not go arbitrarily high. In fact, if our calculated β and θ are perfectly accurate, then $\delta[i] \to 0$, so the car will (eventually) drive straight!

One question remains – what if our calculated β and θ are *not* perfectly accurate? The answer turns out to be that there is some small steady-state discrepancy that your δ will converge to. You will see how to quantify this in next week's homework.

6. Miscellaneous Practice Problems for Midterm

(a) You are given the graph in Figure 3. **Express the coordinates of vectors $\vec{v}$ and $\vec{w}$ in both Cartesian (x, y) and Polar $(re^{j\theta})$ forms.**
You may use the atan2() or $\tan^{-1}$ function for angle (θ) as necessary.

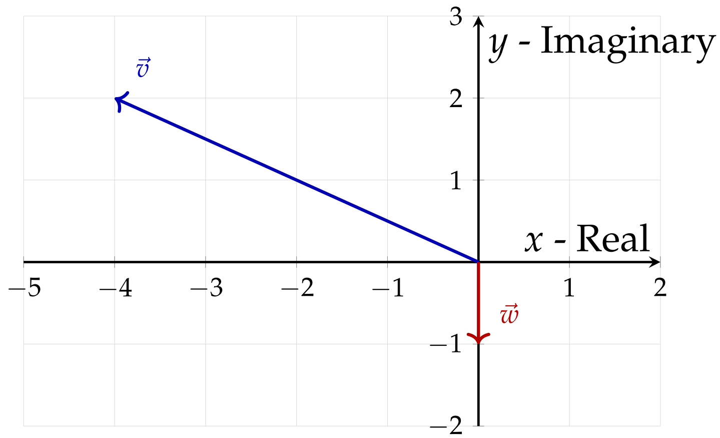


Figure 3: Vectors in the $\mathbf{x} - \mathbf{y}$ plane

i. Label $\vec{v}$ with its corresponding Cartesian (x, y) and Polar $(re^{j\theta})$ coordinates, in the given form.
Solution: Vector $\vec{v}$ Cartesian = $(-4, 2)$

$\text{atan2}(b, a)$ measures the angle (phase) of complex number $a + bj$, which equals to:

$$\text{atan2}(b, a) = \begin{cases} \tan^{-1}(\frac{b}{a}) & \text{if } a > 0, \\ \tan^{-1}(\frac{b}{a}) + \pi & \text{if } a < 0 \text{ and } b \geq 0, \\ \tan^{-1}(\frac{b}{a}) - \pi & \text{if } a < 0 \text{ and } b < 0, \\ +\frac{\pi}{2} & \text{if } a = 0 \text{ and } b > 0, \\ -\frac{\pi}{2} & \text{if } a = 0 \text{ and } b < 0, \\ \text{undefined} & \text{if } a = 0 \text{ and } b = 0. \end{cases} \tag{103}$$

Therefore, Vector $\vec{v}$ Polar = $\sqrt{20}e^{j\,\text{atan2}(2,-4)} \equiv 2\sqrt{5}e^{j(\tan^{-1}(-\frac{1}{2})+\pi)}$

ii. Label $\vec{w}$ with its corresponding Cartesian (x, y) and Polar $(re^{j\theta})$ coordinates, in the given form.
Solution: Vector $\vec{w}$ Cartesian = $(0, -1)$

Vector $\vec{w}$ Polar $= 1e^{-j\frac{\pi}{2}}$

(b) You are given an input voltage signal below:

$$v_{\text{in}}(t) = -1.5\sin\left(\omega t - \frac{\pi}{3}\right). \tag{104}$$

Convert the signal of eq. (104) to its phasor representation. That is, find $\widetilde{V}_{\text{in}}$.
Solution:

$$\widetilde{V}_{\text{in}} = -0.75e^{-j\frac{5\pi}{6}} \tag{105}$$

We can use Euler's formulae here, which states that:

$$\cos(x) = \frac{1}{2}\left(e^{jx} + e^{-jx}\right), \tag{106}$$

$$\sin(x) = \frac{1}{2\mathrm{j}}\left(\mathrm{e}^{\mathrm{j}x} - \mathrm{e}^{-\mathrm{j}x}\right). \tag{107}$$

There are many ways to proceed. All of them should count for credit and will give you the same answer.

One way is to remember that a sine is just a phase-shifted cosine. This would let us applying the second of these formulae as follows:

$$\begin{aligned}
v_{\text{in}}(t) &= -1.5\sin\left(\omega t - \frac{\pi}{3}\right) &(108)\\
&= -1.5\cos\left(\omega t - \frac{\pi}{3} - \frac{\pi}{2}\right) &(109)\\
&= -1.5\cos\left(\omega t - \frac{5\pi}{6}\right) &(110)\\
&= -1.5\cdot\frac{1}{2}\left(\mathrm{e}^{\mathrm{j}\left(\omega t - \frac{5\pi}{6}\right)} + \mathrm{e}^{-\mathrm{j}\left(\omega t - \frac{5\pi}{6}\right)}\right) &(111)\\
&= -0.75\left(\mathrm{e}^{\mathrm{j}\omega t}\mathrm{e}^{-\mathrm{j}\frac{5\pi}{6}} + \mathrm{e}^{-\mathrm{j}\omega t}\mathrm{e}^{\mathrm{j}\frac{5\pi}{6}}\right) &(112)\\
&= \left(-0.75\mathrm{e}^{-\mathrm{j}\frac{5\pi}{6}}\right)\mathrm{e}^{\mathrm{j}\omega t} + \left(-0.75\mathrm{e}^{\mathrm{j}\frac{5\pi}{6}}\right)\mathrm{e}^{-\mathrm{j}\omega t} &(113)
\end{aligned}$$

When we have a term of the form $u(t) = \widetilde{U}\mathrm{e}^{\mathrm{j}\omega t} + \overline{\widetilde{U}}\mathrm{e}^{-\mathrm{j}\omega t}$, we denote $\widetilde{U}$ as the phasor for the time-domain signal. So, by pattern matching:

$$\widetilde{V}_{\text{in}} = -0.75\mathrm{e}^{-\mathrm{j}\frac{5\pi}{6}} \tag{114}$$

We also could have proceeded using the second formula eq. (107) as follows:

$$\begin{aligned}
v_{\text{in}}(t) &= -1.5\sin\left(\omega t - \frac{\pi}{3}\right) &(115)\\
&= -1.5\cdot\frac{1}{2j}\left(\mathrm{e}^{\mathrm{j}\left(\omega t - \frac{\pi}{3}\right)} - \mathrm{e}^{-\mathrm{j}\left(\omega t - \frac{\pi}{3}\right)}\right) &(116)\\
&= 0.75\mathrm{j}\left(\mathrm{e}^{\mathrm{j}\omega t}\mathrm{e}^{-\mathrm{j}\frac{\pi}{3}} - \mathrm{e}^{-\mathrm{j}\omega t}\mathrm{e}^{\mathrm{j}\frac{\pi}{3}}\right) &(117)\\
&= 0.75\mathrm{e}^{\mathrm{j}\frac{\pi}{2}}\left(\mathrm{e}^{\mathrm{j}\omega t}\mathrm{e}^{-\mathrm{j}\frac{\pi}{3}} - \mathrm{e}^{-\mathrm{j}\omega t}\mathrm{e}^{\mathrm{j}\frac{\pi}{3}}\right) &(118)\\
&= 0.75\left(\mathrm{e}^{\mathrm{j}\omega t}\mathrm{e}^{\mathrm{j}\left(\frac{\pi}{2} - \frac{\pi}{3}\right)} - \mathrm{e}^{-\mathrm{j}\omega t}\mathrm{e}^{\mathrm{j}\left(\frac{\pi}{2} + \frac{\pi}{3}\right)}\right) &(119)\\
&= 0.75\left(\mathrm{e}^{\mathrm{j}\omega t}\mathrm{e}^{\mathrm{j}\left(\frac{\pi}{2} - \frac{\pi}{3}\right)} + \mathrm{e}^{-\mathrm{j}\pi}\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{e}^{\mathrm{j}\left(\frac{\pi}{2} + \frac{\pi}{3}\right)}\right) &(120)\\
&= 0.75\left(\mathrm{e}^{\mathrm{j}\omega t}\mathrm{e}^{\mathrm{j}\frac{\pi}{6}} + \mathrm{e}^{-\mathrm{j}\omega t}\mathrm{e}^{-\mathrm{j}\frac{\pi}{6}}\right) &(121)
\end{aligned}$$

which gives us

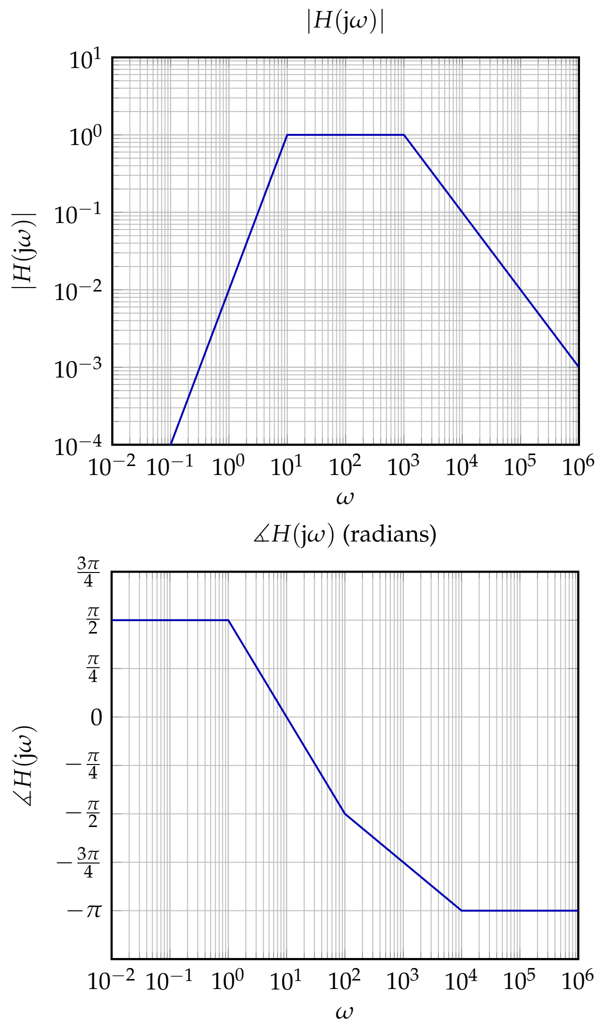
$$\widetilde{V}_{\text{in}} = 0.75\mathrm{e}^{\mathrm{j}\frac{\pi}{6}}. \tag{122}$$

This is the same answer and is in a sense, more standard in its form because all of the phase is showing up where you expect to see it, instead of π hiding in the minus sign up front.

(c) You decided to analyze the transfer function of a band-pass filter, and have generated the following Bode plots for $H(\mathrm{j}\omega)$. **If your input voltage signal is**

$$v_{\text{in}}(t) = 4\sin\left(\omega_s t + \frac{2\pi}{3}\right) \tag{123}$$

where $\omega_s = 1 \times 10^4 \,\frac{\text{rad}}{\text{s}}$, **what is the approximate value of** $v_{\text{out}}(t)$ **based on the Bode plots**? *Since the original transfer function is not provided, you cannot numerically compute the exact values of magnitude and phase. Just read the approximate values from the Bode plot.*



Solution: We first need to represent our input signal in phasor form. Recognize that we must first rewrite the sinusoidal transient function in cosine form:

$$v_{\text{in}}(t) = 4\sin\left(10^4 t + \frac{2\pi}{3}\right) = 4\cos\left(10^4 t + \frac{2\pi}{3} - \frac{\pi}{2}\right) = 4\cos\left(10^4 t + \frac{\pi}{6}\right) \tag{124}$$

We can then write the input phasor as:

$$\widetilde{V}_{\text{in}} = 2e^{j\frac{\pi}{6}}. \tag{125}$$

Alternatively, we can convert our original sine function to a phasor using Euler's formula:

$$v_{\text{in}}(t) = 4\sin\left(10^4 t + \frac{2\pi}{3}\right) \tag{126}$$

$$= 4 \cdot \frac{1}{2j}\left(e^{j\left(10^4 t + \frac{2\pi}{3}\right)} - e^{-j\left(10^4 t + \frac{2\pi}{3}\right)}\right) \tag{127}$$

$$= 2(-j)\left(e^{j10^4 t}e^{j\frac{2\pi}{3}} - e^{-j10^4 t}e^{-j\frac{2\pi}{3}}\right) \tag{128}$$

$$= 2e^{-j\frac{\pi}{2}}\left(e^{j10^4 t}e^{j\frac{2\pi}{3}} - e^{-j10^4 t}e^{-j\frac{2\pi}{3}}\right) \tag{129}$$

$$= 2e^{-j\frac{\pi}{2}}\left(e^{j10^4 t}e^{j\frac{2\pi}{3}} + (-1)e^{-j10^4 t}e^{-j\frac{2\pi}{3}}\right) \tag{130}$$

$$= 2e^{-j\frac{\pi}{2}}\left(e^{j10^4t}e^{j\frac{2\pi}{3}} + e^{j\pi}e^{-j10^4t}e^{-j\frac{2\pi}{3}}\right) \tag{131}$$
$$= 2\left(e^{j10^4t}e^{j(\frac{2\pi}{3}-\frac{\pi}{2})} + e^{-j10^4t}e^{j(-\frac{2\pi}{3}+\pi-\frac{\pi}{2})}\right) \tag{132}$$
$$= 2\left(e^{j10^4t}e^{j(\frac{2\pi}{3}-\frac{\pi}{2})} + e^{-j10^4t}e^{j(-\frac{2\pi}{3}+\frac{\pi}{2})}\right) \tag{133}$$
$$= 2\left(e^{j10^4t}e^{j(\frac{2\pi}{3}-\frac{\pi}{2})} + e^{-j10^4t}e^{-j(\frac{2\pi}{3}-\frac{\pi}{2})}\right) \tag{134}$$
$$= 2\left(e^{j10^4t}e^{j\frac{\pi}{6}} + e^{-j10^4t}e^{-j\frac{\pi}{6}}\right) \tag{135}$$

which then yields a phasor representation:

$$\widetilde{V}_{\text{in}} = 2e^{j\frac{\pi}{6}} \tag{136}$$

as before.

Next, we need to find the phasor representation of our transfer function, $H(j\omega)$, at the frequency of interest, $\omega = 10^4$ rad/s. The Magnitude Bode Plot reveals that at $\omega = 10^4$ rad/s, the value is $|H(j\omega)| = 10^{-1} = 0.1$. The Phase Bode Plot reveals that at $\omega = 10^4$ rad/s, the value is $\angle H(j\omega) = -\pi$ radians = $-180°$. Therefore, we can write the transfer function as a phasor:

$$H(j\omega) = 0.1e^{-j\pi}. \tag{137}$$

We know that in the phasor domain, the output voltage phasor is the input voltage phasor multiplied by the transfer function phasor. That is:

$$\widetilde{V}_{\text{out}} = H(j\omega) \cdot \widetilde{V}_{\text{in}} \tag{138}$$
$$= |H(j\omega)|e^{j\angle H(j\omega)} \cdot |\widetilde{V}_{\text{in}}|e^{j\angle \widetilde{V}_{\text{in}}} \tag{139}$$
$$= |H(j\omega)||\widetilde{V}_{\text{in}}|e^{j(\angle H(j\omega)+\angle \widetilde{V}_{\text{in}})} \tag{140}$$

In other words,

$$|\widetilde{V}_{\text{out}}| = |H(j\omega)||\widetilde{V}_{\text{in}}| \tag{141}$$
$$= 0.1 \cdot 2 \tag{142}$$
$$= 0.2 \tag{143}$$
$$\angle \widetilde{V}_{\text{out}} = \angle H(j\omega) + \angle \widetilde{V}_{\text{in}} \tag{144}$$
$$= -\pi + \frac{\pi}{6} \tag{145}$$
$$= -\frac{5\pi}{6} \tag{146}$$

Therefore our output phasor is:

$$\widetilde{V}_{\text{out}} = 0.2e^{-j\frac{5\pi}{6}}. \tag{147}$$

We can then convert $\widetilde{V}_{\text{out}}$ from the phasor domain to the time domain to arrive at our desired solution:

$$v_{\text{out}}(t) = 2|\widetilde{V}_{\text{out}}|\cos\left(\omega_s t + \angle \widetilde{V}_{\text{out}}\right) \tag{148}$$
$$= 0.4\cos\left(10^4 t - \frac{5\pi}{6}\right). \tag{149}$$

(d) Assume that the overall transfer function of a new filter, $H(j\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}}$, is given by

$$H(j\omega) = \left(\frac{1}{1 + j\frac{\omega}{\omega_{c1}}} + \frac{j\frac{\omega}{\omega_{c2}}}{1 + j\frac{\omega}{\omega_{c2}}} \right), \tag{150}$$

where $\omega_{c2} = 100\omega_{c1}$. **Qualitatively describe the magnitude of the transfer function $|H(j\omega)|$ in three regions: frequencies below ω_{c1}, frequencies between ω_{c1} and ω_{c2}, and frequencies above ω_{c2}. Identify the filter type by explaining what it is doing qualitatively** (for example, a low-pass filter passes low frequencies but does not pass high frequencies).

Solution: We can qualitatively analyze the behavior of the transfer function by evaluating the transfer function at $\omega \to 0$, $\omega = 10\,\omega_{c1}$, and $\omega \to \infty$.

Note that the first term of $H(j\omega)$ is a low pass filter, and we can denote it as $H_{LPF}(j\omega)$. Similarly, the second term of $H(j\omega)$ is a high pass filter and we can denote it as $H_{HPF}(j\omega)$, so

$$H(j\omega) = H_{LPF}(j\omega) + H_{HPF}(j\omega) \tag{151}$$

When $\omega \to 0$, we know that the LPF will be approximately 1 and the HPF will be approximately 0. Thus the overall sum $H(j\omega)$ will be approximately 1 so the magnitude will be about 1.

When $\omega \to \infty$, we know the LPF will be approximately 0 and the HPF will be approximately 1, so the overall sum and magnitude is still approximately 1.

Finally when $\omega = 10\omega_{c1} = \frac{1}{10}\omega_{c2}$, we are above the cutoff of the low pass and below the cutoff of the high pass, meaning we are in the attentuation region of both filters and so both filters will be much less than 1. Numerically,

$$H_{LPF}(j10\omega_{c1}) = \frac{1}{1+10j} \implies |H_{LPF}(j10\omega_{c1})| \ll 1 \tag{152}$$

$$H_{HPF}(j\frac{1}{10}\omega_{c2}) = \frac{.1j}{1+.1j} \implies |H_{HPF}(j\frac{1}{10}\omega_{c2})| \ll 1 \tag{153}$$

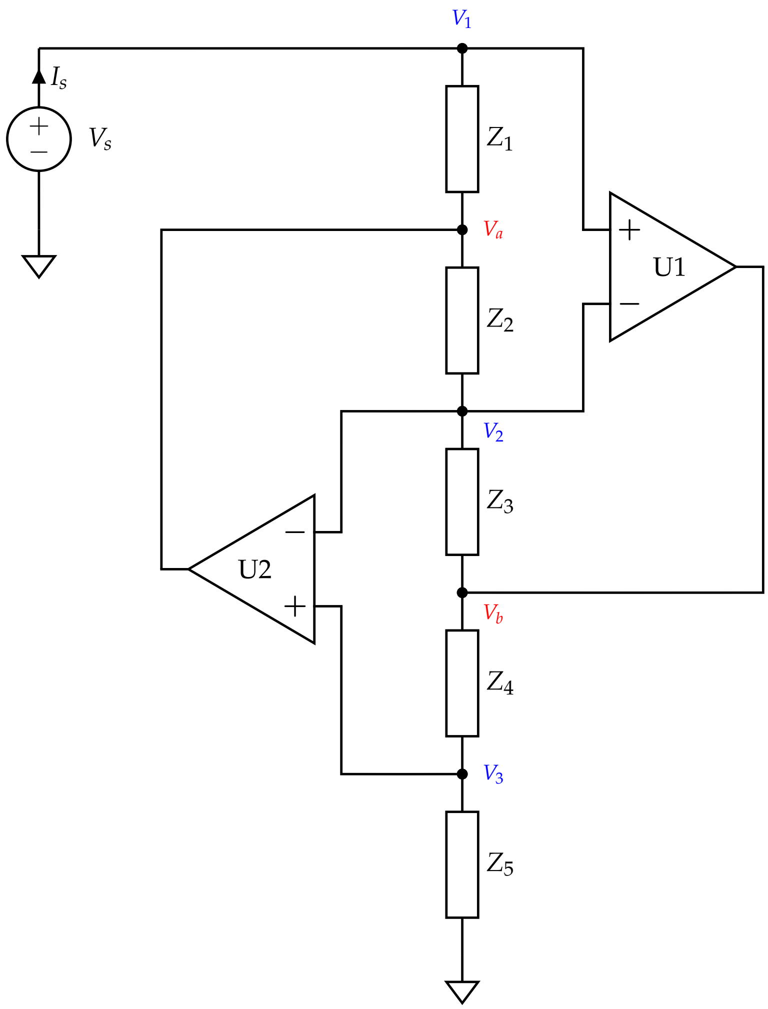
and so the total magnitude $|H(j\omega)| \ll 1$ as well.

Therefore, this filter attenuates the frequencies between ω_{c1} and ω_{c2} and passes frequencies outside of this range. This can be thought of as the opposite of a band-pass filter, and is commonly called a *band-stop* filter. (You don't need to know the name for credit, just how the filter acts on the various frequency ranges).

7. Generalized Impedance Converter

"Active inductors" are circuits that make a capacitor act like an inductor with the help of active devices such as transistors and op amps. This can be advantageous when the circuit requires inductors but are significantly larger and non-ideal compared to capacitors and other elements. The tradeoff is that the active devices consume power, but this may be an acceptable design tradeoff. There are many ways of building an an active inductor. We are going to analyze one example: the generalized impedance converter (also known as an *Antoniou Gyrator*).

The schematic of a generalized impedance converter is shown below. Consider the circuit in the phasor domain. All the voltages and currents in the problem are phasors and all the Z_i are impedances.



(a) Treat all the opamps as being in negative feedback and therefore following the Golden Rules. **What are the voltages at V_1, V_2, and V_3 in terms of V_s?**

Solution: Since we are told the op amps are in negative feedback, the golden rules apply. Specifically, the positive and negative input terminals form a virtual short, i.e. $V_+ = V_-$ for both op amps. Therefore:

$$V_1 = V_2 = V_3 = V_s \tag{154}$$

(b) **Express I_s in terms of V_s, V_a, Z_1.**

Solution:

$$I_s = \frac{V_s - V_a}{Z_1} \tag{155}$$

(c) The input impedance seen by the source looking into the circuit is defined as $Z_{in} = \frac{V_s}{I_s}$. Note that this is true because there are no independent sources in the rest of the circuit (op amps are dependent sources). **Find Z_{in} in terms of Z_1, Z_2, Z_3, Z_4, Z_5.**

Solution: (Long version:)
We can first find V_b in terms of V_s, Z_4 and Z_5 as follows:

$$\frac{V_b - V_3}{Z_4} = \frac{V_3}{Z_5} \tag{156}$$

$$\frac{V_b}{Z_4} = V_3\left(\frac{1}{Z_5} + \frac{1}{Z_4}\right) = V_s\left(\frac{1}{Z_5} + \frac{1}{Z_4}\right) \tag{157}$$

$$V_b = V_s\left(1 + \frac{Z_4}{Z_5}\right) \tag{158}$$

Once we express V_b in terms of V_s, Z_4 and Z_5, we can express V_a in terms of Z_2, Z_3, Z_4, Z_5 as follows:

$$\frac{V_a - V_2}{Z_2} = \frac{V_2 - V_b}{Z_3} \tag{159}$$

$$\text{LHS: } \frac{V_a - V_2}{Z_2} = \frac{V_a - V_s}{Z_2} \tag{160}$$

$$\text{RHS: } \frac{V_2 - V_b}{Z_3} = \frac{V_s - V_b}{Z_3} = \frac{V_s}{Z_3}\left(-\frac{Z_4}{Z_5}\right) \tag{161}$$

$$V_a = V_s\left(1 - \frac{Z_2 \cdot Z_4}{Z_3 \cdot Z_5}\right) \tag{162}$$

Using the answer from part (b), we get:

$$I_s = \frac{V_s - V_a}{Z_1} = V_s\left(\frac{Z_2 \cdot Z_4}{Z_1 \cdot Z_3 \cdot Z_5}\right) \tag{163}$$

$$\implies Z_{in} = \frac{V_s}{I_s} = \frac{Z_1 \cdot Z_3 \cdot Z_5}{Z_2 \cdot Z_4} \tag{164}$$

(Short version:)
Since $V_1 = V_2 = V_3 = V_s$, we do know that

$$\frac{V_b - V_s}{Z_4} = \frac{1}{Z_5}V_s \quad \rightarrow \quad V_b - V_s = \frac{Z_4}{Z_5}V_s \tag{165}$$

$$\frac{V_a - V_s}{Z_2} = \frac{V_s - V_b}{Z_3} = \frac{-Z_4}{Z_3 \cdot Z_5}V_s \rightarrow V_a - V_s = \frac{-Z_2 \cdot Z_4}{Z_3 \cdot Z_5}V_s \tag{166}$$

$$I_s = \frac{V_s - V_a}{Z_1} = \frac{Z_2 \cdot Z_4}{Z_1 \cdot Z_3 \cdot Z_5}V_s \tag{167}$$

$$Z_{in} = \frac{V_s}{I_s} = \frac{V_s}{\left(\frac{Z_2 \cdot Z_4}{Z_1 \cdot Z_3 \cdot Z_5}V_s\right)} \tag{168}$$

$$\therefore Z_{in} = \frac{Z_1 \cdot Z_3 \cdot Z_5}{Z_2 \cdot Z_4} \tag{169}$$

(d) Assume the following:

$$Z_1 = R_1 \tag{170}$$

$$Z_2 = \frac{1}{j\omega C_2} \tag{171}$$

$$Z_3 = R_3 \tag{172}$$

$$Z_4 = R_4 \tag{173}$$

$$Z_5 = R_5 \tag{174}$$

Evaluate Z_{in} for the above case.

Solution:

$$Z_{in} = \frac{Z_1 \cdot Z_3 \cdot Z_5}{Z_2 \cdot Z_4} = \frac{R_1 \cdot R_3 \cdot R_5}{\frac{1}{j\omega C_2 \cdot R_4}} = j\omega\left(\frac{R_1 \cdot R_3 \cdot R_5 \cdot C_2}{R_4}\right) \tag{175}$$

(e) You should have found that Z_{in} is inductive, i.e. $Z_{in} = j\omega L_{eq}$ where L_{eq} is the equivalent inductance. **What is L_{eq} in terms of R_1, C_2, R_3, R_4, and R_5?**

Solution: We recognize that Z_{in} is indeed inductive since it is a positive imaginary number like of the form of $j\omega L_{eq}$. We then get $L_{eq} = \frac{R_1 \cdot R_3 \cdot R_5 \cdot C_2}{R_4}$.

8. (OPTIONAL) Make Your Own Problem.

Write your own problem about content covered in the course thus far, and provide a thorough solution to it.

NOTE: This can be a totally new problem, a modification on an existing problem, or a Jupyter part for a problem that previously didn't have one. Please cite all sources for anything (including course material) that you used as inspiration.

NOTE: High-quality problems may be used as inspiration for the problems we choose to put on future homeworks or exams.

9. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!

We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) **What sources (if any) did you use as you worked through the homework?**

(b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

(c) **Roughly how many total hours did you work on this homework? Write it down here where you'll need to remember it for the self-grade form.**

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