Homework 6

This homework is due on Friday, March 4, 2022, at 11:59PM. Self-grades and HW Resubmissions are due on the following Friday, March 11, 2022, at 11:59PM.

1. **Survey**

   Please fill out the following online surveys.

   (a) **Mask survey(link)**: We want to understand student sentiment about the mask mandate lifting on 03/07 and how this may affect them.
   NOTE: We STRONGLY RECOMMEND that students continue wearing their masks after the mandate is lifted - please be mindful and empathetic that those around you may not necessarily be comfortable if you are maskless.

   (b) **(OPTIONAL) Note Feedback(link)**: This survey is totally anonymous and made with the intent to improve our notes for students (especially for future semesters).

2. **Reading Lecture Notes**

Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: Note 7, Note 8, Note 9, Note 10, and Note 11.

(a) In the magnitude Bode plot for a first-order-low-pass filter drawn on a log-log graph (i.e., $|H(\mathrm{j}\omega)|$ is plotted on a log scale), **what is the slope of the straight line approximation at frequencies higher than the cut-off frequency? What is the slope for a third-order-low-pass filter (three identical first-order-low-pass filters cascaded with unity gain buffers)?**

Solution: For frequency $\omega \gg \omega_o$, where ω_o is the cut-off frequency, the magnitude of the transfer function of a first-order-low-pass filter is approximated as

$$|H_1(\mathrm{j}\omega)| = \frac{\omega_o}{\omega} \tag{1}$$

$$\log|H_1(\mathrm{j}\omega)| = \log(\omega_o) - \log(\omega) \tag{2}$$

Hence the slope is -1.

For a third-order-low-pass filter,

$$|H_3(\mathrm{j}\omega)| = \left(\frac{\omega_o}{\omega}\right)^3 \tag{3}$$

$$\log|H_3(j\omega)| = 3\log(\omega_o) - 3\log(\omega) \tag{4}$$

Hence the slope is -3.

If the question is misread and the Bode plot on dB scale with $20\log(H(\mathrm{j}\omega))$ is used instead, then the slopes will be -20 for first-order and -60 for third-order.

(b) If you know a filter has an attenuation of $|H(j\omega_0)| = 0.95$ at ω_0, **what would the attenuation or gain be when you cascade 10 of these filters at ω_0?** If you know this same filter has a gain of $|H(j\omega_1)| = 1.1$ at some different angular frequency ω_1, **what would the gain be when you cascade 10 of these filters at ω_1?**

Solution: With a cascade of 10 copies of the first filter, we should expect to see at ω_0 an attenuation of $|H(j\omega_0)|^{10} = 0.95^{10} \approx 0.60$. With a cascade of 10 copies of the second filter, we should see a gain of $|H(j\omega_1)|^{10} = 1.1^{10} \approx 2.60$.

While you were not asked for this, consider that the significance of this effect is that cascading of filters repeatedly matters even for regions that are close to outputting a magnitude of ~ 1. We see a similar phenomenon in discrete time stability where powers of numbers less than 1 or greater than 1 can shrink or grow and thus must be considered in design.

(c) Suppose we had a continuous-time differential equation model

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}_c(t) = A_c\vec{x}_c(t) + B_c\vec{u}_c(t) \tag{5}$$

and we wanted to discretize it to get a model of the form

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + B_d\vec{u}_d[i] \tag{6}$$

where $\vec{x}_d[i] = \vec{x}_c(i\Delta)$ and the input $\vec{u}_c(t)$ is generated to be piecewise constant so $\vec{u}_d[i]$ is the value of $\vec{u}_c(t)$ for the whole time interval $t \in [i\Delta, (i+1)\Delta)$.

Describe the steps that you would take to get A_d and B_d from A_c and B_c. Feel free to assume that A_c is diagonalizable.

Solution: The first conceptual step is to solve the matrix/vector differential equation via diagonalization. With the assumption that A_c is diagonalizable, we can change coordinates such that $\vec{x}_c(t) = V\vec{y}_c(t)$, where V is the matrix containing the eigenvectors of A_c. By writing the differential equation in the $\vec{y}_c(t)$ coordinates, we will get a set of scalar differential equations $\frac{\mathrm{d}}{\mathrm{d}t}y_{c,k}(t) = \lambda_k y_{c,k}(t) + (V^{-1}B_c\vec{u}_c(t))_k$. We know the form of this solution in the scalar case:

$y_{c,k}(t) = e^{\lambda_k\Delta(t-t_0)}y_{c,k}(t_0) + \int_{t_0}^{t} e^{\lambda_k(t-\theta)}(V^{-1}B_c\vec{u}_c(\theta))_k d\theta$. Then, to convert to discrete time, we set $t_0 = i\Delta$ and $t = (i+1)\Delta$, and using the fact that $\vec{u}_c(t)$ is constant over this interval:

$$y_{d,k}[i+1] = e^{\lambda_k\Delta}y_{d,k}[i] + \frac{e^{\lambda_k\Delta}-1}{\lambda_k}(V^{-1}B_c\vec{u}_d[i])_k$$

Then, we stack these solutions back into matrix/vector form (using definitions of $e^{\Lambda\Delta}$ and Λ as in Discussion 6A.):

$$\vec{y}_d[i+1] = e^{\Lambda\Delta}\vec{y}_d[i] + \Lambda^{-1}(e^{\Lambda\Delta}-I)V^{-1}B_c\vec{u}_d[i]$$

and convert back to our original coordinates:

$$\vec{x}_d[i+1] = Ve^{\Lambda\Delta}V^{-1}\vec{x}_d[i] + V\Lambda^{-1}(e^{\Lambda\Delta}-I)V^{-1}B_c\vec{u}_d[i]$$

Then $A_d = Ve^{\Lambda\Delta}V^{-1}$ and $B_d = V\Lambda^{-1}(e^{\Lambda\Delta}-I)V^{-1}B_c$. For a detailed solution, see Problem 1 in Discussion 6A.

(d) **What relevance does the following property of matrix multiplication have to system identification of vector systems by means of least-squares?**

$$AB = A\begin{bmatrix} | & & | \\ \vec{b}_1 & \cdots & \vec{b}_N \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ A\vec{b}_1 & \cdots & A\vec{b}_N \\ | & & | \end{bmatrix} = C = \begin{bmatrix} | & & | \\ \vec{c}_1 & \cdots & \vec{c}_N \\ | & & | \end{bmatrix}$$

Solution: Our system identification problem amounts to the following: we wish to solve for P in $DP \approx S$, where P is a matrix unknown of parameters and D and S are known matrices of data. Matrix multiplication applying column-wise and being able to state that $A\vec{b}_i = \vec{c}_i$ implies that in our system identification problem, we have $D\vec{p}_i = \vec{s}_i$ where $\vec{p}_i$ and $\vec{s}_i$ are columns of P and S respectively. This allows us to use least squares separately for each column pair, and that the matrix unknown problem is solvable with the tools we have. If we use least squares on all column pairs, $(\vec{p}_i, \vec{s}_i)$, separately and regroup our data, using the same matrix multiplication property, system identification amounts to a single multiplication: $(D^\top D)^{-1}D^\top S = \widehat{P}$.

(e) **What is a condition for a discrete time scalar system to be BIBO stable? What is a condition on the eigenvalues for a discrete time vector system to be BIBO stable? What is a condition on the eigenvalues for a continuous time system to be BIBO stable?**

Solution: A discrete time scalar system is stable if the factor λ in $x[t+1] = \lambda x[t]$ satisfies $|\lambda| < 1$. Similarly, for discrete time vector systems is stable if the matrix A in $\vec{x}[t+1] = A\vec{x}[t]$ has all of its eigenvalues $|\lambda| < 1$. Finally, a continuous time vector system is stable if the matrix A in $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t)$ has all of its eigenvalues satisfying $\mathrm{Re}\{\lambda\} < 0$.

3. Bandpass Filter: Lowpass and Highpass Cascade

In lecture, you heard about how you can go through the design of a bandpass filter by cascading lowpass and highpass filters via buffers (op-amps in unity-gain negative feedback to prevent loading effects). In this problem, you will do this for yourself.

Consider an input signal that is composed of the superposition of:

- $A_p := 20\,\text{mV}$ level pure tone at frequency $f_p := 60\,\text{Hz}$ and phase ϕ_p corresponding to power line noise.
- $A_v := 1\,\text{mV}$ level pure tone at frequency $f_v := 600\,\text{Hz}$ and phase ϕ_v corresponding to a voice signal.
- $A_f := 10\,\text{mV}$ level pure tone at frequency $f_f := 60\,\text{kHz}$ and phase ϕ_f corresponding to fluorescent light control electronics noise.

We would like to keep the 600 Hz tone, which could correspond to a voice signal.

NOTE: The phases ϕ are symbolic – we do not provide numerical values – but the amplitudes A are not symbolic.

(a) **Write the $V_{\text{in}}(t)$ that describes the above input in time domain, in the following format:**.

$$V_{\text{in}}(t) = A_p \cos(2\pi f_p t + \phi_p) + A_v \cos(2\pi f_v t + \phi_v) + A_f \cos\left(2\pi f_f t + \phi_f\right) \tag{7}$$

Solution: Each pure tone in the input signal corresponds to a cosine wave. All of these tones can be added up:

$$\begin{aligned} V_{\text{in}}(t) &= (20 \times 10^{-3}) \cos((2\pi \cdot 60)t + \phi_p) &(8)\\ &+ (1 \times 10^{-3}) \cos((2\pi \cdot 600)t + \phi_v) &(9)\\ &+ (10 \times 10^{-3}) \cos\left((2\pi \cdot 60000)t + \phi_f\right) &(10)\end{aligned}$$

NOTE: Unfortunately, earlier in the week, there was an error where we previously asked you to ignore the phases. If you did so, give yourself full points in the self-grade.

(b) **What are the angular frequencies (i.e., $\omega_p, \omega_v, \omega_f$) involved and the phasors associated with each tone?** Remember that the frequencies of the tones are provided in Hz. To convert these frequencies to angular frequencies, we use $\omega = 2\pi f$.

NOTE: This scenario is common in applications; usually, the data collected is in "regular" frequencies, but the analysis requires angular frequencies.

Solution:

$$\text{Power line:} \quad 376.99\,\frac{\text{rad}}{\text{s}}$$
$$\text{Voice:} \quad 3769.9\,\frac{\text{rad}}{\text{s}}$$
$$\text{Fluorescent light:} \quad 376\,991.1\,\frac{\text{rad}}{\text{s}}$$

Recall that the phasor is defined as the term in front of $e^{j\omega t}$ in the complex exponential expansion, and that $V\cos(\omega t) = \frac{V}{2}e^{j\omega t} + \frac{V}{2}e^{-j\omega t}$.

$$\text{Power line:} \quad \widetilde{V}_{\text{in},p} = \frac{1}{2}(20 \times 10^{-3})e^{j\phi_p} = 10e^{j\phi_p} \times 10^{-3}\text{V}$$
$$\text{Voice:} \quad \widetilde{V}_{\text{in},v} = \frac{1}{2}(1 \times 10^{-3})e^{j\phi_v} = 5e^{j\phi_v} \times 10^{-4}\text{V}$$
$$\text{Fluorescent light:} \quad \widetilde{V}_{\text{in},f} = \frac{1}{2}(10 \times 10^{-3})e^{j\phi_f} = 5e^{j\phi_f} \times 10^{-3}\text{V}$$

Note that from this part and beyond, we will be computing many values for $\widetilde{V}_{\text{in}}, H(j\omega), \omega_p, \omega_v, \omega_f$, etc.. When self grading, as long as your calculations are correct, do not mark yourself down for how you round if it is different from what is here in the solutions, as long as your solutions are consistent with your previous answers.

(c) To achieve your goal of keeping the voice tone but rejecting the noise from the power-lines and fluorescent lights, **at what frequency do you want to have the cutoff frequency for the lowpass filters**?

(HINT: To arrive at a unique solution consider computing the geometric mean (the analogous quantity to the arithmetic mean on a log scale) of the two frequencies of interest.)

Solution: There are a wide range of possible cutoff-frequencies you can choose for the lowpass filter. You can choose any value in between the voice frequency and the fluorescent light frequency and get some filtering (and get credit). Your specific choice depends on the tradeoffs you may want to take. For example, if you choose a cutoff frequency close to the voice frequency, you will suppress the fluorescent light frequency well, but will also start to attenuate the voice frequency a little. If you choose a cutoff frequency close to the fluorescent light frequency, you will minimize attenuation of the voice frequency, but also have very little suppression of the fluorescent light frequency. Choosing this value in a principled way depends on more contextual information.

One reasonable solution (as was mentioned by the hint) is to choose a frequency that is the geometric mean of the voice and fluorescent frequencies. The geometric mean puts the cutoff frequency half way between the voice and fluorescent frequencies on a log scale:

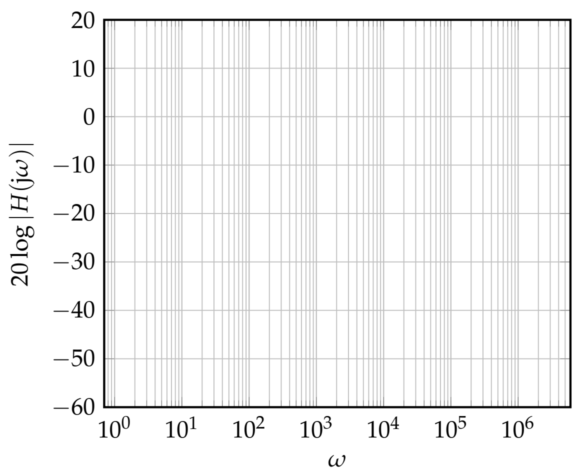
$$f_c = \sqrt{f_v \cdot f_f} = \sqrt{600 \cdot 60000}\,\mathrm{Hz} = 6000\,\mathrm{Hz} \qquad \omega_c = 2\pi f_c = 2\pi \cdot 6000\,\frac{\mathrm{rad}}{\mathrm{s}} = 37\,699.1\,\frac{\mathrm{rad}}{\mathrm{s}} \tag{11}$$

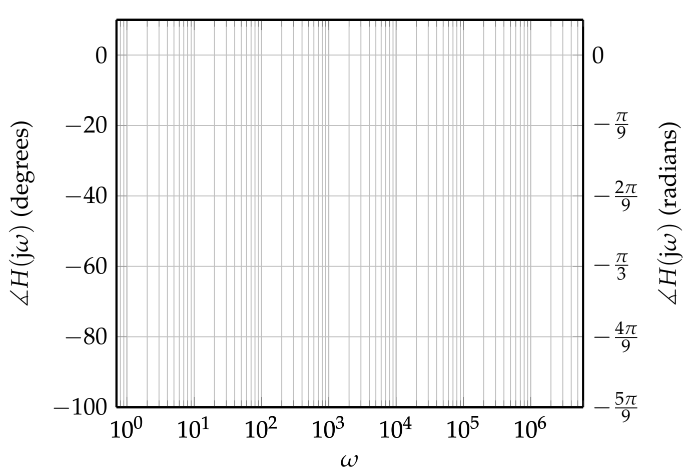
A second reasonable way to select a cutoff frequency is to note that since 600 Hz and 60 kHz are fairly far apart, we can simply choose a frequency 10x the voice frequency. This will minimize attenuation of the voice tone while still suppressing the fluorescent light tone:

$$f_c = f_v \cdot 10 = 6\,\mathrm{kHz} \qquad \omega_c = 2\pi \cdot f_c = 2\pi \cdot 6000\,\frac{\mathrm{rad}}{\mathrm{s}} = 37\,699.1\,\frac{\mathrm{rad}}{\mathrm{s}} \tag{12}$$

Coincidentally, this is the same frequency as the geometric mean approach for the numbers given here. In practice, in the absence of anything else, one uses the second approach (put the cutoff 10x away from the desirable frequencies) when the desirable and undesirable frequencies are very far apart, and the first approach (geometric means) when they are closer together.

(d) **Draw the Bode plot (straight-line approximations to the transfer function) for the magnitude (using $20\log|H(\mathrm{j}\omega)|$) and phase of the lowpass filter.**





Solution: We note that the y-axis is $20\log|H(\mathrm{j}\omega)|$. Therefore, when we hit a cut-off frequency, the role-off slope will be -20 dB instead of -1, which means that per each decade of movement on the ω-axis, our y-axis drops by 20 units.

The magnitude is given by $|H(\mathrm{j}\omega)| = \frac{1}{\sqrt{1+\frac{\omega^2}{\omega_c^2}}}$ where $\omega_c = 37\,699.1\,\frac{\mathrm{rad}}{\mathrm{s}}$. Hence we have the following properties for the magnitude Bode plot approximation:

i. At $\omega < \omega_c$, the magnitude is $20 \log(1) = 0\,\text{dB}$.
ii. At $\omega > \omega_c$, the magnitude drops by 20 dB per decade of increase in ω.

The phase is given by

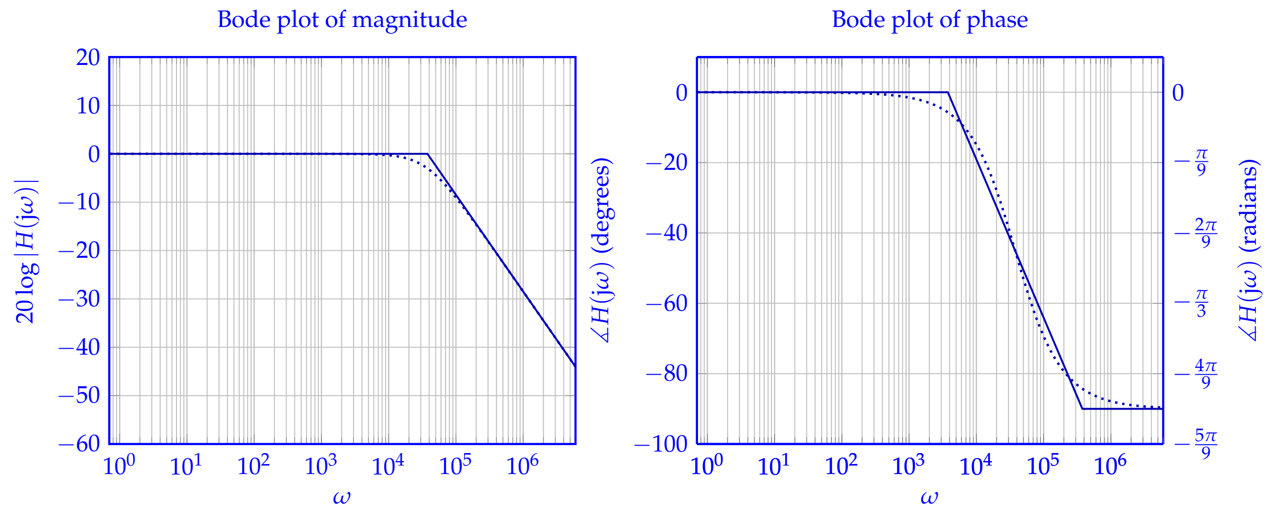
$$\measuredangle(H(\mathrm{j}\omega)) = \measuredangle(1) - \measuredangle\left(1 + \frac{\mathrm{j}\cdot\omega}{\omega_c}\right) \tag{13}$$

$$= 0^\circ - \operatorname{atan2}\left(\frac{\omega}{\omega_c}, 1\right) \tag{14}$$

Hence we have the following properties for the phase Bode plot:

i. At $\omega < \frac{\omega_c}{10}$, the phase is $-\operatorname{atan2}(0, 1) = 0^\circ$.
ii. At $\omega > 10\omega_c$, the phase is $-\operatorname{atan2}(\infty, 1) = -90^\circ$.
iii. At $\omega = \omega_c$, the phase is $-\operatorname{atan2}(1, 1) = -45^\circ$.

Hence the magnitude and phase Bode plots look as follows:



Note that the plots also have the accurate transfer function magnitude and phase in dotted lines. **This is not required by the problem** — it is there just to help you understand how good the Bode approximations are.

In practice, when hand-sketching such plots, one just draws the straight lines. When one has a computer, one can just plot the actual functions. Engineering is about doing whatever is easier but still lets you achieve your objectives.

(e) To achieve your goal of keeping the voice tone but rejecting the noise from the power-lines and fluorescent lights, **at what frequency do you want to have the cutoff frequency for the highpass filters**?

(HINT: To arrive at a unique solution consider computing the geometric mean (the analogous quantity to the arithmetic mean on a log scale) of the two frequencies of interest.)

Solution: There are a wide range of possible cutoff-frequencies you can choose for the highpass filter. The trade-offs are that if you choose a cutoff frequency close to the voice frequency, you will suppress the power line frequency well, but will also start to attenuate the voice frequency a little. If you choose a cutoff frequency close to the power line frequency, you will minimize attenuation of the voice frequency, but also have very little suppression of the power line frequency. Choosing this value in a principled way depends on more contextual information.

Since 60 Hz and 600 Hz are fairly close together, one reasonable way to select a cutoff frequency is to choose the geometric mean of the two frequencies (as was mentioned in the hint):

$$f_c = \sqrt{f_p f_v} = \sqrt{60 \cdot 600}\,\text{Hz} \approx 189.7\,\text{Hz} \qquad \omega_c = 2\pi f_c = 2\pi \cdot 189\,\frac{\text{rad}}{\text{s}} = 1187.5\,\frac{\text{rad}}{\text{s}} \tag{15}$$

(f) **Draw the Bode plot (straight-line approximations to the transfer function) for the magnitude (using $20\log|H(j\omega)|$) and phase of the highpass filter.**

Solution: Similar to the previous Bode plot, the slopes will be 20 dB instead of 1.

The magnitude is given by $|H(j\omega)| = \frac{1}{\sqrt{1+\frac{\omega_c^2}{\omega^2}}}$ where $\omega_c = 1187.5\ \frac{\text{rad}}{\text{s}}$. Hence we have the following properties for the magnitude Bode plot approximation:

i. At $\omega > \omega_c$, the magnitude is $20\log(1) = 0\,\text{dB}$.

ii. At $\omega < \omega_c$, the magnitude drops by 20 dB per decade of decrease in ω.

The phase is given by

$$\measuredangle(H(j\omega)) = \measuredangle(1) - \measuredangle\left(1 - \frac{j\cdot\omega_c}{\omega}\right) \tag{16}$$

$$= 0^\circ - \text{atan2}\left(-\frac{\omega_c}{\omega}, 1\right) \tag{17}$$

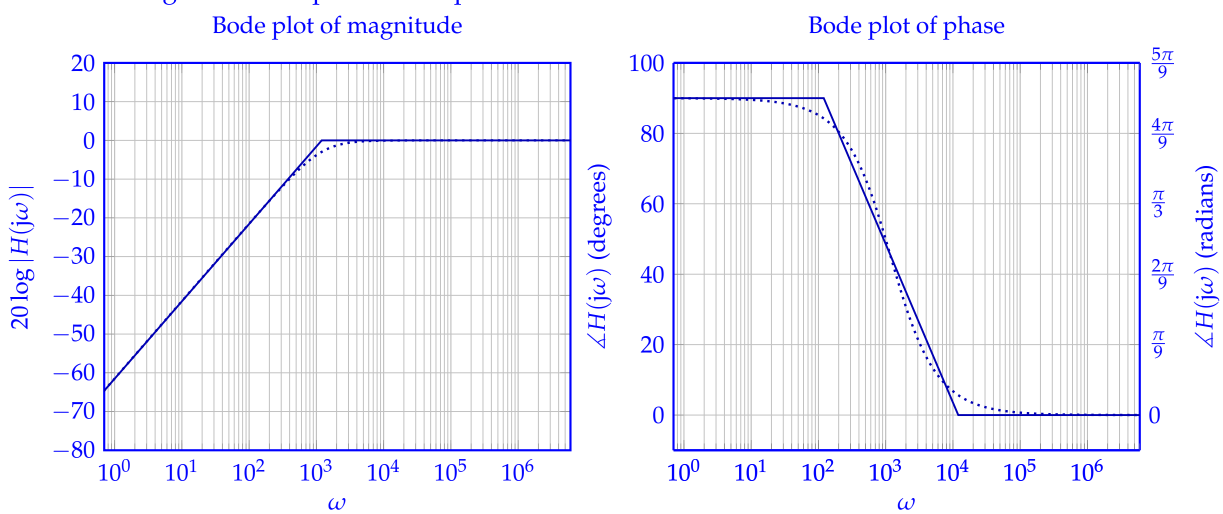
Hence we have the following properties for the phase Bode plot:

i. At $\omega < \frac{\omega_c}{10}$, the phase is $-\text{atan2}(-\infty, 1) = 90^\circ$.

ii. At $\omega > 10\omega_c$, the phase is $-\text{atan2}(0, 1) = 0^\circ$.

iii. At $\omega = \omega_c$, the phase is $-\text{atan2}(-1, 1) = 45^\circ$.

Hence the magnitude and phase Bode plots look as follows:



Note that the plots also have the accurate transfer function magnitude and phase in dotted lines. **This is not required by the problem** — it is there just to help you understand how good the Bode approximations are.

(g) For the following questions, assume your cut-off frequencies for lowpass and highpass are 6 kHz and 189 Hz respectively. Suppose that you only had 1 µF capacitors to use. **What resistance values would you choose for your highpass and lowpass filters so that they have the desired cutoff frequencies?**

Solution: For the highpass filter, the cutoff frequency is $f_c = \frac{1}{2\pi RC}$. Solving for R:

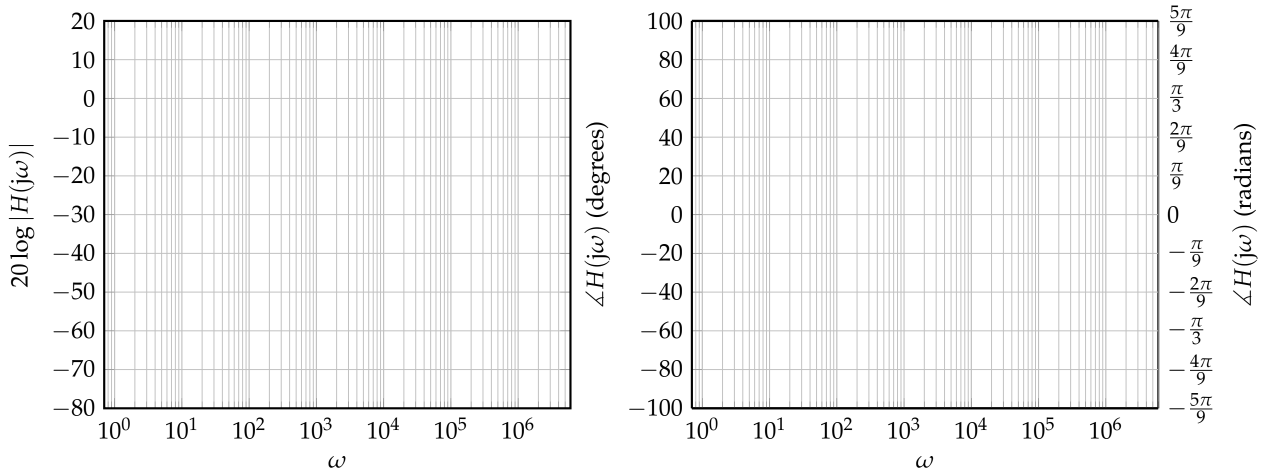
$$R = \frac{1}{2\pi f_c C} \tag{18}$$

In our case, we selected $f_c = 189\,\text{Hz}$, so $R_{\text{HP}} = \frac{1}{(1\times10^{-6})(2\pi\cdot189)} = 842\,\Omega$.

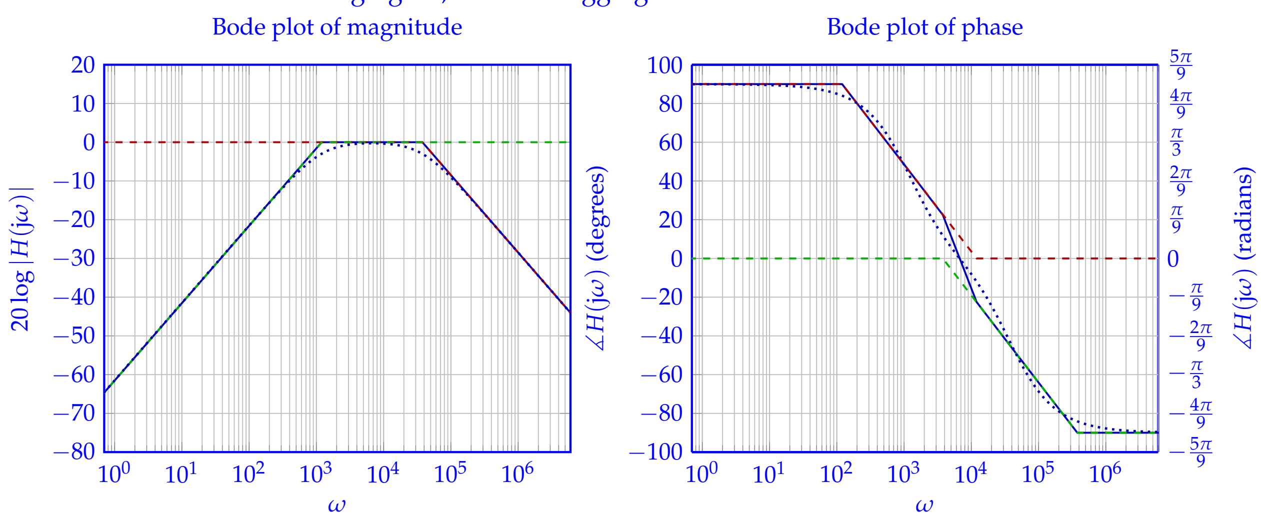
For the RC lowpass filter, the cutoff frequency expression is the same ($f_c = \frac{1}{2\pi RC}$), so $R_{\text{LP}} = \frac{1}{(1\times10^{-6})(2\pi\cdot6000)} = 26.5\,\Omega$.

(h) The overall bandpass filter that is created by cascading the lowpass and highpass with ideal buffers in between. **Draw the Bode plot (straight-line approximations to the transfer function) for the magnitude and phase of the overall bandpass transfer function.**

(HINT: You should think about how the Bode plot of a cascade of two filters can be derived based on the Bode plots of the lower-level filters.)



Solution: Since the bandpass filter is a cascade of the lowpass and highpass filters, for the bandpass Bode plot we can add the two previous plots. The previous Bode plots are drawn in the dashed format in the following figure, and their aggregate is drawn in solid blue.



Note that the plots also have the accurate transfer function magnitude and phase in dotted lines. **This is not required by the problem —** it is there just to help you understand how good the Bode approximations are.

(i) Suppose that the bandpass filter does not have enough suppression at 60 Hz and 60 kHz. You decide to use a cascade of three bandpass filters (with unity-gain buffers in between) (as shown in Figures 1 and 2). **What are the phasors for each of the frequency tones after all three bandpass filters?**

(HINT: Remember how you determined the transfer function of the bandpass filter from the transfer functions of the lowpass and highpass filters.)

Feel free to use a computer to help you evaluate both the magnitudes and the phases here.

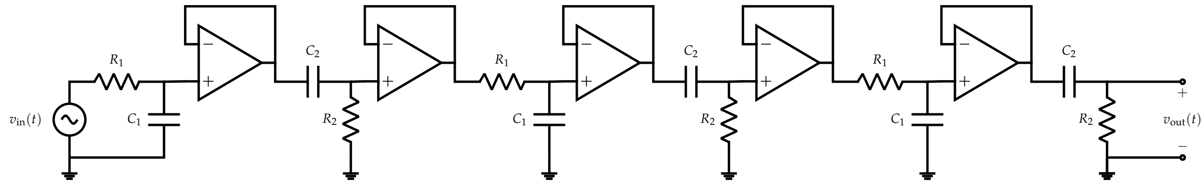


Figure 1: "Time-domain" circuit: Cascade of the three bandpass filters, using buffers to avoid loading.

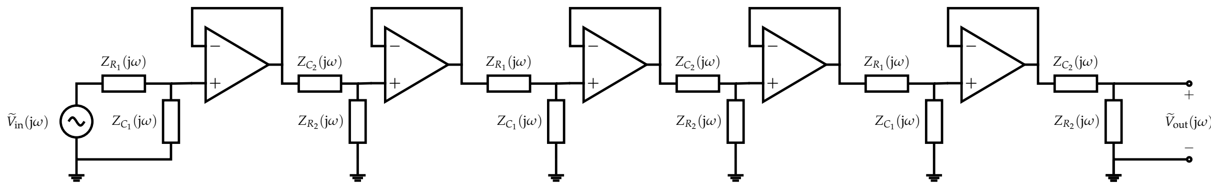


Figure 2: "Phasor-domain" circuit: Cascade of the three bandpass filters, using buffers to avoid loading.

Solution: Since each of the bandpass filters are isolated from each other with unity-gain buffers, the total transfer function is simply the single filter transfer function cubed:

$$H_{\text{total}}(\mathrm{j}\omega) = H_{\text{BP}}(\mathrm{j}\omega)^3 = \left(\frac{1}{1+\mathrm{j}\frac{\omega}{\omega_c^{\text{LP}}}} \times \frac{\mathrm{j}\frac{\omega}{\omega_c^{\text{HP}}}}{1+\mathrm{j}\frac{\omega}{\omega_c^{\text{HP}}}}\right)^3 \tag{19}$$

Remember that $H(\mathrm{j}\omega)$ can be expressed in polar form as well:

$$H_{\text{BP}}(\mathrm{j}\omega) = |H(\mathrm{j}\omega)|\mathrm{e}^{\mathrm{j}\measuredangle H(\mathrm{j}\omega)} \tag{20}$$

$$H_{\text{total}}(\mathrm{j}\omega) = |H(\mathrm{j}\omega)|^3\mathrm{e}^{\mathrm{j}3\measuredangle H(\mathrm{j}\omega)} \tag{21}$$

Plugging in our chosen cutoff frequency for the bandpass filter and evaluating the magnitude and phase of our transfer function with a computer:

$$\text{Power line:} \quad H_{\text{BP}}(\mathrm{j}\omega_p) = 0.30\mathrm{e}^{\mathrm{j}1.26}$$
$$\text{Voice:} \quad H_{\text{BP}}(\mathrm{j}\omega_v) = 0.95\mathrm{e}^{\mathrm{j}0.21}$$
$$\text{Fluorescent light:} \quad H_{\text{BP}}(\mathrm{j}\omega_f) = 0.099\mathrm{e}^{-\mathrm{j}1.48}$$

Then, the output phasors can be calculated from the total transfer function multiplied by the initial phasors for each frequency:

$$\widetilde{V}_{\text{out}} = H_{\text{total}}(\mathrm{j}\omega) \cdot \widetilde{V}_{\text{in}} \tag{22}$$

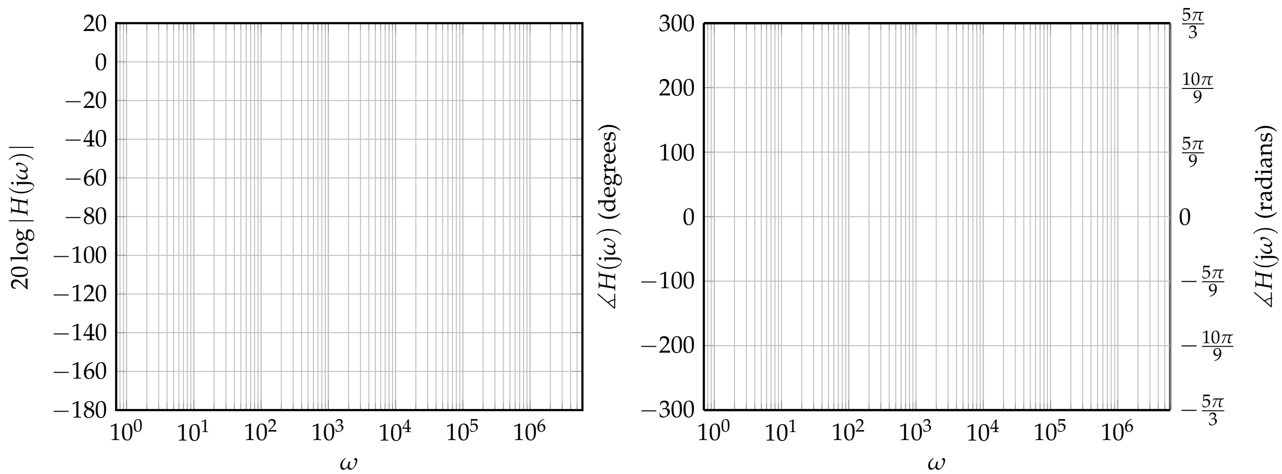
Power line:
$$\widetilde{V}_{\text{out},p} = H_{\text{BP}}(\mathrm{j}\omega_p)^3 \cdot \widetilde{V}_{\text{in},p} = 0.30^3\mathrm{e}^{\mathrm{j}3\cdot 1.26} \cdot 10\mathrm{e}^{\mathrm{j}\phi_p} \times 10^{-3} = 0.00027\mathrm{e}^{\mathrm{j}(3.76+\phi_p)}$$
Voice:
$$\widetilde{V}_{\text{out},v} = H_{\text{BP}}(\mathrm{j}\omega_v)^3 \cdot \widetilde{V}_{\text{in},v} = 0.95^3\mathrm{e}^{\mathrm{j}3\cdot 0.21} \cdot 5\mathrm{e}^{\mathrm{j}\phi_v} \times 10^{-4} = 0.00043\mathrm{e}^{\mathrm{j}(0.62+\phi_v)}$$
Fluorescent light:
$$\widetilde{V}_{\text{out},f} = H_{\text{BP}}(\mathrm{j}\omega_f)^3 \cdot \widetilde{V}_{\text{in},f} = 0.099^3\mathrm{e}^{-\mathrm{j}3\cdot 1.48} \cdot 5\mathrm{e}^{\mathrm{j}\phi_f} \times 10^{-3} = 4.9 \times 10^{-6}\mathrm{e}^{-\mathrm{j}(4.40-\phi_f)}$$

Compared to the original signals, we see that the voice signal has not been significantly attenuated, while the power line and fluorescent light noise have been significantly attenuated.

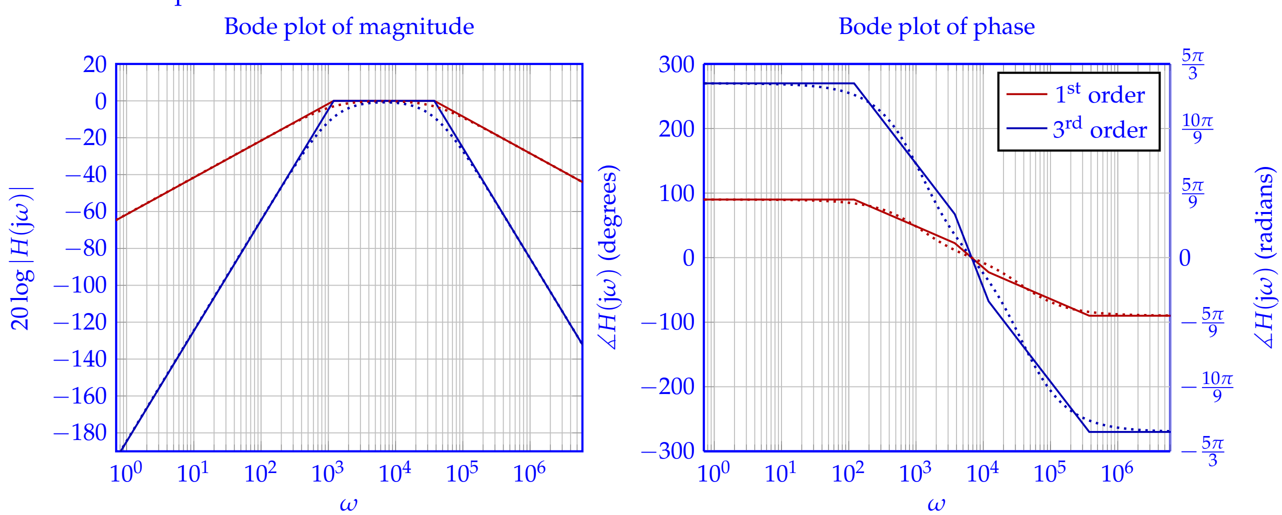
Sidenote: If you directly calculate the phase of $H_{\text{total}}(\mathrm{j}\omega)$ at ω_p or ω_f, you may notice that you get a different answer (in fact, your answer and the answer here will be different by a multiple of 2π). This is due to *phase wrapping*, which constrains phase offsets to a range of $(-\pi, \pi]$ radians.

Phase is often constrained to this range because of convention — there are infintely many phase offsets that correspond to the same signal (any multiple of 2π added to a given phase results in the same signal), so we typically choose the one within this range. However, in this solution, we leave the phase unwrapped so as to avoid confusion.

(j) **Draw the Bode plots (straight-line approximations to the transfer function) for the magnitude and phase of the 3^{rd} order bandpass filter.** To highlight the difference between the 3^{rd} and 1^{st} order filters, please draw both Bode plots on a single figure.



Solution: A bandpass with order $N = 3$ is just the cascade of 3 bandpass filters. Therefore, we can plot the original bandpass filter plot and multiply each point by 3 to get the Bode plot for the 3^{rd}-order bandpass filter.



Note that the plots also have the accurate transfer function magnitude and phase in dotted lines. **This is not required by the problem** — it is there just to help you understand how good the Bode approximations are.

(k) **Write the final time domain voltage waveform that would be present after the filter.**

Solution: From the previous part, we have phasors for each tone at the output of the filter. We can simply take each of these phasors, and convert them back to the time domain.

$$
\begin{aligned}
V_{\text{out}}(t) &= 0.00055\cos\big((2\pi \cdot 60)t + 3.77 + \phi_p\big) && (23)\\
&\quad + 0.00085\cos\big((2\pi \cdot 600)t + 0.62 + \phi_v\big) && (24)
\end{aligned}
$$

$$+9.85 \times 10^{-6} \cos\big((2\pi \cdot 60000)t + \phi_p - 4.40\big) \tag{25}$$

(l) The included Jupyter notebook `filter_cascade.ipynb` sets up the same problem described above. In the notebook, you can use the slider bars to play around with:

- highpass cutoff frequency f_{highpass} (i.e., the knee frequency of the highpass filters)
- lowpass cutoff frequency f_{lowpass} (i.e., the knee frequency of the lowpass filters)
- Filter order N. Filter order means the number of lowpass filters and highpass filters that are used in a row. Here, N means that there are N lowpass filters and N highpass filters, so the overall order of the entire filter is actually $2N$.

The notebook will plot the magnitude and phase, the input voltage waveform, and the output waveform at the end of the filter.

Play around with the values for the highpass and lowpass cutoff frequencies, and N.

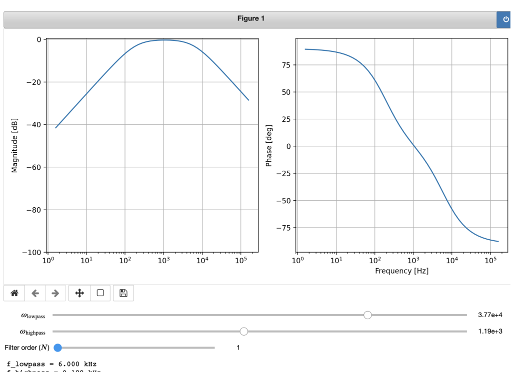
Observe the waveforms at the output of the filter. **Comment on the values of f_{lowpass}, f_{highpass}, and N that you can use to successfully isolate the desired 600 Hz tone. What happens if you keep f_{lowpass} and f_{highpass} constant, and just increase N?**

Solution: As discussed in the problem, we want to choose f_{lowpass} and f_{highpass} so that:

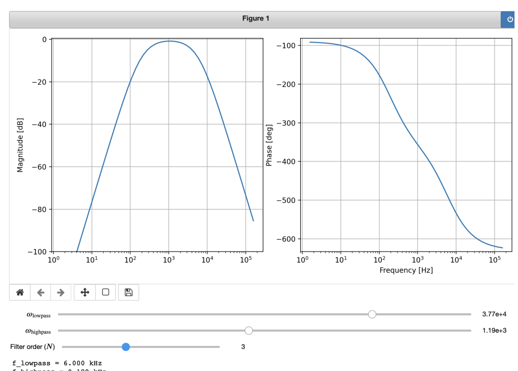
$$f_{\text{power\_line}} < f_{\text{highpass}} < f_{\text{voice}} \qquad f_{\text{voice}} < f_{\text{lowpass}} < f_{\text{fluorescent}} \tag{26}$$

If the highpass or lowpass corners get too close to the frequency we want to keep (600 Hz) then the voice tone starts to get attenuated.

As the filter order increases, we see that the magnitude plot attenuation gets larger.

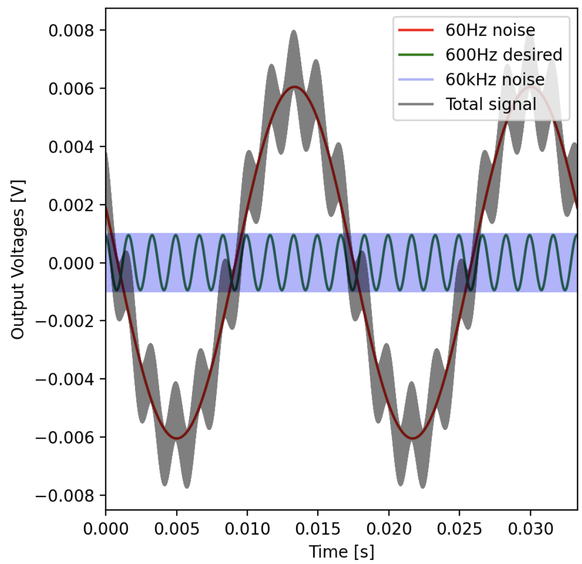


(a) Magnitude and Phase when filter order is 1

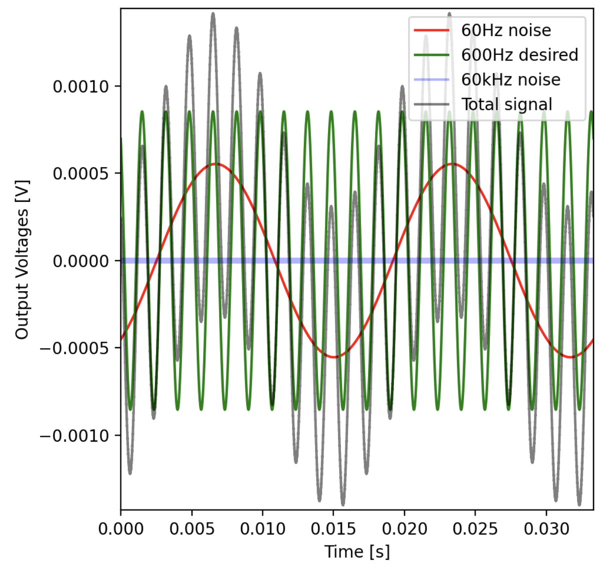


(b) Magnitude and Phase when filter order is 3. Note that the y-axis is more negative, indicating more attenuation.

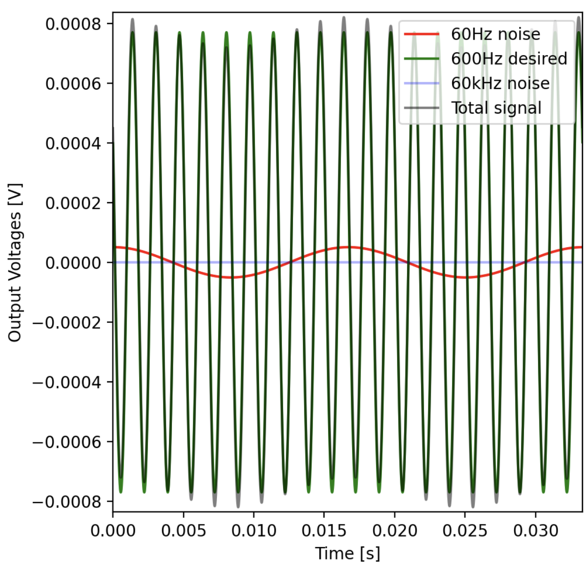
As filter order increases, we see that the overall output voltage waveform (the black line) starts to look more and more like the green voice signal waveform. This is desirable, as it means we are filtering out the unwanted power line noise and fluorescent light noise, which dominate the output voltage when $N = 1$. However, note that the amplitude of the output voltage also decreases as N increases. This is because even though the filters are not supposed to attenuate the 600 Hz voice signal, they still do slightly.



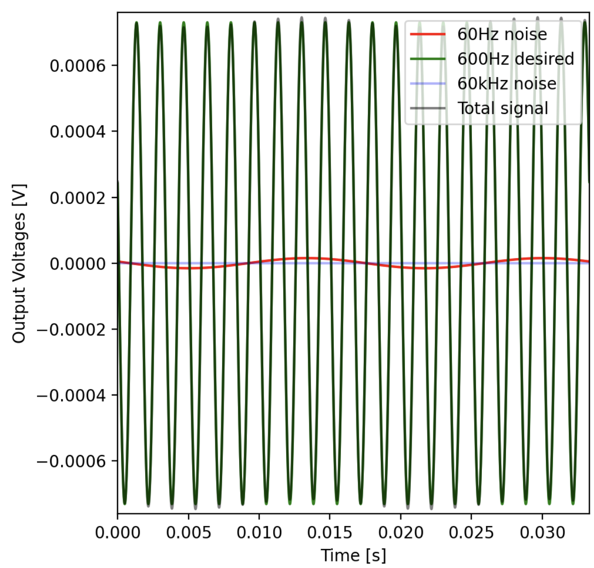
(a) Output voltage for $N = 1$. Note how 60 Hz noise dominates the output signal.



(b) Output voltage for $N = 3$. Note how we start to see the desired green 600 Hz signal, but the red 60 Hz signal is still fairly large. The 60 kHz signal has been almost fully attenuated.



(c) Output voltage for $N = 5$. At this point, the overall output voltage (black) tracks the green 600 Hz signal almost exactly. There is still a slight ripple due to the red 60 Hz noise. Note the amplitude is around 0.75 mV, which is smaller than the 0.88 mV of the 600 Hz signal in the $N = 3$ case.



(d) Output voltage for $N = 6$. At this point, we have successfully isolated the desired green 600 Hz signal. Almost no ripple due to the red 60 Hz noise is present.

4. (PRACTICE) Time-Domain: Boost Converter

Below is a circuit called a Boost Converter. The boost converter is a simple version of a circuit that is very common in battery-operated devices. While a single alkaline battery has a voltage that varies from 1.6 V when it is fresh to 0.8 V at end of life, most electronic devices require a constant supply voltage that stays within ±10% of a certain nominal value. That nominal value is often greater than 1.6 V. So how do we power electronics with batteries? And in particular, how do we power electronics that wants 3 V or 5 V to function when our battery is providing significantly less than that?

The boost converter is a commonly used circuit to provide that stable supply voltage, even from a battery whose voltage is slowly decreasing as it discharges. With proper choice of components and proper control of the gate voltage V_G of the transistor, power from the battery can be converted to higher voltages with almost perfect efficiency.

Consider the following circuit:

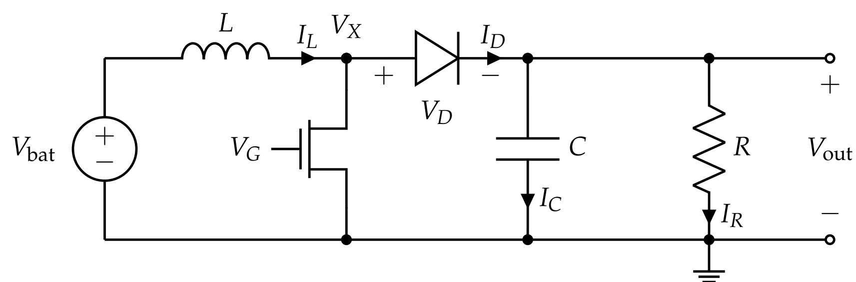


Figure 5: Boost converter schematic.

with

$$V_{\text{bat}} = 1\,\text{V} \tag{27}$$
$$L = 1\,\mu\text{H} \tag{28}$$
$$C = 1\,\text{mF} \tag{29}$$
$$R = 1\,\text{k}\Omega. \tag{30}$$

We model the electronic component that is consuming power as a resistor R with voltage V_{out} across it. Assume that the gate voltage at the NMOS is low (and thus the NMOS is an open circuit) for $t < 0$.

This circuit introduces a new nonlinear circuit element called a diode. The symbol for a diode is shown in Fig. 6. A diode has a certain voltage across of it, V_D. The direction of this diode voltage is important: the + side of the diode voltage is at the base of the triangle and the - side of the diode voltage is at the vertical bar. A diode allows current to flow only in one direction, from the + side to the - side (the triangle of the diode symbol points towards the direction of current). The diode conducts current only when the diode voltage V_D is greater than some threshold. Otherwise, no current flows through it.

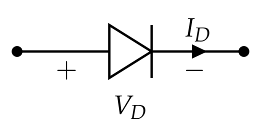


Figure 6: Symbol of a diode. The arrow denotes the direction that current is allowed to flow.

In the real world, this diode voltage threshold is some positive value, usually 0.7V, and the current through it is some function of V_D. For this problem, let's pretend we have an ideal diode. That is, if $V_D \geq 0$, then the diode allows any amount of current it needs to, i.e. a wire (short circuit). Conversely, if $V_D < 0$, the diode conducts no current, i.e. an open circuit. Fig. 7 shows the current-voltage (I-V) characteristic curves for both a real and ideal diode.

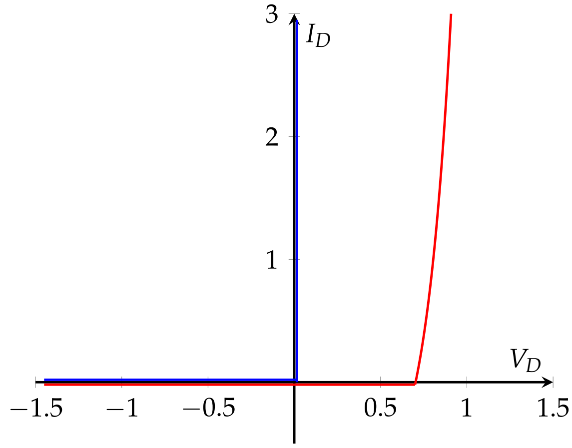
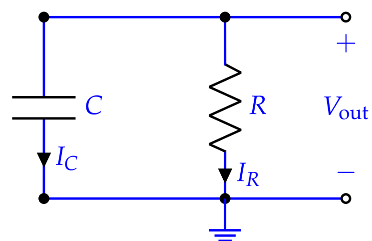


Figure 7: I-V curves for a real diode (red) and an ideal diode (blue). The real diode has a threshold voltage of 0.7V; the ideal diode has a threshold voltage of 0V. The real diode has a current that depends on the voltage when on; the ideal diode acts as a short (0Ω impedance) when on. Both conduct no current when off.

(a) Assume that there is no current flowing through the diode — treat it like an open circuit. **Calculate the time constant of the RC circuit. How long will it take for the output voltage to decay by** 10% **from any non-zero value?** If the voltage on the capacitor is 3 V at $t = 0$, **what is the current in the resistor at time** $t = 0$**? What is the voltage in the capacitor for** $t > 0$**?**

Solution: If there is no current flowing through the diode, we can treat it like an open circuit. So we can redraw the circuit as:



Writing out the KCL equation and substituting in for voltage:

$$I_C = -I_R \tag{31}$$

$$C\frac{\mathrm{d}V_{\text{out}}(t)}{\mathrm{d}t} = -\frac{V_{\text{out}}(t)}{R} \tag{32}$$

$$\frac{\mathrm{d}V_{\text{out}}(t)}{\mathrm{d}t} = -\frac{V_{\text{out}}(t)}{RC} \tag{33}$$

$$\tag{34}$$

Solving this differential equation gives

$$V_{\text{out}}(t) = K\mathrm{e}^{-t/RC}, \tag{35}$$

where K is some constant. We find K by applying our initial condition that $V_{\text{out}}(t = 0)$ is some non-zero value.

$$V_{\text{out}}(0) = K\mathrm{e}^{-0/RC} = K \tag{36}$$

Giving:

$$V_{\text{out}}(t) = V_{\text{out}}(0)\mathrm{e}^{-t/RC}, \tag{37}$$

The RC time constant is therefore:

$$\tau = RC \tag{38}$$

Plugging in for the values of R and C:

$$\tau = RC = 1\,\mathrm{k}\Omega \cdot 1\,\mathrm{mF} = 1\,\mathrm{s} \tag{39}$$

The RC time constant is $\tau = 1\,\mathrm{s}$.
We can calculate how long it takes for the output voltage to decay by 10% from any $V_{\mathrm{out}}(0) > 0$ by solving the following for t:

$$V_{\mathrm{out}}(t) = V_{\mathrm{out}}(0)\mathrm{e}^{-t/\tau} = 0.9 \cdot V_{\mathrm{out}}(0) \tag{40}$$

$$\mathrm{e}^{-t/\tau} = 0.9 \tag{41}$$

$$t = -\ln(0.9) \cdot \tau \approx 0.1\,\mathrm{s} \tag{42}$$

It will take the output voltage $\approx 0.1\,\mathrm{s}$ to decay by 10% from any non-zero value.
If the voltage on the capacitor at $t = 0$ is $3\,\mathrm{V}$, $V_{\mathrm{out}}(0) = 3$ V, giving $V_{\mathrm{out}}(t) = 3\mathrm{e}^{-t/1\,\mathrm{s}}$ V. $V_{\mathrm{out}}(t)$ is the voltage across the resistor and is the voltage across the capacitor.
The current through the resistor is therefore given by:

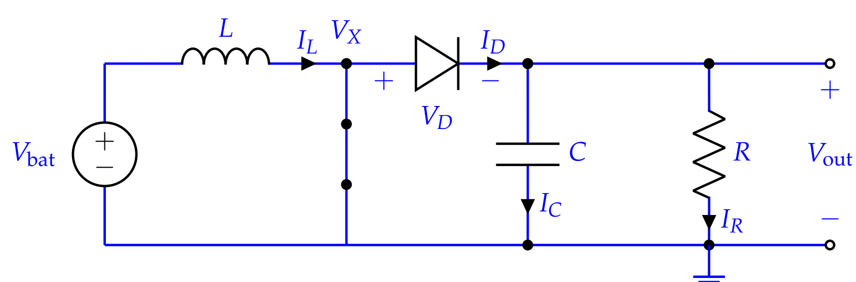
$$I_R(t) = \frac{V_{\mathrm{out}}(t)}{R} = 3 \cdot 10^{-3}\mathrm{e}^{-t/1\,\mathrm{s}}\ \mathrm{A} = 3\mathrm{e}^{-t/1\,\mathrm{s}}\ \mathrm{mA} \tag{43}$$

At time $t = 0$ seconds, $I_R(0) = 3\,\mathrm{mA}$.
The voltage across the capacitor is given by:

$$V_{\mathrm{out}}(t) = 3\mathrm{e}^{-t/1\,\mathrm{s}}\ \mathrm{V} \tag{44}$$

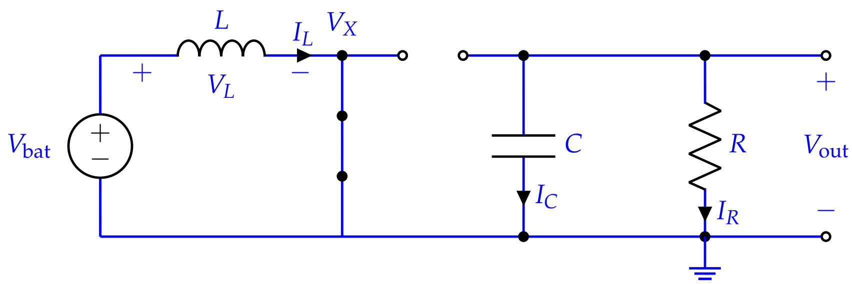
(b) Let's assume that at $t = 0$, the voltage on the capacitor is $3\,\mathrm{V}$. At $t = 0$, the gate voltage on the NMOS V_G goes high, turning the transistor on (effectively shorting that circuit path). **What is the rate of change of current in the inductor? How long does it take for the current in the inductor to increase to $100\,\mathrm{mA}$?**

Solution: We can redraw the circuit for this situation, replacing the transistor with a short (since it is on):



At t=0, we are given that $V_{out} = 3V$ and we observe that $V_X = 0V$. We calculate $V_D = V_X - V_{out} = 0V - 3V = -3V$ is less than the diode threshold $0V$, hence the diode is not conducting. We can therefore replace the diode with an open circuit:
We can also calculate

$$V_L = V_{\mathrm{bat}} - V_X = V_{\mathrm{bat}} - 0V = V_{\mathrm{bat}} \tag{45}$$



We also know the relationship between the current and voltage of an inductor as:

$$V_L = L\frac{dI_L}{dt} \tag{46}$$

Therefore the rate of change at time $t = 0$ is:

$$\frac{dI_L}{dt} = \frac{V_{bat}}{L} = \frac{1\,\text{V}}{1\,\mu\text{H}} = 1 \times 10^6\,\frac{\text{A}}{\text{s}} \tag{47}$$

We can solve this equation for $I_L(t)$ by recognizing this is a constant rate of change, i.e. the constant slope of $I_L(t)$. Using this, along with the initial condition $I_L(0)$, gives us:

$$I_L(t) = \frac{V_{bat}}{L}t + I_L(0) \tag{48}$$

We can find $I_L(0)$ by evaluating KCL at node V_X at time $t = 0$. Since the diode is off right before and after the switch, the diode current $I_D = 0$. Since right after the switch, $V_X = 0V$, this means the current going through the transistor, $I_{NMOS} = 0$. By KCL, we have:

$$I_L(t = 0) = I_D(t = 0) + I_{NMOS}(t = 0) \tag{49}$$
$$= 0 + 0 = 0 \tag{50}$$

Therefore, our complete equation is:

$$I_L(t) = \frac{V_{bat}}{L}t \tag{51}$$

To find the time when $I_L(t) = 100$ mA:

$$t = \frac{L}{V_{bat}} \cdot 100\text{mA} = \frac{1\,\mu\text{H}}{1\,\text{V}} \cdot 100\text{mA} = 0.1\mu\text{s} \tag{52}$$

The current through the inductor reaches 100 mA at $t = 0.1\mu$s.

(c) In the time that the inductor current changes to 100mA, **how much does the capacitor voltage decay in that time?** Round your answer to 7 decimal places. Use your equation for $V_{out}(t)$ from part (a) and the final time you calculated in part (b).

Solution: When the capacitor starts discharging at $t = 0$ (the same time that the NMOS turns on) from an initial voltage $V_{out}(0) = 3$ V, the voltage across it is

$$V_{out}(t) = (3\,\text{V})e^{-t/\tau}. \tag{53}$$

At $t = 0.1\mu$s, the voltage across the capacitor will be

$$V_{out}(0.1\mu\text{s}) = (3\,\text{V})e^{-0.1\mu\text{s}/1\,\text{s}} \tag{54}$$
$$= (3\,\text{V})e^{-10^{-7}} \tag{55}$$
$$\approx (3\,\text{V})(0.999999900000005) \tag{56}$$

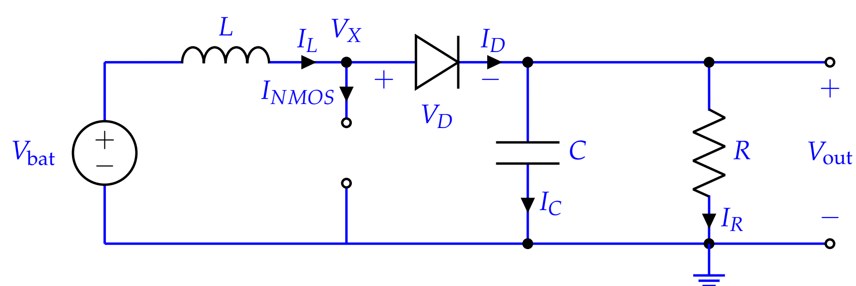
$$\approx 2.999\,999\,7\,\text{V}. \tag{57}$$

In other words, while the inductor current increases, the capacitor only discharged 0.3 µV. This is a small voltage drop compared to 3 V and 1 V. In fact, it is so small, we consider it negligible: we can consider the capacitor voltage, and hence the output voltage, as constant across this time at $V_C = 3V$. Intuitively, this makes sense: we have a very large capacitor, and the larger the capacitor, the slower the voltage across it changes (Q=CV).

(d) You should have found in the previous part that the output voltage remains very close to 3V. This is due to the large output capacitor not allowing the voltage to change quickly. In fact, the output voltage remains so close to 3V that we approximate it as constant. In other words, in the short time that the inductor current changes to 100 mA, we approximate $V_{out} \approx 3V$ throughout. Now let's say that when the inductor current reaches 100 mA, the transistor gate voltage goes low and the transistor turns off. **Is the diode on or off? What is the node voltage V_X?** *(HINT: What is the value of the inductor current right after the transistor switches off? Conduct KCL at node V_X.)*

Solution: The key here is to recognize that the inductor current does not change instantaneously. Compare this to a capacitor: a capacitor's voltage does not change instantaneously due to the stored charge across its metal plates; similarly an inductor's current does not change instantaneously due to the "stored" magnetic flux in its coils. Therefore, $I_L(t = 0.1\mu s) = 100\,\text{mA}$ right before and after the transistor switches off. The rest of the circuit must allow for this current via KCL/KVL.

Let's re-draw the circuit right after the switch, at $t = 0.1\mu s$. We know the transistor is off, so we can replace it as an open circuit. We don't know the state of the diode - that's what we're trying to figure out - so let's leave it as is:



If we conduct KCL at node V_X:

$$I_L(t) = I_{NMOS}(t) + I_D(t) \tag{58}$$

If we solve for this KCL right after the NMOS turns off at $t = 0.1\mu s$, we know from our reasoning above that $I_L(t = 0.1\mu s) = 100\,\text{mA}$. Since the NMOS is off, $I_{NMOS}(t = 0.1\mu s) = 0$. Therefore:

$$I_L(t = 0.1\mu s) = I_{NMOS}(t = 0.1\mu s) + I_D(t = 0.1\mu s) \tag{59}$$
$$= 0 + I_D(t = 0.1\mu s) \tag{60}$$

i.e.

$$I_D(t = 0.1\mu s) = I_L(t = 0.1\mu s) = 100\,\text{mA} \tag{61}$$

Thus, by KCL, the diode must be on and conducting a current equal to the inductor's current.

Now that we know the diode is conducting, how do we solve for V_X? Since we have an ideal diode, it conducts as much current as needed when $V_D \geq 0V$, i.e. the diode acts as a short/wire. In other words, $V_D = 0$ is enough for the diode to let the inductor current pass and satisfy KCL at node V_X.

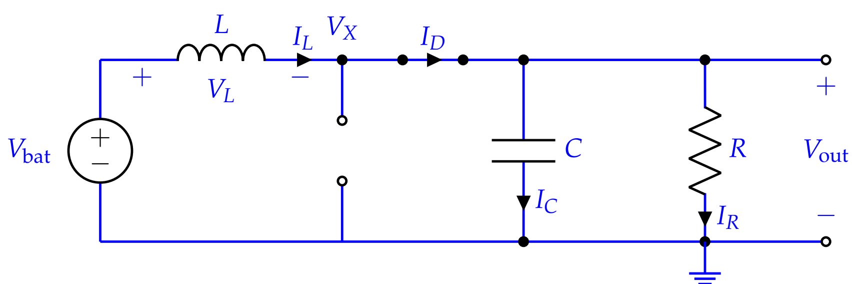
We know that $V_D = V_X - V_{out}$, and we determined from the previous part that we can approximate $V_{out} \approx 3V$. Hence:

$$V_D = V_X - V_{out} = 0 \implies V_X = V_{out} = 3V \tag{62}$$

The astute reader may notice: V_X suddenly changed from 0V before the switch to 3V after the switch, and since V_{bat} did not change, the inductor voltage $V_L = V_{bat} - V_X$ suddenly changed from 1V to -2V. If the inductor current cannot instantaneously change, why can its voltage? Recall that V_L is related to the rate of change in I_L, not I_L itself ($V_L = L\frac{\mathrm{d}I_L}{\mathrm{d}t}$). You can have a large rate of change on the inductor current without changing the current itself (e.g. the current changes directions) – the voltage V_L simply reflects the change. The analogy to a capacitor is having a sudden change in the capacitor voltage - requiring a large change in capacitor current - without changing the voltage itself much ($I_C = C\frac{\mathrm{d}V_C}{\mathrm{d}t}$). A more physical explanation is that the energy stored in the inductor is proportional to its magnetic field and thus current, not voltage. An instantaneous change in inductor current would mean an instantaneous change in energy, i.e. generating or consuming infinite power, which is not possible. The voltage across it is not relevant to the inductor's stored energy. Similarly, capacitor's stored energy is proportional to its electric field and thus voltage, which is why a capacitor's voltage cannot instantaneously change. This physical explanation will make more sense in more advanced physics and electromagnetism courses.

(e) After the transistor turns off from the previous part at $t = 0.1\mu s$, **what is the rate of change of current in the inductor? How long does it take the current in the inductor to go to zero (i.e. for the inductor to fully discharge)?** Assume the inductor discharges much faster than the capacitor, i.e. we can still approximate the output voltage as constant $V_{out} = 3V$.

Solution: We found in the previous part that the diode is turned on after the transistor turns off at $t = 0.1\mu s$. We can redraw the circuit by replacing the diode with a short (the transistor remains open):



The voltage across the inductor is:

$$V_L = V_{bat} - V_X = V_{bat} - V_{out} = 1\,\mathrm{V} - 3\,\mathrm{V} = -2\,\mathrm{V}. \tag{63}$$

The inductor current rate of change is therefore:

$$\frac{\mathrm{d}I_L}{\mathrm{d}t} = \frac{V_L}{L} = -2 \times 10^6\,\frac{\mathrm{A}}{\mathrm{s}}. \tag{64}$$

Note that our initial time here is $t = 0.1\mu s$, not $t = 0$. We need to shift our time variable by $0.1\mu s$. It helps to introduce a dummy time variable, $\tilde{t}$, where $\tilde{t} = t - 0.1\,\mu s$ is a new time reference frame.

Solving for $I_L(\tilde{t})$ and noting that $I_L(t = 0.1\mu s) = I_L(\tilde{t} = 0) = 100 \times 10^{-3}\,\mathrm{A}$:

$$I_L(\tilde{t}) = \frac{V_L}{L}\tilde{t} + c \tag{65}$$

$$I_L(\tilde{t} = 0) = 100 \times 10^{-3}\,\mathrm{A} = -2 \times 10^6\,\frac{\mathrm{A}}{\mathrm{s}} \cdot 0\,\mathrm{s} + c = c \tag{66}$$

$$c = 100 \times 10^{-3}\,\text{A} \tag{67}$$
$$I_L(\tilde{t}) = -2 \times 10^6\,\frac{\text{A}}{\text{s}}\tilde{t} + 100 \times 10^{-3}\,\text{A} \tag{68}$$

To find the time where the current is zero, we solve for $\tilde{t}$:

$$I_L(\tilde{t}) = \frac{V_L}{L}\tilde{t} + 100 \times 10^{-3}\,\text{A} = 0\,\text{A} \tag{69}$$
$$\tilde{t} = \frac{-100 \times 10^{-3}\,\text{A}}{-2 \times 10^6\,\frac{\text{A}}{\text{s}}} = 5 \times 10^{-8}\,\text{s} \tag{70}$$

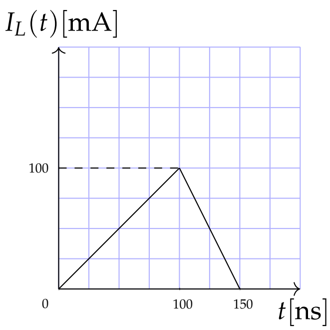
It takes $0.05\,\mu\text{s}$ for the current in the inductor to go to zero.

Once the inductor current falls to zero, it cannot go negative - that would imply the current through the inductor and hence diode changes direction. But remember the diode only allows the current to flow in one direction. Therefore, once the inductor current reaches zero, there will be no more change in the inductor current and the inductor has discharged all its energy.

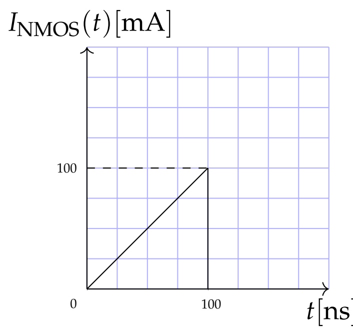
Just to sanity check: $0.05\,\mu\text{s}$ is still a very short amount of time for the capacitor to discharge any meaningful voltage, so the approximation that the capacitor voltage does not meaningfully drop while the inductor is discharging is valid.

(f) **Sketch the following currents vs time: $I_L(t)$, $I_{\text{NMOS}}(t)$, $I_{\text{D}}(t)$ for a single inductor charge/discharge cycle described above. For each plot, plot current on the vertical axis in mA and time on the horizontal axis in ns from t=0ns to 200ns.**

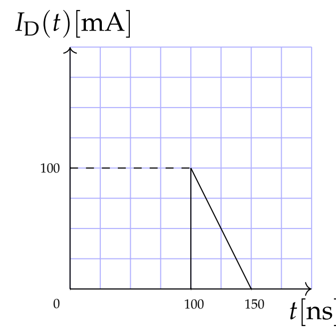
Solution:



(a) Inductor Current: $I_L(t)$



(b) NMOS Current: $I_{\text{NMOS}}(t)$



(c) Diode Current: $I_{\text{D}}(t)$

Intuition:

- $I_L(t)$: For the first 100 ns, the current is charging linearly from V_{bat}. At $t = 100$ ns, the NMOS turns off, and the inductor starts discharging for 50 ns, at which point the current returns to zero.
- $I_{\text{NMOS}}(t)$: The NMOS only conduct current when it is on, that is, for the first 100 ns. During this time, it is the only branch current connected to the inductor, so by KCL its current is equal to that of the inductor as it goes from zero to 100 mA. As soon as the switch opens at $t = 100$ ns, its current immediately goes to zero.
- $I_{\text{D}}(t)$: When the NMOS turns off, all of the inductor current suddenly is redirected through the diode. Once this inductor/diode current goes to zero, the diode prevents the current from changing direction and become negative. Thus, the current remains at 0.

(g) The goal of this Boost converter circuit is to maintain an average output voltage of V_{out} at 3 V. We've seen this is done by turning the transistor on and off at the right times, allowing the inductor to energize and discharge its current through the diode to the capacitor which itself is losing energy to the load. To maintain the constant output voltage on average, over long periods of time (where the capacitor decay would become noticeable if left on its own), we need the average current through the diode to bring enough charge to replenish the capacitor. Recall that we model the load as a $1k\Omega$ resistor, and we want to maintain an output voltage of $V_{\text{out}} = 3\,\text{V}$. We want to determine how quickly we need to toggle the transistor to achieve this.

What is the average current drawn by the resistive load? Using your plot of $I_D(t)$ from the previous part, calculate the charge that flows through the diode in a single cycle. At what frequency does the transistor need to cycle through the on/off phases to supply the current consumed by the resistor? Recall that the charge Q supplied by a current over a time interval T is the time integral of that current, i.e. the area under the graph. The average current I_{avg} for that supplied charge is $I_{avg} = \frac{Q}{T}$. Your job is to find T and consequently the frequency $f = \frac{1}{T}$ that you toggle the transistor on and off.

Solution: First, the resistor draws an average current of $I_R = \frac{V_{out}}{R} = \frac{3\,\text{V}}{1k\Omega} = 3mA$. This is the current that the load draws from the capacitor, so this is what the diode/inductor must replenish to the capacitor.

The total amount of charge that flows through the diode can be found by integrating the current over the cycle time. Based on the sketch from the previous part, we can graphically integrate $i_{\text{diode}}(t)$ to calculate the total charge by finding the area of the "current spike" triangle. If we take the time to discharge as the "base" of the triangle, and the peak current as the "height", then the area is equal to

$$Q = \frac{1}{2}bh = \frac{1}{2}(100\,\text{mA})(0.05\,\mu\text{s}) = 2.5\,\text{nC} \tag{71}$$

Now for the frequency. Suppose that we're performing the charge-discharge cycle once every T seconds; that is, T is the period corresponding to the frequency we choose. Then the average current is

$$i_{\text{avg}} = \frac{Q}{T} = \frac{2.5\,\text{nC}}{T}, \tag{72}$$

meaning we should choose

$$T = \frac{Q}{i_{\text{avg}}} = \frac{2.5\,\text{nC}}{3\,\text{mA}} = \frac{5}{6}\,\mu\text{s}. \tag{73}$$

The frequency is just the inverse of the period, so the frequency we should choose is

$$f = \frac{1}{T} = \frac{1}{\frac{5}{6}\,\mu\text{s}} = 1.2\,\text{MHz}. \tag{74}$$

(h) **(PRACTICE)** The included Jupyter notebook `boost_converter.ipynb` sets up the same problem described above. In the notebook, you can use the slider bars to play around with:

- The cycle frequency for the transistor
- The component values R, L, and C
- The battery voltage V_{bat}

How does the inductor current curve in part (e) change if the battery voltage changes? What is the effect of a larger or smaller capacitor? A larger or smaller load?

Solution: As the battery voltage V_{bat} decreases from 1.6 V to 0.8 V, we can see that the output voltage remains constant, but the inductor takes longer to charge when the transistor is on (i.e., takes longer to reach the 100 mA threshold where the transistor turns off). This behavior is intuitive since we know that the slope of $I_L(t)$ is proportional to V_{bat}. Changing the capacitance does not dramatically change the output voltage behavior, but does influence the minor fluctuations of $V_C(t)$. With a larger capacitance, the capacitor voltage decays more slowly when the diode is off and the capacitor charges more slowly when the diode is on (i.e., when the inductor is discharging). Again, this behavior is intuitive as changing C results in a different time constant in the exponential term of $V_C(t)$. Similarly, changing the load (denoted by the resistance R) changes the behavior of the small fluctuations in $V_C(t)$ (causing the fluctuations to slowly increase or decrease over time for a given frequency). Note that changing the frequency can offset the change in load.

In real world power electronics, interesting circuits like this one are combined with control algorithms that allow the system to adapt to changing situations vis-a-vis the battery level and variations in how much power is being drawn by the load.

5. Circuit Design

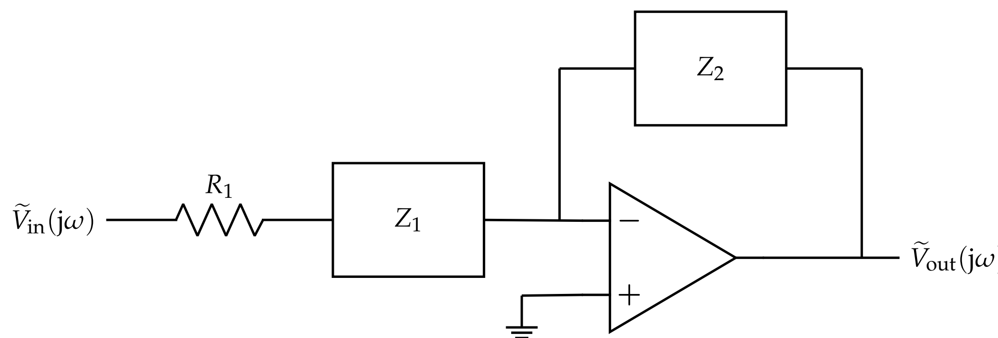
In this problem, you will find a circuit where several components have been left *blank* for you to fill in.

Assume that the op-amp is *ideal*. A special note on op amps in frequency domain analysis: The op-amps you learned about in 16A can be used in exactly the same way for setting up differential equations and even Phasor analysis in 16B. Treat them as ideal op-amps and invoke the Golden Rules.

You have at your disposal *only one of each* of the following components (not including R_1):

(a) an open circuit **(b)** a short circuit **(c)** a resistor (you choose from the values $R = 1\,\text{k}\Omega, 10\,\text{k}\Omega, 20\,\text{k}\Omega$) **(d)** a capacitor (you choose from the values $C = 0.5\,\mu\text{F}, 1\,\mu\text{F}, 2\,\mu\text{F}$)

Consider the circuit below. The labeled voltages $\widetilde{V}_{\text{in}}(\text{j}\omega)$ and $\widetilde{V}_{\text{out}}(\text{j}\omega)$ are the phasor representations of $v_{\text{in}}(t)$ and $v_{\text{out}}(t)$ respectively, where $v_{\text{in}}(t)$ has the form $v_{\text{in}}(t) = v_0 \cos(\omega t + \phi)$. The transfer function $H(\text{j}\omega)$ is defined as $H(\text{j}\omega) = \frac{\widetilde{V}_{\text{out}}(\text{j}\omega)}{\widetilde{V}_{\text{in}}(\text{j}\omega)}$.



(a) Let $Z_1(\text{j}\omega)$ and $Z_2(\text{j}\omega)$ are the impedances of the boxes shown in the circuit diagram. **Write the expression of the transfer function $H(\text{j}\omega)$.**

Solution: The circuit is in the inverting amplifier configuration. Hence the transfer function is given by

$$H(\text{j}\omega) = -\frac{Z_2}{R_1 + Z_1} \tag{75}$$

(b) Let R_1 be $1\,\text{k}\Omega$. We have to find Z_1 and Z_2, such that the circuit's transfer function $H(\text{j}\omega)$ has the following properties:

- It is a high-pass filter.
- $|H(\text{j}\infty)| = 10$.
- $|H(\text{j}10^3)| = \sqrt{50}$.

Using the fact that the circuit is a high pass filter, infer the components (we will find values later) of Z_1 and Z_2. Write the transfer function $H(\text{j}\omega)$ using these components.

Hint: Try method of elimination: figure out what Z_2 cannot be. Once you find what Z_2 is, what does Z_1 have to be for the circuit to be a filter?

Solution: The circuit should be a high-pass filter, so Z_2 cannot be a short circuit or a capacitor, otherwise $\widetilde{V}_{\text{out}}(\text{j}\infty) = 0$. Also Z_2 cannot be open circuit as that will break the negative feedback. So Z_2 is a resistor, i.e. $Z_2 = R$.

Since Z_2 cannot be a capacitor, Z_1 must be a capacitor. Otherwise, we would not have any frequency dependent impedance in the circuit, which means it wouldn't be a filter. So $Z_1 = \frac{1}{\mathrm{j}\omega C}$. Hence the transfer function is

$$H(\mathrm{j}\omega) = -\frac{R}{R_1 + \frac{1}{\mathrm{j}\omega C}} \tag{76}$$

$$= -\frac{R}{R_1} \cdot \frac{1}{1 - \frac{\mathrm{j}}{\omega C R_1}} \tag{77}$$

Observe that $|H(\mathrm{j}0)| = 0$ and $|H(\mathrm{j}\infty)| = \frac{R}{R_1}$, so it is a high-pass filter.

(c) **Now use the facts that $|H(\mathrm{j}\infty)| = 10$ and $R_1 = 1\,\mathrm{k}\Omega$ to find the component value of Z_2.**

Solution: From $|H(\mathrm{j}\infty)| = 10 = \frac{R}{R_1}$, we know that $R = 10R_1 = 10\,\mathrm{k}\Omega$, which is one of the options for resistor values.

(d) **Finally use the fact that $|H(\mathrm{j}10^3)| = \sqrt{50}$ and the values of R_1 and Z_2 to find the component value of Z_1.**

Solution: We can now write the transfer function as

$$H(\mathrm{j}\omega) = -10\frac{1}{1 - \frac{\mathrm{j}}{1000\omega C}} \tag{78}$$

We get

$$|H(\mathrm{j}10^3)| = 10\frac{1}{\sqrt{1 + \frac{1}{(10^6 C)^2}}} = \sqrt{50} \tag{79}$$

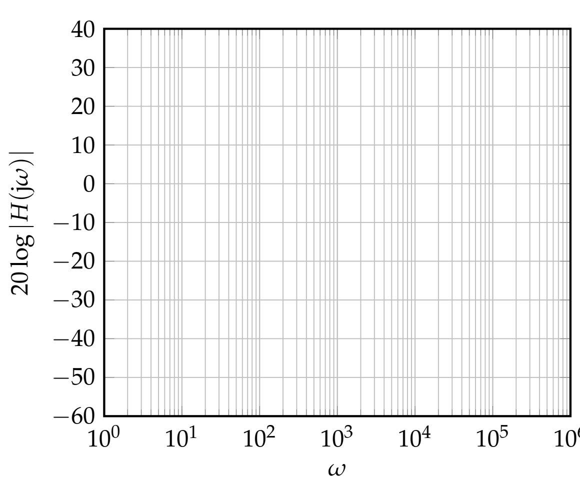
$$\implies 100\frac{1}{1 + \frac{1}{(10^6 C)^2}} = 50 \tag{80}$$

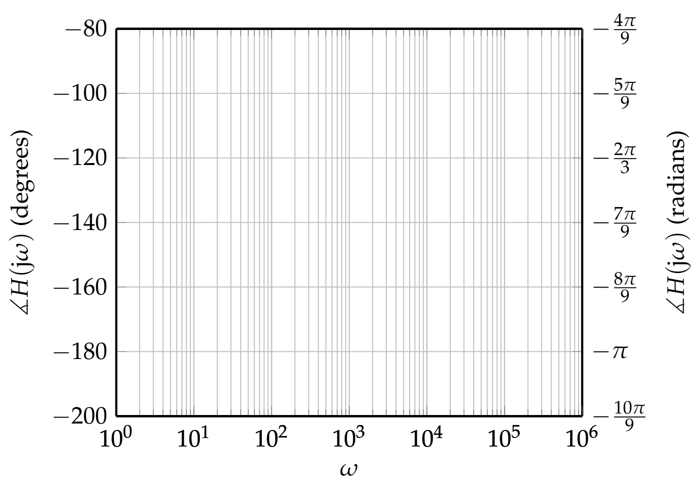
$$\implies \frac{1}{1 + \frac{1}{(10^6 C)^2}} = \frac{1}{2} \tag{81}$$

$$\implies \frac{1}{(10^6 C)^2} = 1 \tag{82}$$

$$\implies C = 1\,\mu\mathrm{F} \tag{83}$$

(e) **Draw the magnitude and phase Bode plots (straight-line approximations to the transfer function) of this transfer function.** Blank plots are provided here for you to use.





Solution: The transfer function is

$$H(\mathrm{j}\omega) = -10\frac{1}{1 - \frac{\mathrm{j}\cdot 1000}{\omega}} \tag{84}$$

This is a first-order-high-pass filter with cut-off frequency $\omega_o = 1000\,\frac{\text{rad}}{\text{s}}$.
The magnitude is given by

$$|H(\mathrm{j}\omega)| = 10\frac{1}{\sqrt{1 + \frac{10^6}{\omega^2}}} \tag{85}$$

Hence we have the following properties for the magnitude Bode plot approximation:

i. At $\omega > \omega_o$, the magnitude is $20\log(10) = 20\,\text{dB}$.
ii. At $\omega < \omega_o$, the magnitude drops by 20 dB per decade of ω.

The phase is given by

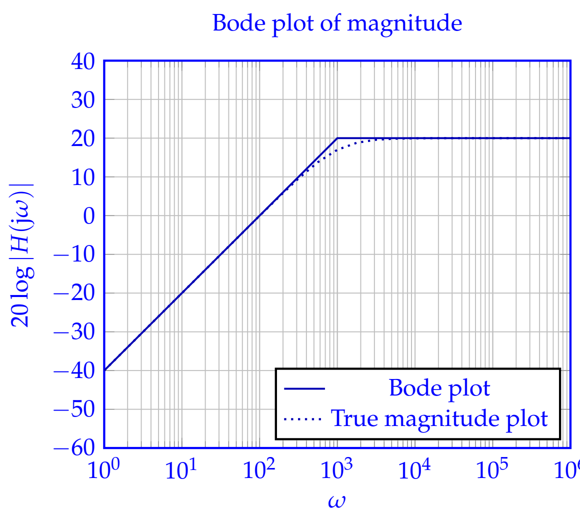
$$\angle(H(\omega)) = \angle(-10) - \angle\left(1 - \frac{\mathrm{j}\cdot 1000}{\omega}\right) \tag{86}$$

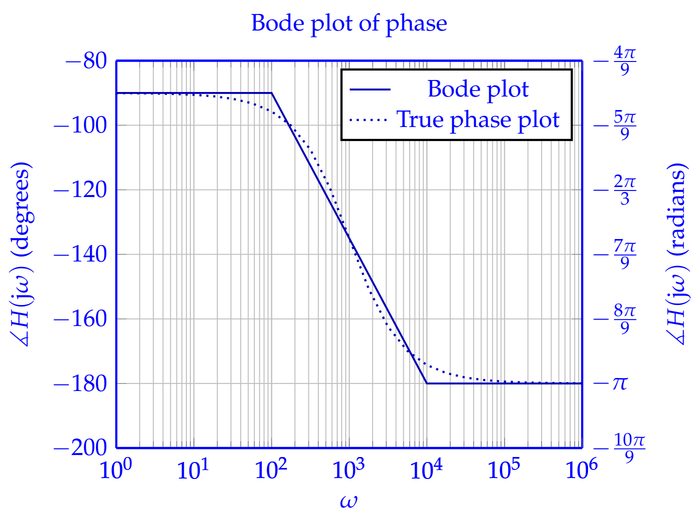
$$= -180^\circ - \operatorname{atan2}\left(-\frac{1000}{\omega}, 1\right) \tag{87}$$

Hence we have the following properties for the phase Bode plot:

i. At $\omega < \frac{\omega_o}{10}$, the phase is $-180^\circ - \operatorname{atan2}(-\infty, 1) = -90^\circ$.
ii. At $\omega > 10\omega_o$, the phase is $-180^\circ - \operatorname{atan2}(-0, 1) = -180^\circ$.
iii. At $\omega = \omega_o$, the phase is $-180^\circ - \operatorname{atan2}(-1, 1) = -135^\circ$.

So the magnitude and phase Bode plots look as follows:





Note that plotting the true magnitude/phase using a computer is not necessary for credit – they are just there to illustrate to you how our Bode plot approximations are close to the true plots at many frequencies. But you can see that there are some frequencies where the approximation is not so good. At these frequencies, if they are relevant to the problem you are doing, you should check with the true plot instead of the approximated one.

6. System Identification

You are given a discrete-time system as a black box. You don't know the specifics of the system but you know that it takes one scalar input and has two states that you can observe. You assume that the system is linear and of the form

$$\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{w}[i], \tag{88}$$

where $\vec{w}[i]$ is an external small unknown disturbance, $u[i]$ is a scalar input, and

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad x[i] = \begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix}. \tag{89}$$

You want to identify the system parameters (a_1, a_2, a_3, a_4, b_1 and b_2) from measured data. However, you can only interact with the system via a black box model, i.e., you can see the states $\vec{x}[t]$ and set the inputs $u[i]$ that allow the system to move to the next state.

(a) You observe that the system has state $\vec{x}[i] = \begin{bmatrix} x_1[i] & x_2[i] \end{bmatrix}^\top$ at time i. You pass input $u[i]$ into the black box and observe the next state of the system: $\vec{x}[i+1] = \begin{bmatrix} x_1[i+1] & x_2[i+1] \end{bmatrix}^\top$.
Write scalar equations for the new states, $x_1[i+1]$ and $x_2[i+1]$. Write these equations in terms of the a_i, b_i, the states $x_1[i]$, $x_2[i]$ and the input $u[i]$. Here, assume that $\vec{w}[i] = \vec{0}$ (i.e., the model is perfect).
Solution:

$$x_1[i+1] = a_1 x_1[i] + a_2 x_2[i] + b_1 u[i] \tag{90}$$
$$x_2[i+1] = a_3 x_1[i] + a_4 x_2[i] + b_2 u[i]. \tag{91}$$

(b) Now we want to identify the system parameters. We observe the system at the start state $\vec{x}[0] = \begin{bmatrix} x_1[0] \\ x_2[0] \end{bmatrix}$. We can then input $u[0]$ and observe the next state $\vec{x}[1] = \begin{bmatrix} x_1[1] \\ x_2[1] \end{bmatrix}$. We can continue this for a sequence of ℓ inputs.
Let us define an ℓ-length trajectory to be an initial condition $\vec{x}[0]$, an input sequence $u[0], \ldots, u[\ell-1]$, and the corresponding states that are produced by the system $x[1], \ldots, x[\ell]$. **Assuming that the model is perfect ($\vec{w}[i] = \vec{0}$), what is the minimum value of ℓ you need to identify the system parameters?**
Solution: There are 6 unknowns so we need 6 equations to properly identify the system. Each additional timestep gives two new equations. To form the 6 equations we need to give the black box $\ell = 3$ inputs. Namely, given inputs $u[0], u[1]$, and $u[2]$, we can see the state at times $t = 0, 1, 2, 3$ to give us our six equations.
Notice that the initial condition on its own gives us no equations because the unknowns we are interested in do not impact the initial condition. They govern the evolution of the system, and hence the states at times $1, 2, 3$ each give us two equations.
Note that having 6 equations is a necessary, but not sufficient, condition for us to be able to invert the system to uniquely determine the system parameters. For example, if $A = I$ and $u[0] = \cdots = u[\ell-1] = 0$, then we would only have two independent equations.

(c) We now remove our assumption that $\vec{w} = 0$. We assume it is small, so the model is approximately correct and we have

$$\vec{x}[i+1] \approx A\vec{x}[i] + Bu[i]. \tag{92}$$

Say we feed in a total of 4 inputs $u[0], \ldots, u[3]$, and observe the states $\vec{x}[0], \ldots, \vec{x}[4]$. To identify the system we need to set up an approximate (because of potential, small, disturbances) matrix equation

$$DP \approx S \tag{93}$$

using the observed values above and the unknown parameters we want to find. Let our parameter vector be

$$P := \begin{bmatrix} \vec{p}_1 & \vec{p}_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \\ b_1 & b_2 \end{bmatrix} \tag{94}$$

Find the corresponding D and S to do system identification. Write both out explicitly.

Solution:

Using eq. (91), we get

$$\begin{bmatrix} x_1[0] & x_2[0] & u[0] \\ x_1[1] & x_2[1] & u[1] \\ x_1[2] & x_2[2] & u[2] \\ x_1[3] & x_2[3] & u[3] \end{bmatrix} \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \\ b_1 & b_2 \end{bmatrix} \approx \begin{bmatrix} x_1[1] & x_2[1] \\ x_1[2] & x_2[2] \\ x_1[3] & x_2[3] \\ x_1[4] & x_2[4] \end{bmatrix} \tag{95}$$

so

$$D = \begin{bmatrix} x_1[0] & x_2[0] & u[0] \\ x_1[1] & x_2[1] & u[1] \\ x_1[2] & x_2[2] & u[2] \\ x_1[3] & x_2[3] & u[3] \end{bmatrix}, \text{ and } S = \begin{bmatrix} x_1[1] & x_2[1] \\ x_1[2] & x_2[2] \\ x_1[3] & x_2[3] \\ x_1[4] & x_2[4] \end{bmatrix}. \tag{96}$$

(d) Now that we have set up $DP \approx S$, we can estimate a_0, a_1, a_2, a_3, b_0, and b_1. **Give an expression for the estimates of $\vec{p}_1$ and $\vec{p}_2$ (which are denoted $\hat{\vec{p}}_1$ and $\hat{\vec{p}}_2$ respectively) in terms of D and S.** Denote the columns of S as $\vec{s}_1$ and $\vec{s}_2$, so we have $S = [\vec{s}_1\ \vec{s}_2]$. Assume that the columns of D are linearly independent. *(HINT: Don't forget that D is not a square matrix. It is taller than it is wide.) (HINT: Can we split $DP = S$ into separate equations for p_1 and p_2?)*

Solution:

Notice that eq. (95) can be split into two matrix equations, one for each of p_1 and p_2:

$$D\vec{p}_1 \approx \vec{s}_1 \tag{97}$$
$$D\vec{p}_2 \approx \vec{s}_2. \tag{98}$$

Since D isn't square, it isn't invertible. However, we can still find $\vec{p}_1$ and $\vec{p}_2$ that best satisfy the equation via least-squares, which gives the solution

$$\hat{\vec{p}}_1 = (D^\top D)^{-1} D^\top \vec{s}_1 \tag{99}$$
$$\hat{\vec{p}}_2 = (D^\top D)^{-1} D^\top \vec{s}_2. \tag{100}$$

Here, $D^\top D$ is invertible (i.e. the solution is well-defined) because the columns of D are linearly independent. This was proved in 16A, but for completeness we include it here.

Assume that the columns of D are linearly independent. Let $\vec{v} \in \mathbb{R}^3$ such that $(D^\top D)\vec{v} = 0$. Then $0 = \vec{v}^\top D^\top D\vec{v} = (D\vec{v})^\top (D\vec{v}) = \|D\vec{v}\|_2^2$, so $D\vec{v} = 0$. Since D has linearly independent columns, then $\vec{v} = 0$. This means that the nullspace of $D^\top D$ is $\{0\}$, so $D^\top D$ must have full rank and is invertible.

7. Identifying systems from their responses to known inputs

In many problems, we have an unknown system, and would like to characterize it. One of the ways of doing so is to observe the system response with different initial conditions (or inputs). This problem is also called system identification. It is a prototypical example of a problem that today is called machine learning — inferring an underlying pattern from data, and doing so well enough to be able to exploit that pattern in some practical setting. Go through the attached Jupyter notebook `demo_system_id.ipynb` and answer the following questions.

(a) In Example 2, we assume that instead of measuring the state $\vec{x}$, we are instead measuring a transformation of the state $\vec{y} = T\vec{x}$ where T is a full rank matrix. Assume that we perform system ID on our observations $\vec{y}[i]$ to recover A_y, B_y such that $\vec{y}[i+1] = A_y\vec{y}[i] + B_y u[i]$. **How do the identified A_y and B_y matrices relate to the original A and B matrices in the dynamics of $\vec{x}$?** Remember that our original state dynamics are $\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i]$.

Hint: The answer is given in the Jupyter notebook but remember to show your work.

Solution: Using our given transformation that $\vec{y} = T\vec{x}$,

$$\vec{y}[i+1] = A_y\vec{y}[i] + B_y u[i] \tag{101}$$
$$T\vec{x}[i+1] = A_y T\vec{x}[i] + B_y u[i] \tag{102}$$
$$\vec{x}[i+1] = T^{-1}A_y T\vec{x}[i] + T^{-1}B_y u[i] \tag{103}$$

Thus, $A = T^{-1}A_y T$ and $B = T^{-1}B_y$. This can be rewritten as $A_y = TAT^{-1}$ and $B_y = TB$, which is exactly what is in the Jupyter notebook.

(b) **Please share your observations on Example 2. Comment on what impact a linear transformation of the state trace has on our ability to perform system identification.**

Solution: It's nice to see that a linear transformation of the state trace does not have a tremendous effect on our ability to perform system identification. Basically, this means that we have some leeway in choosing what data to observe in practice. In a circuit, for example, we would typically choose capacitor voltages and inductor currents as our state variables. However, it's difficult to make current measurements in real time in a non-invasive way, so we would prefer for our observations to consist of only voltages. As long as we can in principle recover the inductor currents as some linear combination of voltages (this is usually possible), then we can just measure those voltages and proceed as normal.

Furthermore, even if an unwanted state transformation occurs, we know that our estimate of the system does not change drastically. As we saw, the estimated eigenvalues are still correct. Furthermore, controllability and observability are preserved in coordinate changes, so the system we identified will have almost all of the same control-theoretic properties as the true system. Believe me, that's a relief!

(c) **Prove that for any full rank transformation matrix T, the eigenvalues of A_y and A from part (a) are the same.**

Solution: Assume that the eigenvalue eigenvector pairs of A are $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2), \ldots, (\lambda_n, \vec{v}_n)$. Then, $A\vec{v}_i = \lambda_i\vec{v}_i$. We claim that $T\vec{v}_i$ will be the eigenvectors of A_y. We can see this with

$$A_y T\vec{v}_i = TAT^{-1}T\vec{v}_i = TA\vec{v}_i = T(\lambda_i\vec{v}_i) \tag{104}$$
$$= \lambda_i T\vec{v}_i \tag{105}$$

Thus, A_y also has eigenvalues λ_i with its corresponding eigenvector being $T\vec{v}_i$.

(d) **Please share your observations on Example 3. Comment on the impact that changing the noise magnitude, number of samples and number of states has on the system identification performance.**

Solution: From playing around with the system and state trace parameters, you may have noticed a few trends:

- Increasing the noise magnitude σ reduced the accuracy of the identified eigenvalues (that is, the identified eigenvalues were farther away from the true ones).
- Increasing the number of samples improved the accuracy of the identification.
- Increasing the number of states has a large impact on how accurate the identification is, for a fixed noise magnitude and number of data points.

The final point is important in practice. It suggests that, if we have some control over how many states we model a system with, and if all other things are equal, then we should choose to have fewer states rather than more, so that our system identification requires less data to be accurate.

This is a point that turns out to be extremely important in machine-learning more generally — we do not necessarily always want the most complicated model.

(e) **Please share your observations on Example 4. Comment on the sample efficiency of this method, i.e. do you need more or less samples for accurate system identification when given scalar observations rather than the entire state vector?**

Solution: This example took you beyond what you have learned in lecture. It involved figuring things out without being able to observe the state and instead just seeing scalar observations of the state.

You probably noticed that identifying a system with scalar observations requires a longer sample trace to be accurate than identifying a system with state observations does. The reason why is pretty clear: you only have one scalar point of data at every time step now, instead of a vector. Nonetheless, it's a really great thing that we can still do system identification with scalar observations at all! In many applications, a scalar output is all that's available.

Having to select a value for n with no prior knowledge introduces more trial-and-error than we would like, but in the method used in this section it's a necessary evil. But is it necessary in general? I shouldn't overly anthropomorphize the data, but it must know how many states there are, since it was generated by the system. So it seems like some knowledge of the "true n" is hiding somewhere in the data, if only we know how to look...

We will return to this issue later in the course.

(f) **Please share your observations on Example 5. Comment on how important the model size is for this setting.**

Solution: This is another example that takes you a bit beyond what you have seen in lecture, but in a natural way. What happens if your model size is wrong? Do you have to get the model size right?

It's interesting to see that the effect of many eigenvalues near the origin can be effectively approximated by just a couple of eigenvalues in a smaller system. Just like example 3, this suggests that there is such a thing as "too many states". Basically, 13 out of the 16 states of this system contribute almost nothing to the behavior of the system– we were able to throw them away and still capture the important behavior.

So what happens because we are ignoring these states and the true complexity of the underlying model? We will hopefully see in a later homework that what this does is contribute to the disturbance that our estimated model experiences. It not only has a disturbance because it didn't estimate the parameters of the model perfectly (say because of observation noises), but also because it chose a model that was simpler than reality. But as long as we can be robust to this disturbance, we are still fine.

8. Motor Driver and System Identification

In the lab project, you will be designing SIXT33N, a mischievous little robot who *might* just do what you want — if you design it correctly. In phases 1 and 2, you will build the **legs** of SIXT33N: you will be designing SIXT33N's wheels and developing a linear model for the car system. The wheels will be driven by two 9-Volt DC motors whose driver control circuit is shown in Figure 10 .

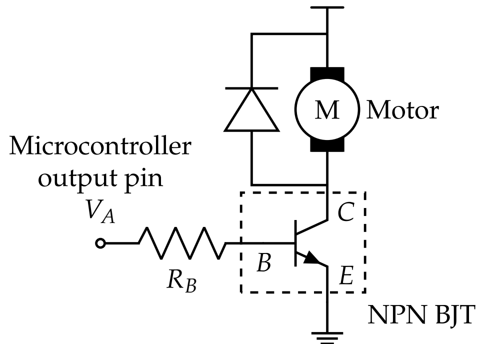


Figure 10: Motor Controller Circuit

There is some minimum voltage required to deliver enough power to the motors to overcome the static friction and start them, but after that point, we treat the motor speed as approximately **linear** with the applied voltage V_A (this will be the basis of the system model you will develop in this problem).

As it is difficult to use a microcontroller (MSP430 in hands-on lab; Arduino in lab sim) to generate a true adjustable DC signal, we will instead make use of its PWM function. A PWM, or pulse-width modulated, signal is a square wave with a variable duty cycle (the proportion of a cycle period for which the power source is turned on, or logic HIGH). PWM is used to digitally change the average voltage delivered to a load by varying the duty cycle. If the frequency is large enough, the on-off switching is imperceptible, but the average voltage delivered to the load changes proportionally with the duty cycle. Hence, changing the duty cycle corresponds to changing the DC voltage supplied to the motor. An example can be seen in figure 11.

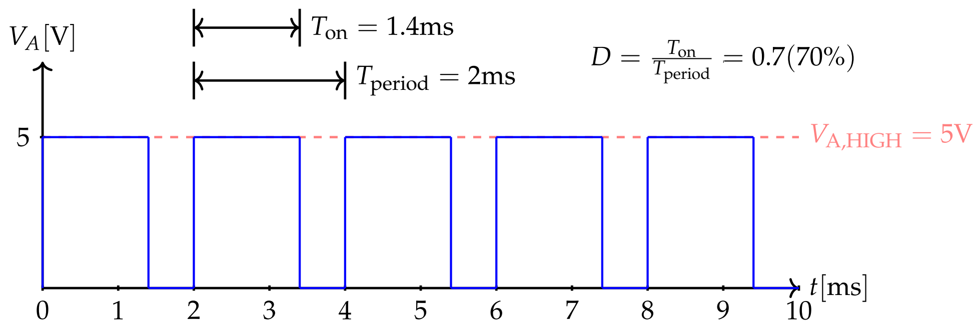


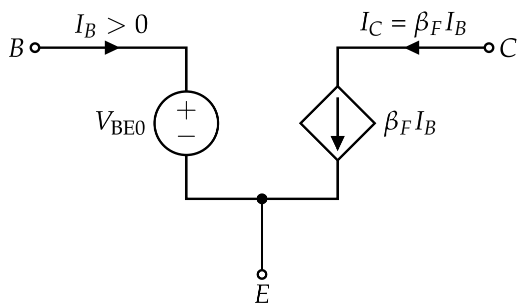
Figure 11: PWM Example with switching frequency 500Hz and 70% duty cycle

The PWM pin (V_A) is connected via a resistor (R_B) to the "Base (B)" of an NPN bipolar junction transistor (BJT). This transistor, in reality, behaves a bit differently from the NMOS with which you are familiar, but for this class, you may assume that it is functionally the same as an NMOS, behaving as a switch. On the BJT, the three terminals are analogous to those of an NMOS: the "Base (B)" is the gate, the "Collector (C)" is the drain, and the "Emitter (E)" is the source.

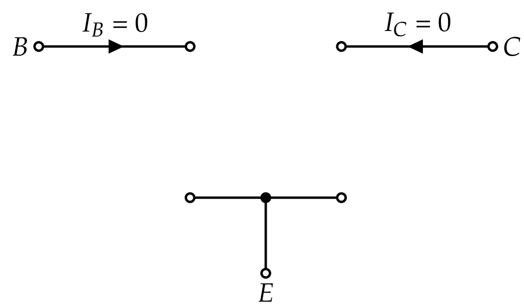
The BJT in Figure 10 is switching between **ON** and **OFF** modes when V_A is HIGH and LOW respectively. The model for both ON and OFF states are shown in Figure 12. When the BJT turns on, V_{BE}

can be modeled as a fixed voltage source with voltage value V_{BE0}. In ON mode, there is a **Current Controlled Current Source** modeled between "Collector (C)" and "Emitter (E)", i.e., current at the "Collector (C)" is an amplified version of current at the "Base (B)" (notice that positive I_B has to flow into the "Base (B)" for the relation $I_C = \beta_F I_B$ to hold). β_F is called the **Common-Emitter Current Gain**.

The diode in parallel with the motor is needed because of the inductive characteristics of the motor. If the motor is on and V_A switches to LOW, the inductive behavior of the motor maintains the current and the diode provides the path to dissipate it as the BJT is turned off. When the BJT turns on, the diode is off so there is no current flow through the diode.



(a) Model of BJT in ON mode (when V_A is logic HIGH)



(b) Model of BJT in OFF mode (when V_A is logic LOW)

Figure 12: Model of NPN BJT in Different Modes

Please use $V_{\text{BE0}} = 0.8\,\text{V}$, $\beta_F = 100$, $V_{A,\text{HIGH}} = 5\,\text{V}$ and $V_{A,\text{LOW}} = 0\,\text{V}$ for all following calculations.

Part 1: Circuit analysis to construct the system model

(a) **Draw the equivalent motor controller circuit when the BJT is ON by substituting in the BJT model from Fig. 12a into Fig. 10. Express I_B and I_C for $V_A = V_{A,\text{HIGH}}$ as a function of R_B.**

Solution: First, we draw the equivalent circuit in figure 13 to analyze the quantities which will appear in it: I_B and I_C.

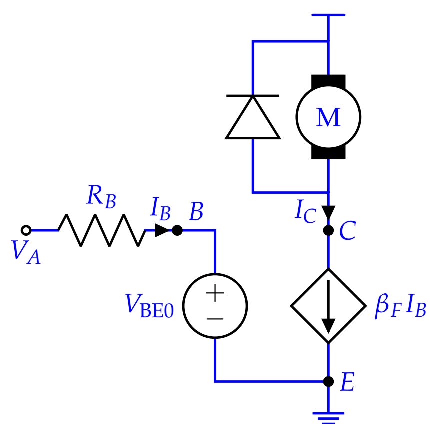


Figure 13: Equivalent circuit with model of BJT in ON state substituted in.

When V_A is HIGH, the voltage across the resistor in Figure 10 is $V_{A,\text{HIGH}} - V_{\text{BE0}}$. This voltage is what determines the the current flowing into the "Base(B)" (I_B) and current flowing into the "Collector (C)" (I_C).

$$I_B = \frac{V_{A,\text{HIGH}} - V_{\text{BE0}}}{R_B} = \frac{5\,\text{V} - 0.8\,\text{V}}{R_B} = \frac{4.2\,\text{V}}{R_B} \tag{106}$$

$$I_C = \beta_F I_B = 100\frac{4.2\,\text{V}}{R_B} = \frac{420\,\text{V}}{R_B} \tag{107}$$

Observe that downward directed current through the motor when V_A is HIGH is equal to the current I_C as the diode will not allow current in the direction from the motor voltage source to the collector (C).

(b) **Draw the equivalent motor controller circuit when the BJT is OFF by substituting in the BJT model from Fig. 12b into Fig. 10. Express I_B and I_C for V_A = $V_{A,\text{LOW}}$ as a function of R_B.**

Solution: We draw the equivalent circuit as before.

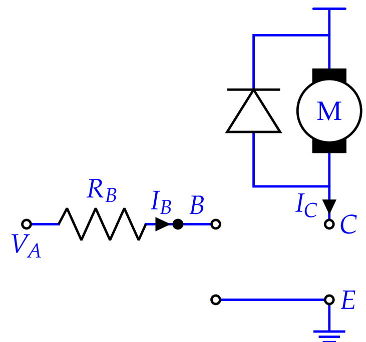


Figure 14: Equivalent circuit with model of BJT in OFF state substituted in.

When V_A is LOW, the BJT is OFF. As shown in Figure 14, the BJT is open from B to E, and from C to E. Thus $I_B = 0$ and $I_C = 0$.

Consider that while I_C can be zero, the current through the motor may not be as the diode conductive path allows for the inductor current of the motor to keep flowing. If we model the motor and diode as a circuit with a small resistor, R, and inductor, L, in series, we see that for a small R, we will have a large time constant $\tau = \frac{L}{R}$, which means that the current does not decay very much through the motor during the time the BJT is in the OFF state.

(c) **Derive the average collector current, I_{AVG}, over one period, T_{period}, of the PWM signal, V_A, as a function of R_B and the duty cycle, D, of the PWM signal.** Hint: The time average of some signal $f(t)$ from time t_0 to t_1 is given as $f_{\text{AVG}} = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} f(\tau)\,\mathrm{d}\tau$. Figure 11 may be useful.

Solution: We compute the time average of I_C from $t = 0$ to $t = T_{\text{period}}$, where T_{period} is the period of the PWM voltage.

$$I_{\text{AVG}} = \frac{1}{T_{\text{period}} - 0}\int_0^{T_{\text{period}}} I_C(\tau)\,\mathrm{d}\tau \tag{108}$$

Since I_C depends on V_A, and V_A switches between two values $V_{A,\text{HIGH}}$ and $V_{\text{A,LOW}}$, we will have corresponding values of current $I_{C,\text{ON}}$ and $I_{C,\text{OFF}}$. If we start at time 0, we spend times $0 \le t \le T_{\text{on}}$ with constant current $I_{C,\text{ON}}$ due to constant voltage $V_A = V_{A,\text{HIGH}}$. This first current $I_{C,\text{ON}}$ is the current we calculated in part (a). At times $T_{\text{on}} \le t \le T_{\text{period}}$, we turn our voltage V_A to its low value, with corresponding constant current $I_{C,\text{OFF}}$. This is the current we calculated in part (b). Our integral becomes the following:

$$\begin{aligned}
I_{\text{AVG}} &= \frac{1}{T_{\text{period}}}\int_0^{T_{\text{period}}} I_C(\tau)\,\mathrm{d}\tau & (109)\\
&= \frac{1}{T_{\text{period}}}\left(\int_0^{T_{\text{on}}} I_{C,\text{ON}}\,\mathrm{d}\tau + \int_{T_{\text{on}}}^{T_{\text{period}}} I_{C,\text{OFF}}\,\mathrm{d}\tau\right) & (110)\\
&= \frac{1}{T_{\text{period}}}\Big(I_{C,\text{ON}}(T_{\text{on}} - 0) + I_{C,\text{OFF}}(T_{\text{period}} - T_{\text{on}})\Big) & (111)
\end{aligned}$$

$$
\begin{aligned}
&= I_{C,\text{ON}} \frac{T_{\text{on}}}{T_{\text{period}}} + I_{C,\text{OFF}} \frac{T_{\text{period}} - T_{\text{on}}}{T_{\text{period}}} && (112)\\
&= I_{C,\text{ON}} D + \cancel{0\,\text{A}(1 - D)} && (113)\\
&= I_{C,\text{ON}} D && (114)\\
&= \frac{420\,\text{V}}{R_B} D && (115)
\end{aligned}
$$

(d) **In the previous part, explain briefly why is it sufficient to take the average over only one period if we are actually interested in the average collector current over multiple periods?**

Solution: It is enough to take the average over one period as this would be the same as the average that would occur over an large integer number of periods and close enough to the average for a large non-integer number of periods. Reasoning of this form is enough.

What follows is a mathematical statement of the intuition above for why we can look at only one period. It is not required but is here for your understanding. The assumption of having a high PWM frequency is relevant here, as a high frequency implies a small period. We should have our period T be small relative to the time we set a certain duty cycle value, so that the time spent in a fraction of a cycle at the very end of our string of full periods does not contribute much to the average and will be negligible. In notation, let the time we average over be from $t = 0$ to $t = T = NT_{\text{period}} + \delta$, where N is an integer and δ is a time interval that is mid-cycle and not through the full period, i.e. $0 \le \delta \le T_{\text{period}}$.

$$
\begin{aligned}
I_{\text{AVG}} &= \frac{1}{T} \int_0^T I_C(t)\, \mathrm{d}\tau && (116)\\
&= \frac{1}{T} \left(\sum_{i=0}^{N-1} \left(\int_{iT_{\text{period}}}^{(i+1)T_{\text{period}}} I_C(t)\, \mathrm{d}\tau \right) + \int_{NT_{\text{period}}}^{NT_{\text{period}}+\delta} I_C(t)\, \mathrm{d}\tau \right) && (117)\\
&= \frac{1}{T} \left(\sum_{i=0}^{N-1} I_{C,\text{ON}} D + I_{C,\text{ON}} \min(\delta, T_{\text{on}}) \right) && (118)\\
&= \frac{1}{T} \left(N I_{C,\text{ON}} T_{\text{on}} + I_{C,\text{ON}} \min(\delta, T_{\text{on}}) \right) && (119)\\
&= \frac{NT_{\text{on}}}{NT_{\text{period}} + \delta} I_{C,\text{ON}} + \frac{\min(\delta, T_{\text{on}})}{NT_{\text{period}} + \delta} I_{C,\text{ON}} && (120)
\end{aligned}
$$

The minimum term comes from that the time mid-cycle δ may be less than or greater than T_{on}, so we may have current on for a shorter interval or have an interval in which we have turned our current off after spending the full time T_{on} with the current on. However, notice that as we make N large, which is to say that our time T we are averaging over is large relative to our period T_{period}, our average value approaches $\frac{T_{\text{on}}}{T_{\text{period}}} I_{C,\text{ON}} = D I_{C,\text{ON}}$, where we have our duty cycle. This is what allows our duty cycle to determine our average current and voltage value, and why we should have a high enough PWM frequency for the system in consideration. If we were to only hold our duty cycle for a short period of time, on par with the period of the PWM signal, we lose this dependence of the average current on duty cycle and would have behavior that is a function of the time we hold our duty cycle for. This is an example of the important idea in control and computing that the parameters of a system (e.g. R, C in transistors, PWM frequency here) are chosen in such a way that trades off performance (usually speed) and reliability.

(e) **If $R_B = 2\,\text{k}\Omega$, what is the average collector current, I_{AVG}, that drives the motor when the duty cycle of the PWM signal is equal to 25%?**

Solution: From part (c), we have that $I_{\text{AVG}} = \frac{420\,\text{V}}{R_B} D$. For $R_B = 2\,\text{k}\Omega$ and $D = 0.25$ (25%), $I_{\text{AVG}} = \frac{420\,\text{V}}{2\,\text{k}\Omega} 0.25 = 52.5\,\text{mA}$.

Part 2: Learning a Car Model from data
To control the car, we need to build a model of the car first. Instead of designing a complex nonlinear model, we will approximate the system with a linear model to work for small perturbations around an equilibrium point. The following model applies separately to each wheel (and associated motor) of the car:

$$v_L[i] = \theta_L u_L[i] - \beta_L \tag{121}$$
$$v_R[i] = \theta_R u_R[i] - \beta_R \tag{122}$$

Notice that this particular model has no state variables since we are measuring velocity directly here. To do system ID, we decide to use the exact same input $u_L[i] = u_R[i] = u[i]$ for both motors. We measure both velocities however.
Meet the variables at play in this model: **(Note: the β here have nothing to do with the previous part.)**

- i - The current timestep of the model. Since we model the car as a discrete-time system, n will advance by 1 on every new sample in the system.
- $v_L[i]$ - The discrete-time velocity (in units of ticks/timestep) of the left wheel, reading from the motor.
- $v_R[i]$ - The discrete-time velocity (in units of ticks/timestep) of the right wheel, reading from the motor.
- $u[i]$ - The input to each wheel. The duty cycle of the PWM signal (V_A), which is the percentage of the square wave's period for which the square wave is HIGH, is mapped to the range $[0, 255]$. Thus, $u[i]$ takes a value in $[0, 255]$ representing the duty cycle. For example, when $u[i] = 255$, the duty cycle is 100 %, and the motor controller just delivers a constant signal at the system's HIGH voltage, delivering the maximum possible power to the motor. When $u[i] = 0$, the duty cycle is 0 %, and the motor controller delivers 0 V. The duty cycle (D) can be written as

$$\text{duty cycle (D)} = \frac{u[i]}{255} \tag{123}$$

- $\theta(\theta_L, \theta_R)$ - Relates change in input to change in velocity. **Its units are ticks/(timestep $\cdot$ duty cycle).** Since our model is linear, we assume that θ is the same for every unit increase in $u[i]$. This is empirically measured using the car. **You will have a separate θ for your left and your right wheel(θ_L, θ_R).**
- $\beta(\beta_L, \beta_R)$ - Similarly to θ, β is dependent upon many physical phenomena, so we will empirically determine it using the car. β represents a constant offset in the velocity of the wheel, and hence **its units are ticks/timestep**. Note that you will also typically have a different β for your left and your right wheel (i.e. $\beta_L \neq \beta_R$). **These β_L and β_R are different from the β_F of the transistor.**

(f) By measuring the car with a PWM signal at different duty cycles, we can collect the velocity data of the left and right wheel, as shown in the following table:

Table 1: The velocity of the left and the right wheel at different duty cycles of PWM signal

Duty Cycle $\times 255$ ($u[i]$)	Velocity of the left wheel ($v_L[i]$)	Velocity of the right wheel ($v_R[i]$)
80	147	127
120	218	187
160	294	253
200	370	317

Since the same input is applied to both the wheels, we can take advantage of the same "horizontal stacking" trick you've seen before to be able to reuse computation. To identify the system we need to setup matrix equations of left and right wheel in the form of:

$$D_{\text{data}} P \approx S \tag{124}$$

where $P = \begin{bmatrix} \theta_L & \theta_R \\ \beta_L & \beta_R \end{bmatrix}$. **Find the matrix D_{data} and matrix S needed to perform system identification to get the matrix of parameters of the left and right wheel, P.**

Solution: We can relate our data points in our table using the model equations eqs. (121) and (122). While we do not know the time indices, we can call them i_1 to i_4. We start with the left wheel speed data. Approximate equality is used due to the possible presence of noise or unmodelled behavior.

$$v_L[i_1] \approx \theta_L u[i_1] - \beta_L \tag{125}$$
$$v_L[i_2] \approx \theta_L u[i_2] - \beta_L \tag{126}$$
$$v_L[i_3] \approx \theta_L u[i_3] - \beta_L \tag{127}$$
$$v_L[i_4] \approx \theta_L u[i_4] - \beta_L \tag{128}$$

Putting the left and right hand sides into vectors, we get the following equation.

$$\underbrace{\begin{bmatrix} v_L[i_1] \\ v_L[i_2] \\ v_L[i_3] \\ v_L[i_4] \end{bmatrix}}_{\vec{s}_L} \approx \begin{bmatrix} \theta_L u[i_1] - \beta_L \\ \theta_L u[i_2] - \beta_L \\ \theta_L u[i_3] - \beta_L \\ \theta_L u[i_4] - \beta_L \end{bmatrix} = \begin{bmatrix} u[i_1] & -1 \\ u[i_2] & -1 \\ u[i_3] & -1 \\ u[i_4] & -1 \end{bmatrix} \underbrace{\begin{bmatrix} \theta_L \\ \beta_L \end{bmatrix}}_{\vec{p}_L} \tag{129}$$

Here, $\vec{s}_L$ is defined as the vector of measured wheel velocities, and $\vec{p}_L$ is the vector of parameters for the left wheel. We will have corresponding vectors $\vec{s}_R$ and $\vec{p}_R$ for the right wheel when writing a similar set of equations with $v_R[i]$, θ_R, and β_R. These equations utilize the same values of $u[i]$.

$$\underbrace{\begin{bmatrix} v_R[i_1] \\ v_R[i_2] \\ v_R[i_3] \\ v_R[i_4] \end{bmatrix}}_{\vec{s}_R} \approx \begin{bmatrix} u[i_1] & -1 \\ u[i_2] & -1 \\ u[i_3] & -1 \\ u[i_4] & -1 \end{bmatrix} \underbrace{\begin{bmatrix} \theta_R \\ \beta_R \end{bmatrix}}_{\vec{p}_R} \tag{130}$$

Since the matrix multiplying the parameter vectors are the same, we can concatenate $\vec{s}_L$, $\vec{s}_R$ and $\vec{p}_L$, $\vec{p}_R$ horizontally in the following way.

$$\underbrace{\begin{bmatrix} | & | \\ \vec{s}_L & \vec{s}_R \\ | & | \end{bmatrix}}_{S} \approx \underbrace{\begin{bmatrix} u[i_1] & -1 \\ u[i_2] & -1 \\ u[i_3] & -1 \\ u[i_4] & -1 \end{bmatrix}}_{D_{\text{data}}} \underbrace{\begin{bmatrix} | & | \\ \vec{p}_L & \vec{p}_R \\ | & | \end{bmatrix}}_{P} \tag{131}$$

Substituting in the numerical values we have our D_{data} and S matrices.

$$D_{\text{data}} = \begin{bmatrix} 80 & -1 \\ 120 & -1 \\ 160 & -1 \\ 200 & -1 \end{bmatrix} \qquad S = \begin{bmatrix} 147 & 127 \\ 218 & 187 \\ 294 & 253 \\ 370 & 317 \end{bmatrix} \tag{132}$$

(g) **Solve the matrix equation $D_{\text{data}}P \approx S$ with least squares to find θ_L, θ_R, β_L, and β_R.** You may use a jupyter notebook for computation.

Solution: Our estimate of our parameter matrix is given by $\widehat{P} = \left(D_{\text{data}}^{\top} D_{\text{data}}\right)^{-1} D_{\text{data}}^{\top} S$. Using a computing tool, we can compute the matrix product or use a built-in least squares function (e.g. NumpPy's `np.linalg.lstsq`). We get the following parameter value estimates: $\widehat{\theta}_L = 1.862$, $\widehat{\theta}_R = 1.59$, $\widehat{\beta}_L = 3.5$, and $\widehat{\beta}_R = 1.6$.

(h) In most advanced systems, we usually use a combination of a physics-based equation and a data-centric approach to build the model. In our case, the velocity of the motor can be written as

$$v[i] = kI_{\text{AVG}}(u[i]) - \beta \tag{133}$$

where $I_{\text{AVG}}(u[i])$ is the average collector current which is the function of the duty cycle that you have already derived in Part 1. k represents the response of your motor speed to the average current. In our simplified motor driver model in Part 1, you have already derived the expression for the I_{AVG} of the motor as a function of the circuit parameters and the duty cycle D. **If we assume that the model from Part 1 holds, determine the resistance ratio $\left(\frac{R_{\text{B,left}}}{R_{\text{B,right}}}\right)$ from the model parameters you identified in Part 2 item (f).** Assume that the left motor and the right motor respond the same, that is, $k_L = k_R$. The only difference is presumed to come from the resistors used.

Solution: First, we investigate how the current affects our model by starting with the velocity equation involving I_{AVG} for just one wheel.

$$v[i] = kI_{\text{AVG}}(u[i]) - \beta \tag{134}$$

$$= k\frac{420\,\text{V}}{R_B}D(u[i]) - \beta \tag{135}$$

After substituting in our average current expression from part (c), the only quantity that can be a function of $u[i]$ is our duty cycle, D. We recall from the introduction to part 2 that $D = \frac{u[i]}{255}$.

$$v[i] = k\frac{420\,\text{V}}{R_B}\left(\frac{u[i]}{255}\right) - \beta \tag{136}$$

$$= \left(k\frac{420\,\text{V}}{255R_B}\right)u[i] - \beta \tag{137}$$

From our earlier learned model equations in eqs. (121) and (122), we can match coefficients and say that $\theta = k\frac{420\,\text{V}}{255R_B}$. Since we identified $\widehat{\theta}_L = 1.862$ and $\widehat{\theta}_R = 1.59$ in part (f), we can take their ratio to see common terms cancel out.

$$\frac{\widehat{\theta}_R}{\widehat{\theta}_L} \approx \frac{k_R\frac{420\,\text{V}}{255R_{\text{B,right}}}}{k_L\frac{420\,\text{V}}{255R_{\text{B,left}}}} = \frac{R_{\text{B,left}}}{R_{\text{B,right}}} \tag{138}$$

$$\frac{1.59}{1.862} \approx \frac{R_{\text{B,left}}}{R_{\text{B,right}}} \tag{139}$$

$$0.85 \approx \frac{R_{\text{B,left}}}{R_{\text{B,right}}} \tag{140}$$

(i) In order for the car to drive straight, the wheels must be moving at the same velocity. However, the data from Table 1 tell us that two motors cannot run at the same velocity if the duty cycles of driving PWM signals are the same. **Based on the model you extracted in Part 2 item (f), if we want the car to drive straight and $u_L = 100$, what should u_R be?**

Solution: To have the car drive straight, we want to find u_R satisfying $v_L = v_R$ for $u_L = 100$. From our identified constants, we can write the following equations.

$$v_L = 1.862u_L - 3.5 \tag{141}$$

$$v_R = 1.59u_R - 1.6 \tag{142}$$

Setting v_L and v_R equal to each other with $u_L = 100$, we get the following value of u_R.

$$1.862(100) - 3.5 = 1.59u_R - 1.6 \implies u_R \approx 115.91 \approx 116 \tag{143}$$

This observation was not required, but let us interpret the results of (f) and this part, (h). The significance of this calculation based on our learned parameters is that this difference in the required u_L and u_R required to have the same wheel velocities is a reflection of the underlying physics of our system. Taking the ratio $\frac{u_L}{u_R} = \frac{100}{116} \approx 0.86$, we find that we have nearly the same ratio as in (f). This is not a coincidence. u_L and u_R are proportional to the average voltages supplied, $V_{\text{A,L,AVG}}$ (left average voltage) and $V_{\text{A,R,AVG}}$ (right average voltage). If the resistance $R_{\text{B,right}}$ is larger than $R_{\text{B,left}}$ per part (f), this implies for the same average voltage applied to both motors, we will see a lower average current driving the right motor, thereby making its speed lower, assuming $k_L = k_R$ in the physical model equation. Note however, the ratios are not exactly equal for the reason of the β_L and β_R. However, because they are small relative to the larger values of $\theta_L u_L$ and $\theta_R u_R$ we see that we will have approximate equality, $\frac{u_L}{u_R} \approx \frac{R_{\text{B,left}}}{R_{\text{B,right}}}$, to move straight with $u_L = 100$.

This is to say that in our systems we will always have to account for the variation in the subparts of our system that are not necessarily identical, matched, or ideal. We made an assumption in part (f) that $k_L = k_R$, and this could introduce another adjustment required.

9. (OPTIONAL) Make Your Own Problem.

Write your own problem about content covered in the course thus far, and provide a thorough solution to it.

NOTE: This can be a totally new problem, a modification on an existing problem, or a Jupyter part for a problem that previously didn't have one. Please cite all sources for anything (including course material) that you used as inspiration.

NOTE: High-quality problems may be used as inspiration for the problems we choose to put on future homeworks or exams.

10. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!

We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) **What sources (if any) did you use as you worked through the homework?**

(b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

(c) **Roughly how many total hours did you work on this homework? Write it down here where you'll need to remember it for the self-grade form.**

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