Homework 6

This homework is due on Friday, March 4, 2022, at 11:59PM. Self-grades and HW Resubmissions are due on the following Friday, March 11, 2022, at 11:59PM.

1. **Survey**

   Please fill out the following online surveys.

   (a) **Mask survey(link)**: We want to understand student sentiment about the mask mandate lifting on 03/07 and how this may affect them.
   NOTE: We STRONGLY RECOMMEND that students continue wearing their masks after the mandate is lifted - please be mindful and empathetic that those around you may not necessarily be comfortable if you are maskless.

   (b) **(OPTIONAL) Note Feedback(link)**: This survey is totally anonymous and made with the intent to improve our notes for students (especially for future semesters).

2. Reading Lecture Notes

Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: Note 7, Note 8, Note 9, Note 10, and Note 11.

(a) In the magnitude Bode plot for a first-order-low-pass filter drawn on a log-log graph (i.e., $|H(\mathrm{j}\omega)|$ is plotted on a log scale), **what is the slope of the straight line approximation at frequencies higher than the cut-off frequency? What is the slope for a third-order-low-pass filter (three identical first-order-low-pass filters cascaded with unity gain buffers)?**

(b) If you know a filter has an attenuation of $|H(j\omega_0)| = 0.95$ at ω_0, **what would the attenuation or gain be when you cascade** 10 **of these filters at** ω_0**?** If you know this same filter has a gain of $|H(j\omega_1)| = 1.1$ at some different angular frequency ω_1, **what would the gain be when you cascade** 10 **of these filters at** ω_1**?**

(c) Suppose we had a continuous-time differential equation model

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}_c(t) = A_c\vec{x}_c(t) + B_c\vec{u}_c(t) \tag{1}$$

and we wanted to discretize it to get a model of the form

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + B_d\vec{u}_d[i] \tag{2}$$

where $\vec{x}_d[i] = \vec{x}_c(i\Delta)$ and the input $\vec{u}_c(t)$ is generated to be piecewise constant so $\vec{u}_d[i]$ is the value of $\vec{u}_c(t)$ for the whole time interval $t \in [i\Delta, (i+1)\Delta)$.

Describe the steps that you would take to get A_d **and** B_d **from** A_c **and** B_c**.** Feel free to assume that A_c is diagonalizable.

(d) **What relevance does the following property of matrix multiplication have to system identification of vector systems by means of least-squares?**

$$AB = A\begin{bmatrix} | & & | \\ \vec{b}_1 & \cdots & \vec{b}_N \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ A\vec{b}_1 & \cdots & A\vec{b}_N \\ | & & | \end{bmatrix} = C = \begin{bmatrix} | & & | \\ \vec{c}_1 & \cdots & \vec{c}_N \\ | & & | \end{bmatrix}$$

(e) **What is a condition for a discrete time scalar system to be BIBO stable? What is a condition on the eigenvalues for a discrete time vector system to be BIBO stable? What is a condition on the eigenvalues for a continuous time system to be BIBO stable?**

3. **Bandpass Filter: Lowpass and Highpass Cascade**

   In lecture, you heard about how you can go through the design of a bandpass filter by cascading lowpass and highpass filters via buffers (op-amps in unity-gain negative feedback to prevent loading effects). In this problem, you will do this for yourself.

   Consider an input signal that is composed of the superposition of:

   - $A_p := 20\,\text{mV}$ level pure tone at frequency $f_p := 60\,\text{Hz}$ and phase ϕ_p corresponding to power line noise.
   - $A_v := 1\,\text{mV}$ level pure tone at frequency $f_v := 600\,\text{Hz}$ and phase ϕ_v corresponding to a voice signal.
   - $A_f := 10\,\text{mV}$ level pure tone at frequency $f_f := 60\,\text{kHz}$ and phase ϕ_f corresponding to fluorescent light control electronics noise.

   We would like to keep the 600 Hz tone, which could correspond to a voice signal.

   NOTE: The phases ϕ are symbolic – we do not provide numerical values – but the amplitudes A are not symbolic.

   (a) **Write the $V_{\text{in}}(t)$ that describes the above input in time domain, in the following format:**.

   $$V_{\text{in}}(t) = A_p \cos(2\pi f_p t + \phi_p) + A_v \cos(2\pi f_v t + \phi_v) + A_f \cos\Big(2\pi f_f t + \phi_f\Big) \tag{3}$$

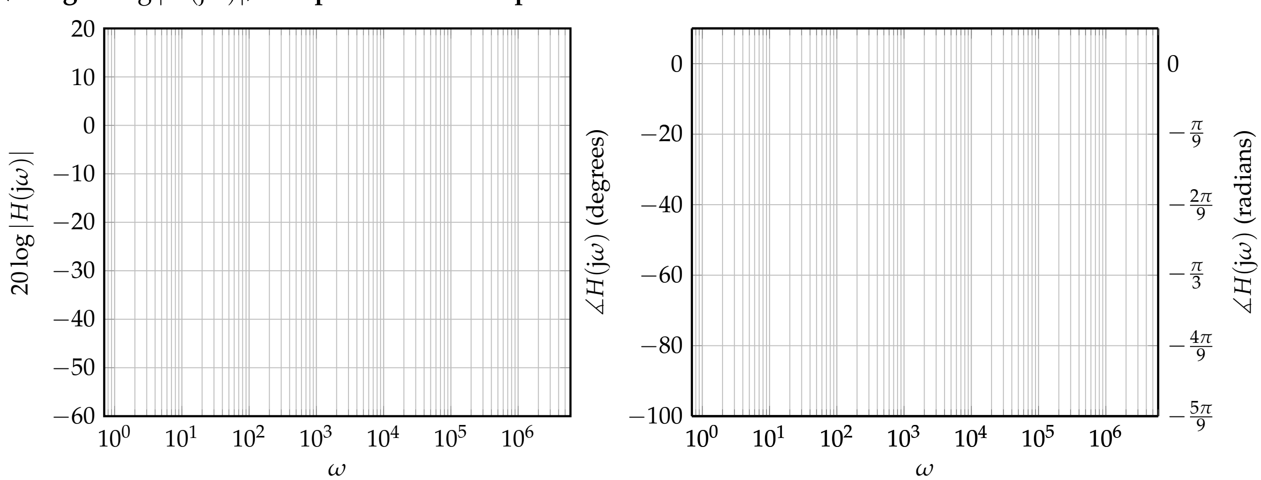
   (b) **What are the angular frequencies (i.e., $\omega_p, \omega_v, \omega_f$) involved and the phasors associated with each tone?** Remember that the frequencies of the tones are provided in Hz. To convert these frequencies to angular frequencies, we use $\omega = 2\pi f$.

   NOTE: This scenario is common in applications; usually, the data collected is in "regular" frequencies, but the analysis requires angular frequencies.

   (c) To achieve your goal of keeping the voice tone but rejecting the noise from the power-lines and fluorescent lights, **at what frequency do you want to have the cutoff frequency for the lowpass filters**?

   (HINT: To arrive at a unique solution consider computing the geometric mean (the analogous quantity to the arithmetic mean on a log scale) of the two frequencies of interest.)

   (d) **Draw the Bode plot (straight-line approximations to the transfer function) for the magnitude (using $20\log|H(j\omega)|$) and phase of the lowpass filter.**



   (e) To achieve your goal of keeping the voice tone but rejecting the noise from the power-lines and fluorescent lights, **at what frequency do you want to have the cutoff frequency for the highpass filters**?

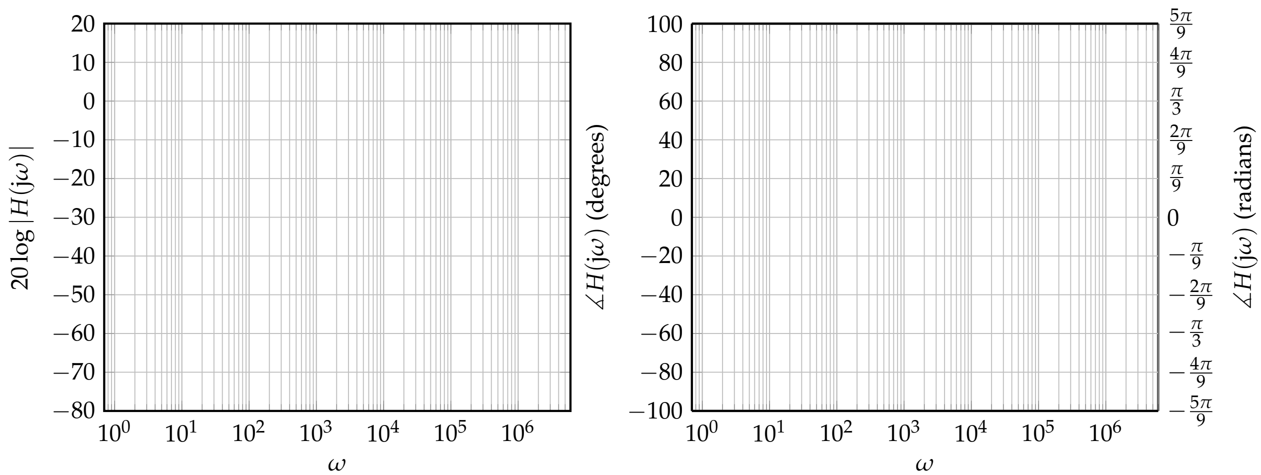
   (HINT: To arrive at a unique solution consider computing the geometric mean (the analogous quantity to the arithmetic mean on a log scale) of the two frequencies of interest.)

(f) **Draw the Bode plot (straight-line approximations to the transfer function) for the magnitude (using $20\log|H(j\omega)|$) and phase of the highpass filter.**

(g) For the following questions, assume your cut-off frequencies for lowpass and highpass are 6 kHz and 189 Hz respectively. Suppose that you only had 1 µF capacitors to use. **What resistance values would you choose for your highpass and lowpass filters so that they have the desired cutoff frequencies?**

(h) The overall bandpass filter that is created by cascading the lowpass and highpass with ideal buffers in between. **Draw the Bode plot (straight-line approximations to the transfer function) for the magnitude and phase of the overall bandpass transfer function.**

(HINT: You should think about how the Bode plot of a cascade of two filters can be derived based on the Bode plots of the lower-level filters.)



(i) Suppose that the bandpass filter does not have enough suppression at 60 Hz and 60 kHz. You decide to use a cascade of three bandpass filters (with unity-gain buffers in between) (as shown in Figures 1 and 2). **What are the phasors for each of the frequency tones after all three bandpass filters?**

(HINT: Remember how you determined the transfer function of the bandpass filter from the transfer functions of the lowpass and highpass filters.)

Feel free to use a computer to help you evaluate both the magnitudes and the phases here.

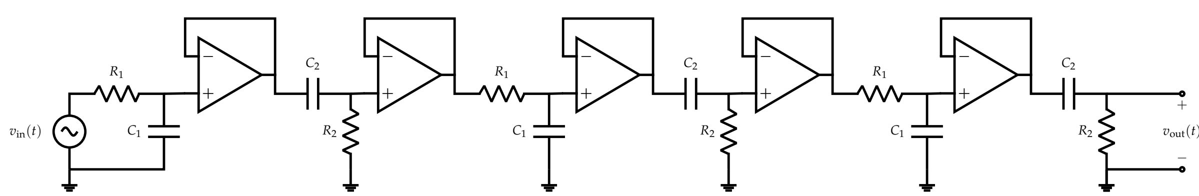


Figure 1: "Time-domain" circuit: Cascade of the three bandpass filters, using buffers to avoid loading.

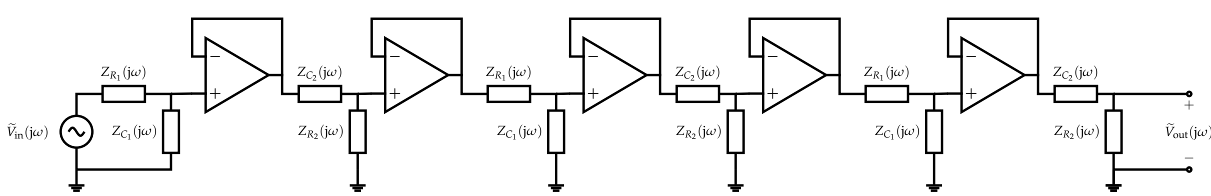
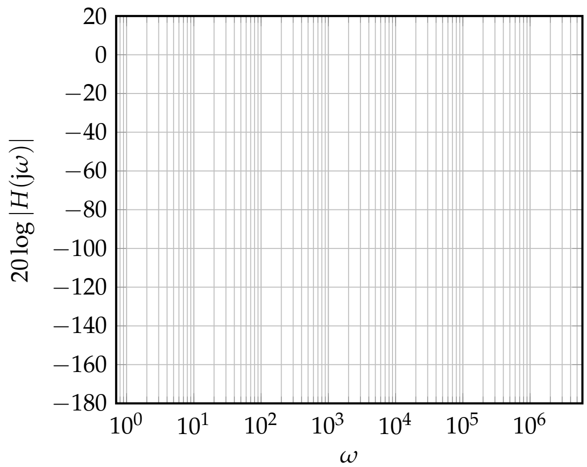
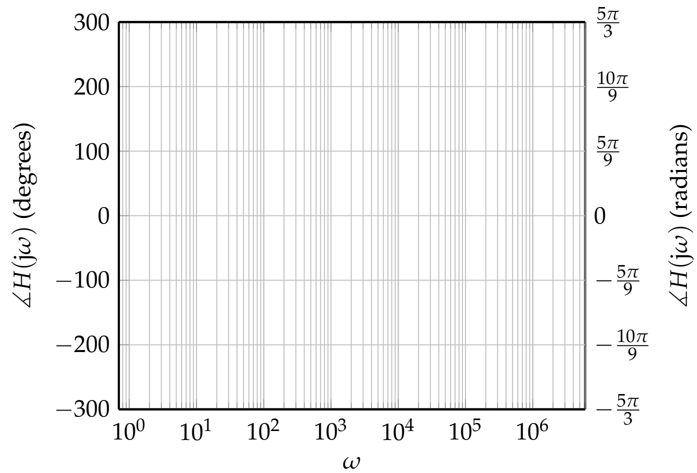


Figure 2: "Phasor-domain" circuit: Cascade of the three bandpass filters, using buffers to avoid loading.

(j) **Draw the Bode plots (straight-line approximations to the transfer function) for the magnitude and phase of the 3rd order bandpass filter.** To highlight the difference between the 3rd and 1st order filters, please draw both Bode plots on a single figure.





(k) **Write the final time domain voltage waveform that would be present after the filter.**

(l) The included Jupyter notebook `filter_cascade.ipynb` sets up the same problem described above. In the notebook, you can use the slider bars to play around with:

- highpass cutoff frequency f_{highpass} (i.e., the knee frequency of the highpass filters)
- lowpass cutoff frequency f_{lowpass} (i.e., the knee frequency of the lowpass filters)
- Filter order N. Filter order means the number of lowpass filters and highpass filters that are used in a row. Here, N means that there are N lowpass filters and N highpass filters, so the overall order of the entire filter is actually $2N$.

The notebook will plot the magnitude and phase, the input voltage waveform, and the output waveform at the end of the filter.

Play around with the values for the highpass and lowpass cutoff frequencies, and N.

Observe the waveforms at the output of the filter. **Comment on the values of f_{lowpass}, f_{highpass}, and N that you can use to successfully isolate the desired 600 Hz tone. What happens if you keep f_{lowpass} and f_{highpass} constant, and just increase N?**

4. (PRACTICE) Time-Domain: Boost Converter

Below is a circuit called a Boost Converter. The boost converter is a simple version of a circuit that is very common in battery-operated devices. While a single alkaline battery has a voltage that varies from 1.6 V when it is fresh to 0.8 V at end of life, most electronic devices require a constant supply voltage that stays within ±10% of a certain nominal value. That nominal value is often greater than 1.6 V. So how do we power electronics with batteries? And in particular, how do we power electronics that wants 3 V or 5 V to function when our battery is providing significantly less than that?

The boost converter is a commonly used circuit to provide that stable supply voltage, even from a battery whose voltage is slowly decreasing as it discharges. With proper choice of components and proper control of the gate voltage V_G of the transistor, power from the battery can be converted to higher voltages with almost perfect efficiency.

Consider the following circuit:

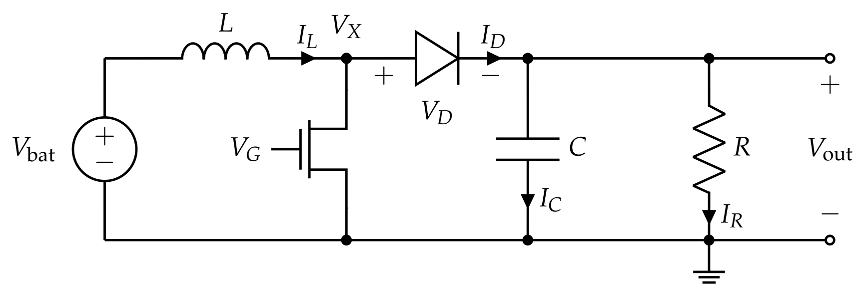


Figure 3: Boost converter schematic.

with

$$V_{\text{bat}} = 1\,\text{V} \tag{4}$$
$$L = 1\,\mu\text{H} \tag{5}$$
$$C = 1\,\text{mF} \tag{6}$$
$$R = 1\,\text{k}\Omega. \tag{7}$$

We model the electronic component that is consuming power as a resistor R with voltage V_{out} across it. Assume that the gate voltage at the NMOS is low (and thus the NMOS is an open circuit) for $t < 0$.

This circuit introduces a new nonlinear circuit element called a diode. The symbol for a diode is shown in Fig. 4. A diode has a certain voltage across of it, V_D. The direction of this diode voltage is important: the + side of the diode voltage is at the base of the triangle and the - side of the diode voltage is at the vertical bar. A diode allows current to flow only in one direction, from the + side to the - side (the triangle of the diode symbol points towards the direction of current). The diode conducts current only when the diode voltage V_D is greater than some threshold. Otherwise, no current flows through it.

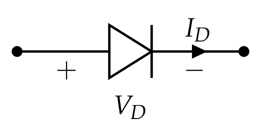


Figure 4: Symbol of a diode. The arrow denotes the direction that current is allowed to flow.

In the real world, this diode voltage threshold is some positive value, usually 0.7V, and the current through it is some function of V_D. For this problem, let's pretend we have an ideal diode. That is, if $V_D \geq 0$, then the diode allows any amount of current it needs to, i.e. a wire (short circuit). Conversely, if $V_D < 0$, the diode conducts no current, i.e. an open circuit. Fig. 5 shows the current-voltage (I-V) characteristic curves for both a real and ideal diode.

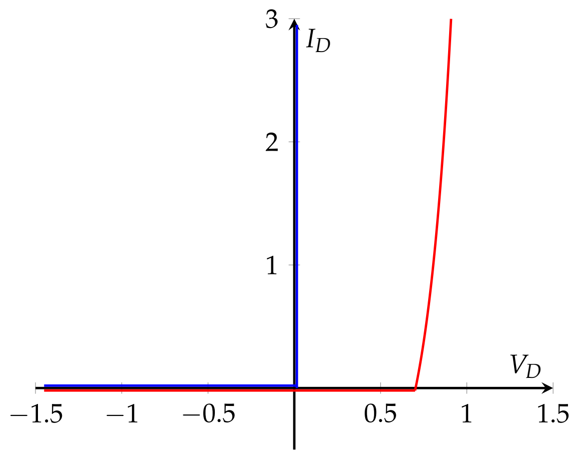


Figure 5: I-V curves for a real diode (red) and an ideal diode (blue). The real diode has a threshold voltage of 0.7V; the ideal diode has a threshold voltage of 0V. The real diode has a current that depends on the voltage when on; the ideal diode acts as a short (0Ω impedance) when on. Both conduct no current when off.

(a) Assume that there is no current flowing through the diode — treat it like an open circuit. **Calculate the time constant of the RC circuit. How long will it take for the output voltage to decay by** 10% **from any non-zero value?** If the voltage on the capacitor is 3 V at $t = 0$, **what is the current in the resistor at time** $t = 0$**? What is the voltage in the capacitor for** $t > 0$**?**

(b) Let's assume that at $t = 0$, the voltage on the capacitor is 3 V. At $t = 0$, the gate voltage on the NMOS V_G goes high, turning the transistor on (effectively shorting that circuit path). **What is the rate of change of current in the inductor? How long does it take for the current in the inductor to increase to** 100 mA**?**

(c) In the time that the inductor current changes to 100mA, **how much does the capacitor voltage decay in that time?** Round your answer to 7 decimal places. Use your equation for $V_{out}(t)$ from part (a) and the final time you calculated in part (b).

(d) You should have found in the previous part that the output voltage remains very close to 3V. This is due to the large output capacitor not allowing the voltage to change quickly. In fact, the output voltage remains so close to 3V that we approximate it as constant. In other words, in the short time that the inductor current changes to 100 mA, we approximate $V_{out} \approx 3V$ throughout. Now let's say that when the inductor current reaches 100 mA, the transistor gate voltage goes low and the transistor turns off. **Is the diode on or off? What is the node voltage** V_X**?** *(HINT: What is the value of the inductor current right after the transistor switches off? Conduct KCL at node* V_X*.)*

(e) After the transistor turns off from the previous part at $t = 0.1\mu s$, **what is the rate of change of current in the inductor? How long does it take the current in the inductor to go to zero (i.e. for the inductor to fully discharge)?** Assume the inductor discharges much faster than the capacitor, i.e. we can still approximate the output voltage as constant $V_{out} = 3V$.

(f) **Sketch the following currents vs time:** $I_L(t)$**,** $I_{\text{NMOS}}(t)$**,** $I_{\text{D}}(t)$ **for a single inductor charge/discharge cycle described above. For each plot, plot current on the vertical axis in mA and time on the horizontal axis in ns from t=0ns to 200ns.**

(g) The goal of this Boost converter circuit is to maintain an average output voltage of V_{out} at 3 V. We've seen this is done by turning the transistor on and off at the right times, allowing the inductor to energize and discharge its current through the diode to the capacitor which itself is losing energy to the load. To maintain the constant output voltage on average, over long periods of time (where the capacitor decay would become noticeable if left on its own), we need the average current through the diode to bring enough charge to replenish the capacitor. Recall that we model the load as a $1k\Omega$ resistor, and we want to maintain an output voltage of $V_{\text{out}} = 3$ V. We want to determine how quickly we need to toggle the transistor to achieve this.

What is the average current drawn by the resistive load? Using your plot of $I_D(t)$ from the previous part, calculate the charge that flows through the diode in a single cycle. At what frequency does the transistor need to cycle through the on/off phases to supply the current consumed by the resistor? Recall that the charge Q supplied by a current over a time interval T is the time integral of that current, i.e. the area under the graph. The average current I_{avg} for that supplied charge is $I_{avg} = \frac{Q}{T}$. Your job is to find T and consequently the frequency $f = \frac{1}{T}$ that you toggle the transistor on and off.

(h) **(PRACTICE)** The included Jupyter notebook `boost_converter.ipynb` sets up the same problem described above. In the notebook, you can use the slider bars to play around with:

- The cycle frequency for the transistor
- The component values R, L, and C
- The battery voltage V_{bat}

How does the inductor current curve in part (e) change if the battery voltage changes? What is the effect of a larger or smaller capacitor? A larger or smaller load?

5. Circuit Design

In this problem, you will find a circuit where several components have been left *blank* for you to fill in.

Assume that the op-amp is *ideal*. A special note on op amps in frequency domain analysis: The op-amps you learned about in 16A can be used in exactly the same way for setting up differential equations and even Phasor analysis in 16B. Treat them as ideal op-amps and invoke the Golden Rules.

You have at your disposal *only one of each* of the following components (not including R_1):

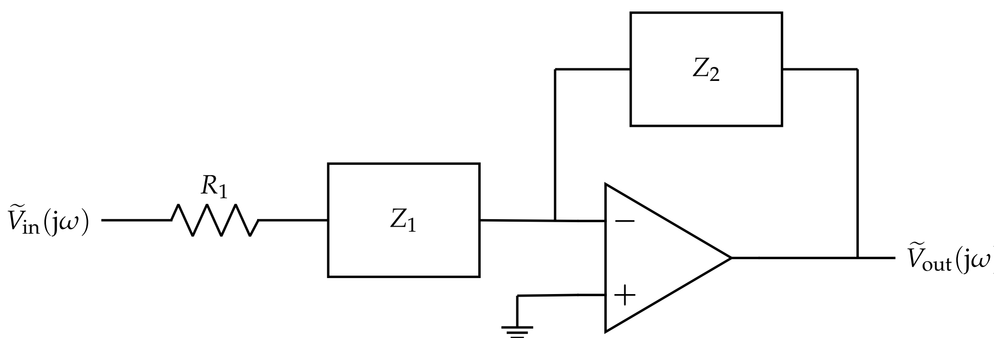
(a) an open circuit

(b) a short circuit

(c) a resistor (you choose from the values $R = 1\,\text{k}\Omega, 10\,\text{k}\Omega, 20\,\text{k}\Omega$)

(d) a capacitor (you choose from the values $C = 0.5\,\mu\text{F}, 1\,\mu\text{F}, 2\,\mu\text{F}$)

Consider the circuit below. The labeled voltages $\widetilde{V}_{\text{in}}(\text{j}\omega)$ and $\widetilde{V}_{\text{out}}(\text{j}\omega)$ are the phasor representations of $v_{\text{in}}(t)$ and $v_{\text{out}}(t)$ respectively, where $v_{\text{in}}(t)$ has the form $v_{\text{in}}(t) = v_0 \cos(\omega t + \phi)$. The transfer function $H(\text{j}\omega)$ is defined as $H(\text{j}\omega) = \frac{\widetilde{V}_{\text{out}}(\text{j}\omega)}{\widetilde{V}_{\text{in}}(\text{j}\omega)}$.



(a) Let $Z_1(\text{j}\omega)$ and $Z_2(\text{j}\omega)$ are the impedances of the boxes shown in the circuit diagram. **Write the expression of the transfer function $H(\text{j}\omega)$.**

(b) Let R_1 be $1\,\text{k}\Omega$. We have to find Z_1 and Z_2, such that the circuit's transfer function $H(\text{j}\omega)$ has the following properties:

- It is a high-pass filter.
- $|H(\text{j}\infty)| = 10$.
- $|H(\text{j}10^3)| = \sqrt{50}$.

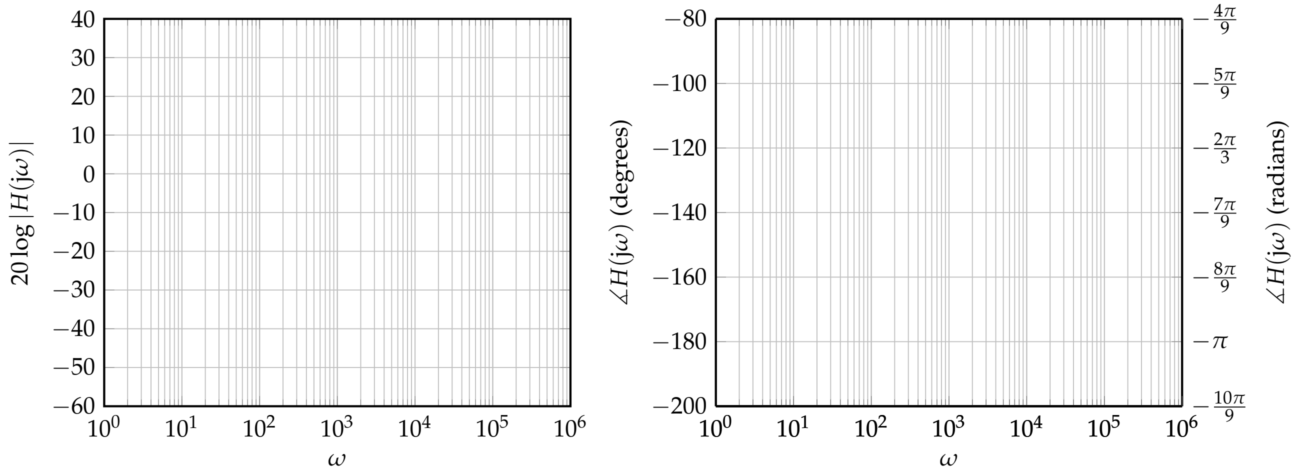
Using the fact that the circuit is a high pass filter, infer the components (we will find values later) of Z_1 and Z_2. Write the transfer function $H(\text{j}\omega)$ using these components.

Hint: Try method of elimination: figure out what Z_2 cannot be. Once you find what Z_2 is, what does Z_1 have to be for the circuit to be a filter?

(c) **Now use the facts that $|H(\text{j}\infty)| = 10$ and $R_1 = 1\,\text{k}\Omega$ to find the component value of Z_2.**

(d) **Finally use the fact that $|H(\text{j}10^3)| = \sqrt{50}$ and the values of R_1 and Z_2 to find the component value of Z_1.**

(e) **Draw the magnitude and phase Bode plots (straight-line approximations to the transfer function) of this transfer function.** Blank plots are provided here for you to use.



40
30
20
10
0
−10
−20
−30
−40
−50
−60
20 log |H(jω)|
10^0 10^1 10^2 10^3 10^4 10^5 10^6
ω
−80
−100
−120
−140
−160
−180
−200
∡H(jω) (degrees)
−4π/9
−5π/9
−2π/3
−7π/9
−8π/9
−π
−10π/9
∡H(jω) (radians)

6. System Identification

You are given a discrete-time system as a black box. You don't know the specifics of the system but you know that it takes one scalar input and has two states that you can observe. You assume that the system is linear and of the form

$$\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{w}[i], \tag{8}$$

where $\vec{w}[i]$ is an external small unknown disturbance, $u[i]$ is a scalar input, and

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad x[i] = \begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix}. \tag{9}$$

You want to identify the system parameters (a_1, a_2, a_3, a_4, b_1 and b_2) from measured data. However, you can only interact with the system via a black box model, i.e., you can see the states $\vec{x}[t]$ and set the inputs $u[i]$ that allow the system to move to the next state.

(a) You observe that the system has state $\vec{x}[i] = \begin{bmatrix} x_1[i] & x_2[i] \end{bmatrix}^\top$ at time i. You pass input $u[i]$ into the black box and observe the next state of the system: $\vec{x}[i+1] = \begin{bmatrix} x_1[i+1] & x_2[i+1] \end{bmatrix}^\top$.
Write scalar equations for the new states, $x_1[i+1]$ **and** $x_2[i+1]$. Write these equations in terms of the a_i, b_i, the states $x_1[i]$, $x_2[i]$ and the input $u[i]$. Here, assume that $\vec{w}[i] = \vec{0}$ (i.e., the model is perfect).

(b) Now we want to identify the system parameters. We observe the system at the start state $\vec{x}[0] = \begin{bmatrix} x_1[0] \\ x_2[0] \end{bmatrix}$. We can then input $u[0]$ and observe the next state $\vec{x}[1] = \begin{bmatrix} x_1[1] \\ x_2[1] \end{bmatrix}$. We can continue this for a sequence of ℓ inputs.
Let us define an ℓ-length trajectory to be an initial condition $\vec{x}[0]$, an input sequence $u[0], \ldots, u[\ell-1]$, and the corresponding states that are produced by the system $x[1], \ldots, x[\ell]$. **Assuming that the model is perfect ($\vec{w}[i] = \vec{0}$), what is the minimum value of ℓ you need to identify the system parameters?**

(c) We now remove our assumption that $\vec{w} = 0$. We assume it is small, so the model is approximately correct and we have

$$\vec{x}[i+1] \approx A\vec{x}[i] + Bu[i]. \tag{10}$$

Say we feed in a total of 4 inputs $u[0], \ldots, u[3]$, and observe the states $\vec{x}[0], \ldots, \vec{x}[4]$. To identify the system we need to set up an approximate (because of potential, small, disturbances) matrix equation

$$DP \approx S \tag{11}$$

using the observed values above and the unknown parameters we want to find. Let our parameter vector be

$$P := \begin{bmatrix} \vec{p}_1 & \vec{p}_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \\ b_1 & b_2 \end{bmatrix} \tag{12}$$

Find the corresponding D and S to do system identification. Write both out explicitly.

(d) Now that we have set up $DP \approx S$, we can estimate a_0, a_1, a_2, a_3, b_0, and b_1. **Give an expression for the estimates of $\vec{p}_1$ and $\vec{p}_2$ (which are denoted $\hat{\vec{p}}_1$ and $\hat{\vec{p}}_2$ respectively) in terms of D and S.** Denote the columns of S as $\vec{s}_1$ and $\vec{s}_2$, so we have $S = [\vec{s}_1 \; \vec{s}_2]$. Assume that the columns of D are linearly independent. *(HINT: Don't forget that D is not a square matrix. It is taller than it is wide.) (HINT: Can we split $DP = S$ into separate equations for p_1 and p_2?)*

7. Identifying systems from their responses to known inputs

In many problems, we have an unknown system, and would like to characterize it. One of the ways of doing so is to observe the system response with different initial conditions (or inputs). This problem is also called system identification. It is a prototypical example of a problem that today is called machine learning — inferring an underlying pattern from data, and doing so well enough to be able to exploit that pattern in some practical setting. Go through the attached Jupyter notebook `demo_system_id.ipynb` and answer the following questions.

(a) In Example 2, we assume that instead of measuring the state $\vec{x}$, we are instead measuring a transformation of the state $\vec{y} = T\vec{x}$ where T is a full rank matrix. Assume that we perform system ID on our observations $\vec{y}[i]$ to recover A_y, B_y such that $\vec{y}[i+1] = A_y\vec{y}[i] + B_y u[i]$. **How do the identified A_y and B_y matrices relate to the original A and B matrices in the dynamics of $\vec{x}$?** Remember that our original state dynamics are $\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i]$.

Hint: The answer is given in the Jupyter notebook but remember to show your work.

(b) **Please share your observations on Example 2. Comment on what impact a linear transformation of the state trace has on our ability to perform system identification.**

(c) **Prove that for any full rank transformation matrix T, the eigenvalues of A_y and A from part (a) are the same.**

(d) **Please share your observations on Example 3. Comment on the impact that changing the noise magnitude, number of samples and number of states has on the system identification performance.**

(e) **Please share your observations on Example 4. Comment on the sample efficiency of this method, i.e. do you need more or less samples for accurate system identification when given scalar observations rather than the entire state vector?**

(f) **Please share your observations on Example 5. Comment on how important the model size is for this setting.**

8. Motor Driver and System Identification

In the lab project, you will be designing SIXT33N, a mischievous little robot who *might* just do what you want — if you design it correctly. In phases 1 and 2, you will build the **legs** of SIXT33N: you will be designing SIXT33N's wheels and developing a linear model for the car system. The wheels will be driven by two 9-Volt DC motors whose driver control circuit is shown in Figure 7 .

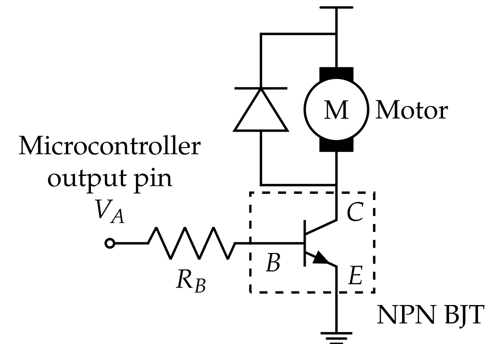


Figure 7: Motor Controller Circuit

There is some minimum voltage required to deliver enough power to the motors to overcome the static friction and start them, but after that point, we treat the motor speed as approximately **linear** with the applied voltage V_A (this will be the basis of the system model you will develop in this problem).

As it is difficult to use a microcontroller (MSP430 in hands-on lab; Arduino in lab sim) to generate a true adjustable DC signal, we will instead make use of its PWM function. A PWM, or pulse-width modulated, signal is a square wave with a variable duty cycle (the proportion of a cycle period for which the power source is turned on, or logic HIGH). PWM is used to digitally change the average voltage delivered to a load by varying the duty cycle. If the frequency is large enough, the on-off switching is imperceptible, but the average voltage delivered to the load changes proportionally with the duty cycle. Hence, changing the duty cycle corresponds to changing the DC voltage supplied to the motor. An example can be seen in figure 8.

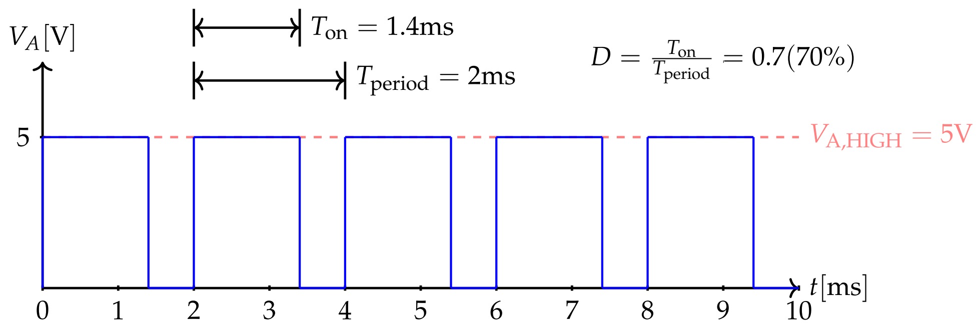


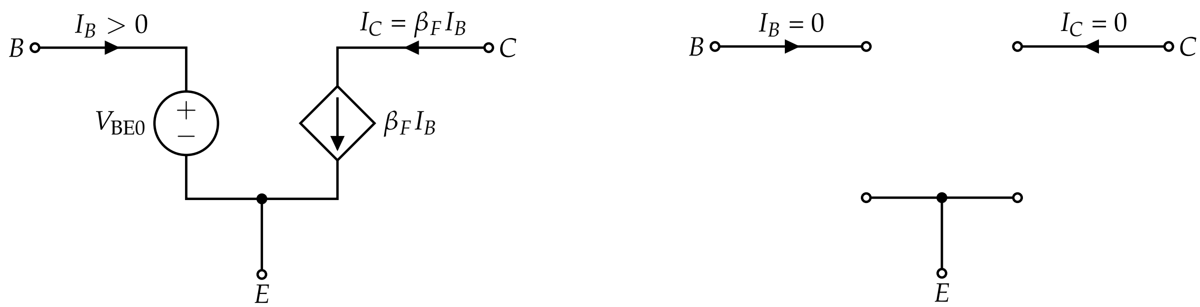
Figure 8: PWM Example with switching frequency 500Hz and 70% duty cycle

The PWM pin (V_A) is connected via a resistor (R_B) to the "Base (B)" of an NPN bipolar junction transistor (BJT). This transistor, in reality, behaves a bit differently from the NMOS with which you are familiar, but for this class, you may assume that it is functionally the same as an NMOS, behaving as a switch. On the BJT, the three terminals are analogous to those of an NMOS: the "Base (B)" is the gate, the "Collector (C)" is the drain, and the "Emitter (E)" is the source.

The BJT in Figure 7 is switching between **ON** and **OFF** modes when V_A is HIGH and LOW respectively. The model for both ON and OFF states are shown in Figure 9. When the BJT turns on, V_{BE}

can be modeled as a fixed voltage source with voltage value V_{BE0}. In ON mode, there is a **Current Controlled Current Source** modeled between "Collector (C)" and "Emitter (E)", i.e., current at the "Collector (C)" is an amplified version of current at the "Base (B)" (notice that positive I_B has to flow into the "Base (B)" for the relation $I_C = \beta_F I_B$ to hold). β_F is called the **Common-Emitter Current Gain**.

The diode in parallel with the motor is needed because of the inductive characteristics of the motor. If the motor is on and V_A switches to LOW, the inductive behavior of the motor maintains the current and the diode provides the path to dissipate it as the BJT is turned off. When the BJT turns on, the diode is off so there is no current flow through the diode.



(a) Model of BJT in ON mode (when V_A is logic HIGH) **(b)** Model of BJT in OFF mode (when V_A is logic LOW)

Figure 9: Model of NPN BJT in Different Modes

Please use $V_{\text{BE0}} = 0.8\,\text{V}$, $\beta_F = 100$, $V_{A,\text{HIGH}} = 5\,\text{V}$ and $V_{A,\text{LOW}} = 0\,\text{V}$ for all following calculations.

Part 1: Circuit analysis to construct the system model

(a) **Draw the equivalent motor controller circuit when the BJT is ON by substituting in the BJT model from Fig. 9a into Fig. 7. Express I_B and I_C for $V_A = V_{A,\text{HIGH}}$ as a function of R_B.**

(b) **Draw the equivalent motor controller circuit when the BJT is OFF by substituting in the BJT model from Fig. 9b into Fig. 7. Express I_B and I_C for $V_A = V_{A,\text{LOW}}$ as a function of R_B.**

(c) **Derive the average collector current, I_{AVG}, over one period, T_{period}, of the PWM signal, V_A, as a function of R_B and the duty cycle, D, of the PWM signal.** Hint: The time average of some signal $f(t)$ from time t_0 to t_1 is given as $f_{\text{AVG}} = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} f(\tau)\,\mathrm{d}\tau$. Figure 8 may be useful.

(d) **In the previous part, explain briefly why is it sufficient to take the average over only one period if we are actually interested in the average collector current over multiple periods?**

(e) **If $R_B = 2\,\text{k}\Omega$, what is the average collector current, I_{AVG}, that drives the motor when the duty cycle of the PWM signal is equal to 25%?**

Part 2: Learning a Car Model from data
To control the car, we need to build a model of the car first. Instead of designing a complex nonlinear model, we will approximate the system with a linear model to work for small perturbations around an equilibrium point. The following model applies separately to each wheel (and associated motor) of the car:

$$v_L[i] = \theta_L u_L[i] - \beta_L \tag{13}$$
$$v_R[i] = \theta_R u_R[i] - \beta_R \tag{14}$$

Notice that this particular model has no state variables since we are measuring velocity directly here. To do system ID, we decide to use the exact same input $u_L[i] = u_R[i] = u[i]$ for both motors. We measure both velocities however.
Meet the variables at play in this model: **(Note: the β here have nothing to do with the previous part.)**

- i - The current timestep of the model. Since we model the car as a discrete-time system, n will advance by 1 on every new sample in the system.
- $v_L[i]$ - The discrete-time velocity (in units of ticks/timestep) of the left wheel, reading from the motor.
- $v_R[i]$ - The discrete-time velocity (in units of ticks/timestep) of the right wheel, reading from the motor.
- $u[i]$ - The input to each wheel. The duty cycle of the PWM signal (V_A), which is the percentage of the square wave's period for which the square wave is HIGH, is mapped to the range $[0, 255]$. Thus, $u[i]$ takes a value in $[0, 255]$ representing the duty cycle. For example, when $u[i] = 255$, the duty cycle is 100 %, and the motor controller just delivers a constant signal at the system's HIGH voltage, delivering the maximum possible power to the motor. When $u[i] = 0$, the duty cycle is 0 %, and the motor controller delivers 0 V. The duty cycle (D) can be written as

$$\text{duty cycle (D)} = \frac{u[i]}{255} \tag{15}$$

- $\theta(\theta_L, \theta_R)$ - Relates change in input to change in velocity. **Its units are ticks/(timestep · duty cycle).** Since our model is linear, we assume that θ is the same for every unit increase in $u[i]$. This is empirically measured using the car. **You will have a separate θ for your left and your right wheel**(θ_L, θ_R)**.**
- $\beta(\beta_L, \beta_R)$ - Similarly to θ, β is dependent upon many physical phenomena, so we will empirically determine it using the car. β represents a constant offset in the velocity of the wheel, and hence **its units are ticks/timestep**. Note that you will also typically have a different β for your left and your right wheel (i.e. $\beta_L \neq \beta_R$). **These β_L and β_R are different from the β_F of the transistor.**

(f) By measuring the car with a PWM signal at different duty cycles, we can collect the velocity data of the left and right wheel, as shown in the following table:

Table 1: The velocity of the left and the right wheel at different duty cycles of PWM signal

Duty Cycle $\times 255$ ($u[i]$)	Velocity of the left wheel ($v_L[i]$)	Velocity of the right wheel ($v_R[i]$)
80	147	127
120	218	187
160	294	253
200	370	317

Since the same input is applied to both the wheels, we can take advantage of the same "horizontal stacking" trick you've seen before to be able to reuse computation. To identify the system we need to setup matrix equations of left and right wheel in the form of:

$$D_{\text{data}} P \approx S \tag{16}$$

where $P = \begin{bmatrix} \theta_L & \theta_R \\ \beta_L & \beta_R \end{bmatrix}$. **Find the matrix D_{data} and matrix S needed to perform system identification to get the matrix of parameters of the left and right wheel, P.**

(g) **Solve the matrix equation $D_{\text{data}} P \approx S$ with least squares to find θ_L, θ_R, β_L, and β_R.** You may use a jupyter notebook for computation.

(h) In most advanced systems, we usually use a combination of a physics-based equation and a data-centric approach to build the model. In our case, the velocity of the motor can be written as

$$v[i] = k I_{\text{AVG}}(u[i]) - \beta \tag{17}$$

where $I_{\text{AVG}}(u[i])$ is the average collector current which is the function of the duty cycle that you have already derived in Part 1. k represents the response of your motor speed to the average current. In our simplified motor driver model in Part 1, you have already derived the expression for the I_{AVG} of the motor as a function of the circuit parameters and the duty cycle D. **If we assume that the model from Part 1 holds, determine the resistance ratio $\left(\frac{R_{\text{B,left}}}{R_{\text{B,right}}}\right)$ from the model parameters you identified in Part 2 item (f).** Assume that the left motor and the right motor respond the same, that is, $k_L = k_R$. The only difference is presumed to come from the resistors used.

(i) In order for the car to drive straight, the wheels must be moving at the same velocity. However, the data from Table 1 tell us that two motors cannot run at the same velocity if the duty cycles of driving PWM signals are the same. **Based on the model you extracted in Part 2 item (f), if we want the car to drive straight and $u_L = 100$, what should u_R be?**

9. **(OPTIONAL) Make Your Own Problem.**

   Write your own problem about content covered in the course thus far, and provide a thorough solution to it.

   NOTE: This can be a totally new problem, a modification on an existing problem, or a Jupyter part for a problem that previously didn't have one. Please cite all sources for anything (including course material) that you used as inspiration.

   NOTE: High-quality problems may be used as inspiration for the problems we choose to put on future homeworks or exams.

10. **Homework Process and Study Group**

    Citing sources and collaborators are an important part of life, including being a student!

    We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

    (a) **What sources (if any) did you use as you worked through the homework?**

    (b) **If you worked with someone on this homework, who did you work with?**
    List names and student ID's. (In case of homework party, you can also just describe the group.)

    (c) **Roughly how many total hours did you work on this homework? Write it down here where you'll need to remember it for the self-grade form.**

Contributors:

- Sidney Buchbinder.
- Anant Sahai.
- Kris Pister.
- Pavan Bhargava.
- Kourosh Hakhamaneshi.
- Ayan Biswas.
- Michael Danielczuk.
- Druv Pai.
- Geoffrey Négiar.
- Alex Devonport.
- Regina Eckert.
- Nathan Lambert.
- Wahid Rahman.
- Yen-Sheng Ho.
- Antroy Roy Chowdhury.
- Nikhil Shinde.
- Ashwin Vangipuram.
- Sally Hui.
- Tanmay Gautam.
- Bozhi Yin.
- Kaitlyn Chan.
- Yi-Hsuan Shih.
- Vladimir Stojanovic.
- Moses Won.