Homework 4

This homework is due on Friday, February 18, 2022 at 11:59PM. Self-grades and HW Resubmissions are due the following Friday, February 25, 2022 at 11:59PM.

1. **Reading Lecture Notes**

   Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: Note 5 and Note 6.

   (a) What is the equation that governs the relationship that an inductor enforces on the voltage across it and the current through it? What is the behavior of an inductor under DC current (i.e. constant current)?

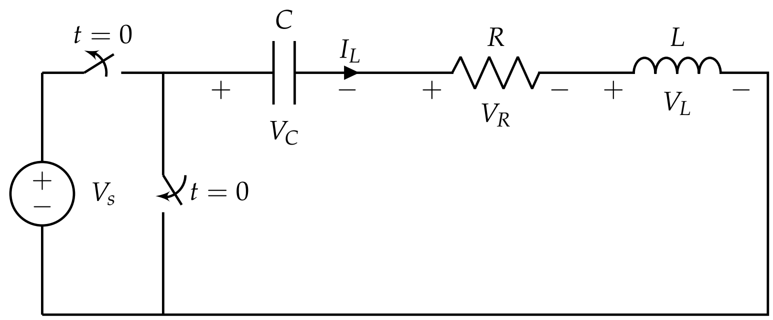
   Solution: An inductor's I-V relationship is $V_L = L\frac{\mathrm{d}I_L}{\mathrm{d}t}$ where L is the inductance of the inductor in Henries. In DC, the current through the inductor isn't changing and so its derivative is 0. Thus, the voltage across the inductor is 0 and it acts as a wire.

   (b) What is the definition of impedance? What quantity is impedance analogous to in the static DC circuit analysis you learned in 16A?

   Solution: The impedance of an element is the ratio of the voltage phasor of that element to the current phasor of that element. It is analogous to resistance in DC circuit analysis.

2. RLC Responses: Initial Part

Consider the following circuit:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

The sequence of problems 2 - 6 combined will try to show you the various RLC system responses and how they relate to changing circuit properties.

(a) We first need to construct our state space system. Our state variables are the current through the inductor $x_1(t) = I_L(t)$ and the voltage across the capacitor $x_2(t) = V_C(t)$ since these are the quantities whose derivatives show up in the system of equations governing our circuit. Now, **show that the system of differential equations in terms of our state variables that describes this circuit for $t \geq 0$ is**

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = -\frac{R}{L}x_1(t) - \frac{1}{L}x_2(t) \tag{1}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = \frac{1}{C}x_1(t). \tag{2}$$

Solution: For this part, we need to find two differential equations, each including a derivative of one of the state variables.

First, let's consider the capacitor equation $I_C(t) = C\frac{\mathrm{d}}{\mathrm{d}t}V_C(t)$. In this circuit, $I_C(t) = I_L(t)$, so we can write

$$I_C(t) = C\frac{\mathrm{d}}{\mathrm{d}t}V_C(t) = I_L(t) \tag{3}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}V_C(t) = \frac{1}{C}I_L(t). \tag{4}$$

If we use the state variable names, we can write this as

$$\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = \frac{1}{C}x_1(t), \tag{5}$$

so now we have one differential equation.

For the other differential equation, consider the voltage drop across the capacitor, resistor and inductor. At $t \geq 0$, the voltage difference between the positive '+' terminal of C and the negative '-' terminal of L is given by

$$V_C(t) + V_R(t) + V_L(t) = 0. \tag{6}$$

Using Ohm's Law $V_R(t) = RI_L(t)$ and the inductor equation $V_L(t) = L\frac{\mathrm{d}}{\mathrm{d}t}I_L(t)$, we can write this as

$$V_C(t) + RI_L(t) + L\frac{\mathrm{d}}{\mathrm{d}t}I_L(t) = 0, \tag{7}$$

which we can rewrite as

$$\frac{\mathrm{d}}{\mathrm{d}t} I_L(t) = -\frac{R}{L} I_L(t) - \frac{1}{L} V_C(t). \tag{8}$$

If we use the state variable names, this becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} x_1(t) = -\frac{R}{L} x_1(t) - \frac{1}{L} x_2(t), \tag{9}$$

and we have a second differential equation.
To summarize, the final system is

$$\frac{\mathrm{d}}{\mathrm{d}t} x_1(t) = -\frac{R}{L} x_1(t) - \frac{1}{L} x_2(t) \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} x_2(t) = \frac{1}{C} x_1(t). \tag{11}$$

(b) **Write the system of equations in vector/matrix form with the vector state variable $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} I_L(t) \\ V_C(t) \end{bmatrix}$.** This should be in the form $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t)$ with a 2×2 matrix A.

Solution: By inspection from the previous part, we have

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} x_1(t) \\ \frac{\mathrm{d}}{\mathrm{d}t} x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{12}$$

which is in the form $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t)$, with

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}. \tag{13}$$

(c) **Show that, for the 2×2 matrix A, the two eigenvalues of A are**

$$\lambda_1 = -\frac{1}{2}\frac{R}{L} + \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}} \tag{14}$$

$$\lambda_2 = -\frac{1}{2}\frac{R}{L} - \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}. \tag{15}$$

(*HINT: The quadratic formula will be involved.*)

Solution: To find the eigenvalues, we'll solve $\det(A - \lambda I) = 0$. In other words, we want to find λ such that

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\frac{R}{L} - \lambda & -\frac{1}{L} \\ \frac{1}{C} & -\lambda \end{bmatrix}\right) \tag{16}$$

$$= -\lambda\left(-\frac{R}{L} - \lambda\right) + \frac{1}{LC} \tag{17}$$

$$= \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0. \tag{18}$$

The quadratic formula gives

$$\lambda = -\frac{1}{2}\frac{R}{L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}} \tag{19}$$

as desired.

(d) **Under what condition on the circuit parameters R, L, C will A have two distinct real eigenvalues?**
Solution: For both eigenvalues to be real and distinct, we need the quantity inside the square root to be positive. In other words, we need

$$\frac{R^2}{L^2} - \frac{4}{LC} > 0, \tag{20}$$

or, equivalently,

$$R > 2\sqrt{\frac{L}{C}}. \tag{21}$$

(e) **Under what condition on the circuit parameters R, L, C will A have two imaginary eigenvalues? What will the eigenvalues be in this case?** **Solution:** The only way for both eigenvalues to be purely imaginary is to have $R = 0$. In this case, the eigenvalues would be

$$\lambda = \pm \mathrm{j}\sqrt{\frac{1}{LC}}. \tag{22}$$

(f) Assuming that the circuit parameters are such that there are a pair of (potentially complex) eigenvalues λ_1, λ_2 so that $\lambda_1 \neq \lambda_2$, **show that the corresponding eigenvectors $\vec{v}_{\lambda_1}, \vec{v}_{\lambda_2}$ are**

$$\vec{v}_{\lambda_1} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_1 C} \end{bmatrix} \quad \text{and} \quad \vec{v}_{\lambda_2} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_2 C} \end{bmatrix}. \tag{23}$$

Solution: We use the definition of an eigenvector and eigenvalue. We want $A\vec{v}_{\lambda_i} = \lambda_i \vec{v}_{\lambda_i}$.
Note that, for any y,

$$A\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} -\frac{y}{L} \\ 0 \end{bmatrix} \tag{24}$$

is not a scalar multiple of $\begin{bmatrix} 0 \\ y \end{bmatrix}$, so no eigenvector is of the form $\begin{bmatrix} 0 \\ y \end{bmatrix}$. Thus they must all be of the form $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ with $y_1 \neq 0$, and we can divide through by y_1 to show that every eigenvector is of the form $\begin{bmatrix} 1 \\ y \end{bmatrix}$ for some y.
Thus,

$$\begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ y \end{bmatrix} = \lambda_i \begin{bmatrix} 1 \\ y \end{bmatrix} = \begin{bmatrix} \lambda_i \\ \lambda_i \cdot y \end{bmatrix} \tag{25}$$

We also know that:

$$\begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ y \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} - \frac{y}{L} \\ \frac{1}{C} \end{bmatrix} \tag{26}$$

Equating the two equations from above gives:

$$\begin{bmatrix} \lambda_i \\ \lambda_i \cdot y \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} - \frac{y}{L} \\ \frac{1}{C} \end{bmatrix}. \tag{27}$$

From the second row we see that $y = \frac{1}{\lambda_i C}$. Now we find the eigenvectors as:

$$\vec{v}_{\lambda_1} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_1 C} \end{bmatrix} \tag{28}$$

$$\vec{v}_{\lambda_2} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_2 C} \end{bmatrix} \tag{29}$$

Alternatively, you can try to use the standard approach of finding the nullspace of $A - \lambda_i I$ to arrive at the same answer as above.

(g) Assuming circuit parameters such that the two eigenvalues of A are distinct, let $V = \begin{bmatrix} \vec{v}_{\lambda_1} & \vec{v}_{\lambda_2} \end{bmatrix}$ be a specific eigenbasis. Consider a coordinate system for which we can write $\vec{x}(t) = V\vec{\widetilde{x}}(t)$. **Show that the $\widetilde{A}$ so that $\frac{\mathrm{d}}{\mathrm{d}t}\vec{\widetilde{x}}(t) = \widetilde{A}\vec{\widetilde{x}}(t)$ is**

$$\widetilde{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \tag{30}$$

(*HINT: Write out the original differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t)$, then use the given change of coordinates to write everything in terms of $\vec{\widetilde{x}}(t)$.*)

Solution: V is given by:

$$V = \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix} \tag{31}$$

We know that V transforms from the $\widetilde{x}$ coordinate frame to the x coordinate frame, V^{-1} transforms back, and A takes gives the relationship from x to $\frac{\mathrm{d}}{\mathrm{d}t}x$.

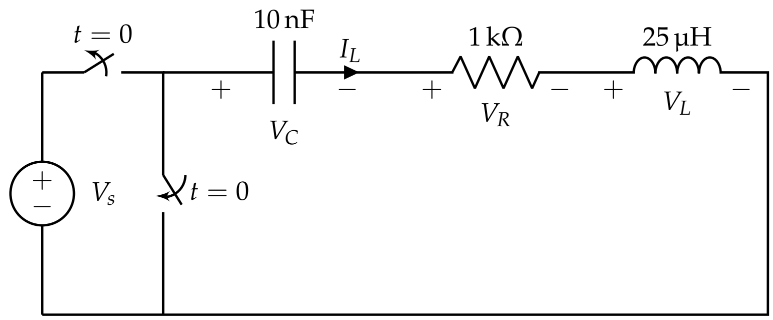
Therefore to go from $\widetilde{x}$ to $\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}$:

$$\begin{aligned}
\widetilde{A} &= V^{-1}AV \qquad (32)\\
&= \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix} \qquad (33)\\
&= \frac{\lambda_1 \lambda_2 C}{\lambda_1 - \lambda_2} \begin{bmatrix} \frac{1}{\lambda_2 C} & -1 \\ -\frac{1}{\lambda_1 C} & 1 \end{bmatrix} \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix} \qquad (34)\\
&= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad (35)
\end{aligned}$$

You didn't have to multiply things out explicitly. You could have just noticed that the eigenvector matrix will diagonalize the A matrix such that $AV = V\Lambda$ or $V^{-1}AV = \Lambda$, as per one of the problems on the last homework.

3. RLC Responses: Overdamped Case

Building on the previous problem, consider the following circuit with specified component values:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2. You may use a calculator or the attached `RLC_Calc.ipynb` Jupyter Notebook for numerical calculations.

(a) Suppose $R = 1\,\text{k}\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1\,\text{V}$. **Find the initial conditions for $\vec{\tilde{x}}(0)$.** Recall that $\vec{\tilde{x}}$ are the eigenbasis coordinates from the first question. **Solution:** First, we must find the initial conditions at $t = 0$. Recall that in steady-state, a capacitor acts as an open circuit. Thus, since the circuit is in steady state before $t = 0$, then no current is flowing in the circuit and the entire voltage drop is across the capacitor. Therefore:

$$x_1(0) = I_L(0) = 0 \tag{36}$$
$$x_2(0) = V_C(0) = V_s = 1 \tag{37}$$

Under these conditions, we can use the circuit component values and plug into the expression for the eigenvalues found in Question 2c to find

$$\lambda_1 = -1.0 \times 10^5, \quad \lambda_2 = -4.0 \times 10^7 \tag{38}$$

Then, using the expression for V^{-1} from Question 2g

$$V^{-1} = \begin{bmatrix} -0.0025 & -0.001 \\ 1.0025 & 0.001 \end{bmatrix} \tag{39}$$

and finally

$$\vec{\tilde{x}}(0) = V^{-1}\vec{x}(0) = \begin{bmatrix} -0.0025 & -0.001 \\ 1.0025 & 0.001 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.001 \\ 0.001 \end{bmatrix}. \tag{40}$$

(b) Using the diagonalized system from 2(g) and continuing the previous part, **find $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ for $t \geq 0$.** **Solution:**
Plugging in for the component values gives:

$$\widetilde{A} = \begin{bmatrix} -1.0 \times 10^5 & 0 \\ 0 & -4.0 \times 10^7 \end{bmatrix} \tag{41}$$

These eigenvalues are the negative reciprocals of the relevant time constants for these modes.

$$\begin{bmatrix} \frac{d}{dt}\tilde{x}_1(t) \\ \frac{d}{dt}\tilde{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1.0 \times 10^5 & 0 \\ 0 & -4.0 \times 10^7 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix}, \tag{42}$$

Therefore:

$$\widetilde{x}_1(t) = K_1 e^{-(1.0\times 10^5)t} \tag{43}$$
$$\widetilde{x}_2(t) = K_2 e^{-(4.0\times 10^7)t} \tag{44}$$

Solving for K_1 and K_2 with the initial condition gives:

$$\widetilde{x}_1(t) = -0.001 e^{-(1.0\times 10^5)t} \tag{45}$$
$$\widetilde{x}_2(t) = 0.001 e^{-(4.0\times 10^7)t} \tag{46}$$

Converting back to the $\vec{x}$ coordinates:

$$\vec{x}(t) = V\vec{\widetilde{x}}(t) = \begin{bmatrix} 1 & 1 \\ -1000 & -2.5 \end{bmatrix} \vec{\widetilde{x}}(t) \tag{47}$$
$$x_1(t) = -0.001 e^{-(1.0\times 10^5)t} + 0.001 e^{-(4.0\times 10^7)t} \tag{48}$$
$$x_2(t) = e^{-(1.0\times 10^5)t} - 0.0025 e^{-(4.0\times 10^7)t}. \tag{49}$$

Note that using the final solution for $x_2(t)$, we get $x_2(0) = 0.9975$. But our original initial condition was $x_2(0) = 1$ in (37). This apparent contradiction is because of rounding errors in the elements of V^{-1} in (39) and V in (47). If we use the 'exact' values of V^{-1} and V (upto machine precision) from `RLC_Calc.ipynb` or any other calculator, we will indeed get $x_2(0) = 1$.

In general, we should always use 'exact' values from the calculator for intermediate steps, and round to significant figures only in the final answer. These solutions round the values in the intermediate steps too just for illustrative purposes.

(c) In the `RLCSliders.ipynb` Jupyter notebook, move the sliders to approximately $R = 1\,\text{k}\Omega$ and $C = 10\,\text{nF}$. **Comment on the graph of $V_C(t)$ and the location of the eigenvalues on the complex plane.**

Solution: $V_C(t)$ looks like a decaying exponential. The eigenvalues lie on the real axis at coordinates $(-1\times 10^5, 0)$ and $(-4\times 10^7, 0)$.

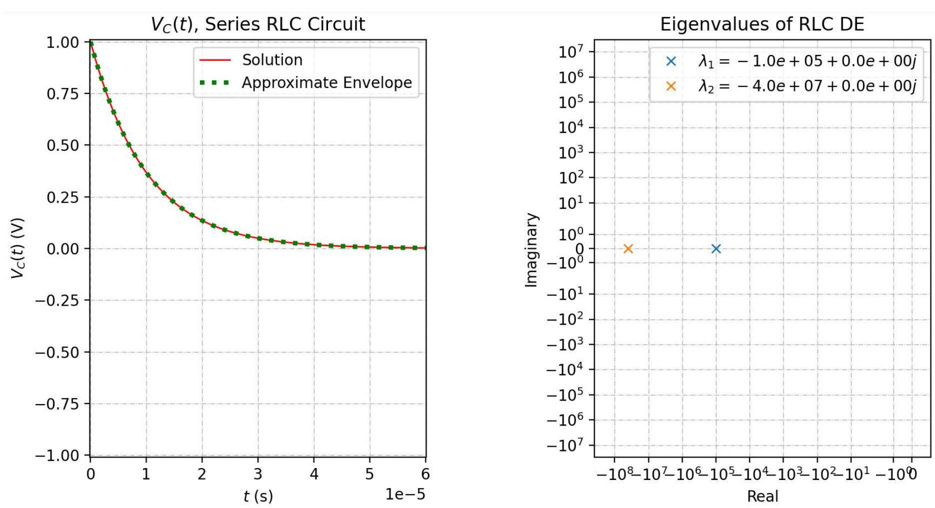
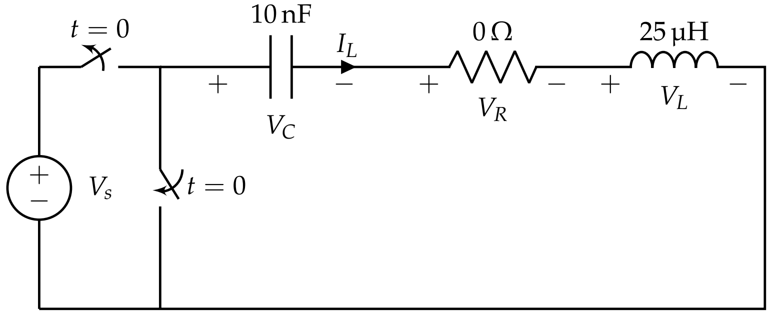


Figure 1: $V_C(t)$ and eigenvalues for overdamped case.

4. RLC Responses: Undamped Case

Building on the previous problem, consider the following circuit with specified component values:



Assume that the capacitor is charged to V_s and there is no current in the inductor for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2. You may use a calculator or the attached `RLC_Calc.ipynb` Jupyter Notebook for numerical calculations.

(a) Suppose $R = 0\,\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1\,\mathrm{V}$. **Find the initial conditions for $\vec{\tilde{x}}(0)$.** Recall that $\vec{\tilde{x}}$ is in the changed "nice" eigenbasis coordinates from the first problem. **Solution:**

Under these conditions, we can solve for

$$\lambda = \pm \mathrm{j}\sqrt{\frac{1}{LC}} = \pm \mathrm{j}\sqrt{\frac{1}{250 \times 10^{-15}}} = \pm \mathrm{j}(2 \times 10^6) \implies \lambda_1 = \mathrm{j}(2 \times 10^6), \quad \lambda_2 = -\mathrm{j}(2 \times 10^6) \tag{50}$$

Using the rule we derived earlier for finding V, we have

$$V = \begin{bmatrix} 1 & 1 \\ -\mathrm{j}50 & \mathrm{j}50 \end{bmatrix} \tag{51}$$

$$V^{-1} = \begin{bmatrix} 0.5 & \mathrm{j}(0.01) \\ 0.5 & -\mathrm{j}(0.01) \end{bmatrix} \tag{52}$$

which lets us say

$$\vec{\tilde{x}}(0) = V^{-1}\vec{x}(0) = \begin{bmatrix} 0.5 & \mathrm{j}(0.01) \\ 0.5 & -\mathrm{j}(0.01) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathrm{j}(0.01) \\ -\mathrm{j}(0.01) \end{bmatrix}. \tag{53}$$

(b) Using the diagonalized system from 2(g) and continuing the previous part, **find $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ for $t \geq 0$.** Remember that your final expressions for $x_1(t)$ and $x_2(t)$ should be real functions (no imaginary terms).

(HINT: Use Euler's formula.)

Solution: Plugging in for the component values gives:

$$\widetilde{A} = \begin{bmatrix} \mathrm{j}(2 \times 10^6) & 0 \\ 0 & -\mathrm{j}(2 \times 10^6) \end{bmatrix} \tag{54}$$

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\tilde{x}_1(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\tilde{x}_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{j}(2 \times 10^6) & 0 \\ 0 & -\mathrm{j}(2 \times 10^6) \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} \tag{55}$$

Therefore:

$$\widetilde{x}_1(t) = K_1 \mathrm{e}^{+\mathrm{j}(2\times 10^6)t} \quad (56)$$
$$\widetilde{x}_2(t) = K_2 \mathrm{e}^{-\mathrm{j}(2\times 10^6)t} \quad (57)$$

Solving for K with the initial condition gives:

$$\widetilde{x}_1(t) = \mathrm{j}(0.01)\mathrm{e}^{+\mathrm{j}(2\times 10^6)t} \quad (58)$$
$$\widetilde{x}_2(t) = -\mathrm{j}(0.01)\mathrm{e}^{-\mathrm{j}(2\times 10^6)t} \quad (59)$$

Converting back to the $\vec{x}$ coordinates:

$$\vec{x}(t) = V\vec{\widetilde{x}}(t) = \begin{bmatrix} 1 & 1 \\ -\mathrm{j}50 & \mathrm{j}50 \end{bmatrix} \begin{bmatrix} \mathrm{j}(0.01)\mathrm{e}^{+\mathrm{j}(2\times 10^6)t} \\ -\mathrm{j}(0.01)\mathrm{e}^{-\mathrm{j}(2\times 10^6)t} \end{bmatrix} \quad (60)$$
$$x_1(t) = \mathrm{j}(0.01)\mathrm{e}^{+\mathrm{j}(2\times 10^6)t} - \mathrm{j}(0.01)\mathrm{e}^{-\mathrm{j}(2\times 10^6)t} = -0.02\sin\left((2\times 10^6)t\right) \quad (61)$$
$$x_2(t) = 0.5\mathrm{e}^{+\mathrm{j}(2\times 10^6)t} + 0.5\mathrm{e}^{-\mathrm{j}(2\times 10^6)t} = \cos\left((2\times 10^6)t\right). \quad (62)$$

(c) In the `RLCSliders.ipynb` Jupyter notebook, move the sliders to approximately $R = 0\,\Omega$ and $C = 10\,\mathrm{nF}$. **Comment on the graph of $V_C(t)$ and the location of the eigenvalues on the complex plane. Do the waveforms for $x_1(t)$ and $x_2(t)$ decay to 0?**

Note: Because there is no resistance, this is called the "undamped" case. **Solution:** No, these waveforms are sinusoids and do not die out over time. The eigenvalues are located on the imaginary axis at coordinates $(0, -2\times 10^6)$ and $(0, 2\times 10^6)$.

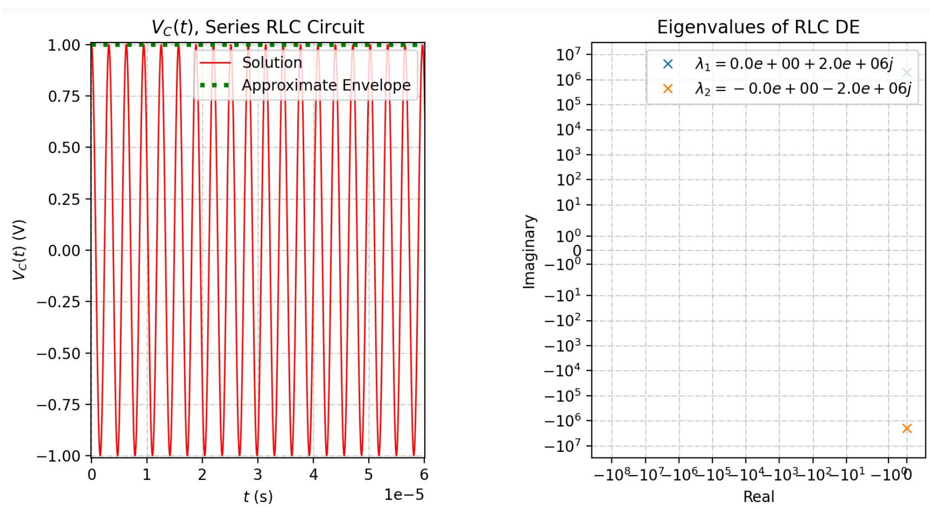


Figure 2: $V_C(t)$ and eigenvalues for undamped case.

Note that if the matrix has eigenvalues which are purely imaginary, the resulting system shows oscillations. Although there is no external energy provided to the system after $t > 0$, the system keeps on oscillating forever. The circuit-theoretical explanation of this is that the total energy of the system remains constant as there is no resistor in the circuit which is responsible for power dissipation. At the beginning the capacitor stores the entire energy while the inductor has 0 energy. As $V_c(t)$ starts decreasing, portion of the capacitor energy gets transferred to the inductor.

When $V_c(t)$ becomes 0, the entire energy of the capacitor gets transferred to the inductor. This process goes on for infinite time resulting in an oscillatory system. The energy waveforms below show how the energy gets transferred from the capacitor to the inductor and vice-versa.

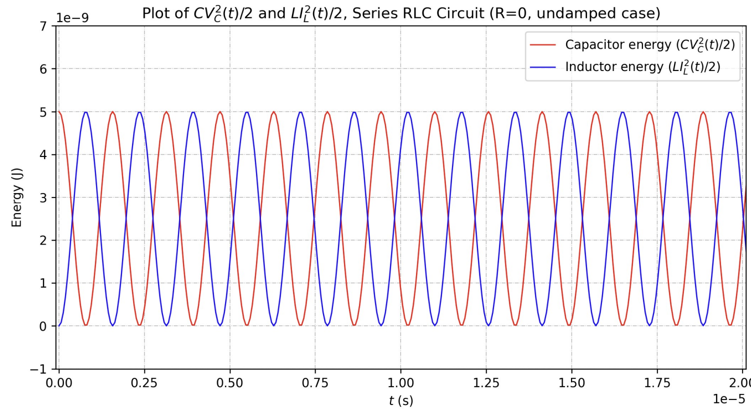
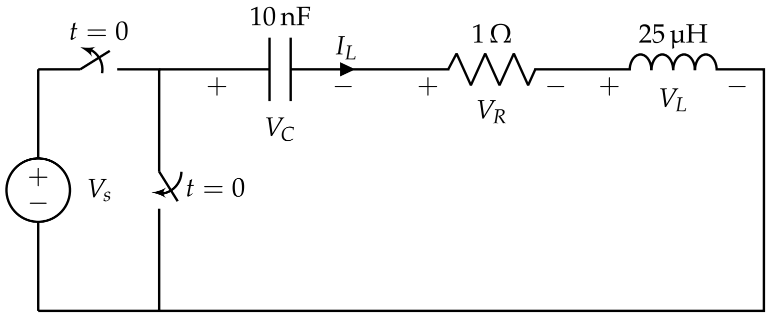


Figure 3: Periodic transfer of stored energy between the capacitor and the inductor.

5. RLC Responses: Underdamped Case

Building on the previous problem, consider the following circuit with specified component values:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2. You may round numbers to make the algebra more simple. You may use a calculator or the attached `RLC_Calc.ipynb` Jupyter Notebook for numerical calculations.

(a) Now suppose that $R = 1\,\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1\,\text{V}$. **Find the initial conditions for $\vec{\tilde{x}}(0)$.** Recall that $\vec{\tilde{x}}$ is in the changed "nice" eigenbasis coordinates from the first problem. **Solution:**

Under these conditions, we can solve for

$$\lambda_1 = -0.02 \times 10^6 + \text{j}(2 \times 10^6), \tag{63}$$

$$\lambda_2 = -0.02 \times 10^6 - \text{j}(2 \times 10^6), \tag{64}$$

$$V = \begin{bmatrix} 1 & 1 \\ \frac{1}{-0.0002+\text{j}(0.02)} & \frac{1}{-0.0002-\text{j}(0.02)} \end{bmatrix} \tag{65}$$

$$V^{-1} = \begin{bmatrix} 0.5 + \text{j}(0.005) & \text{j}(0.01) \\ .5 - \text{j}(0.005) & -\text{j}(0.01) \end{bmatrix} \tag{66}$$

$$\vec{\tilde{x}}(0) = V^{-1}\vec{x}(0) = V^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \text{j}(0.01) \\ -\text{j}(0.01) \end{bmatrix} \tag{67}$$

(b) Using the diagonalized system from 2(g) and continuing the previous part, **find** $x_1(t) = I_L(t)$ **and** $x_2(t) = V_C(t)$ **for** $t \geq 0$. Remember that your final expressions for $x_1(t)$ and $x_2(t)$ should be real functions (no imaginary terms).

(HINT: Remember that $\text{e}^{a+\text{j}b} = \text{e}^a\text{e}^{\text{j}b}$. Use Euler's formula.)

Solution:

$$\tilde{A} = \begin{bmatrix} -0.02 \times 10^6 + \text{j}(2 \times 10^6) & 0 \\ 0 & -0.02 \times 10^6 - \text{j}(2 \times 10^6) \end{bmatrix} \tag{68}$$

$$\begin{bmatrix} \frac{\text{d}}{\text{d}t}\tilde{x}_1(t) \\ \frac{\text{d}}{\text{d}t}\tilde{x}_2(t) \end{bmatrix} = \begin{bmatrix} -0.02 \times 10^6 + \text{j}(2 \times 10^6) & 0 \\ 0 & -0.02 \times 10^6 - \text{j}(2 \times 10^6) \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} \tag{69}$$

Therefore:

$$\tilde{x}_1(t) = K_1\text{e}^{(-0.02\times 10^6 + \text{j}(2\times 10^6))t} \tag{70}$$

$$\widetilde{x}_2(t) = K_2 e^{(-0.02\times 10^6 - j(2\times 10^6))t}. \tag{71}$$

Solving for K with the initial condition gives:

$$\widetilde{x}_1(t) = j(0.01)e^{(-0.02\times 10^6 + j2\times 10^6)t} \tag{72}$$
$$\widetilde{x}_2(t) = -j(0.01)e^{(-0.02\times 10^6 - j(2\times 10^6))t}. \tag{73}$$

Converting back to the $(\vec{x})$ coordinates:

$$\vec{x}(t) = V\vec{\widetilde{x}}(t) = \begin{bmatrix} 1 & 1 \\ -0.5 - j50 & -0.5 + j50 \end{bmatrix} \begin{bmatrix} j(0.01)e^{(-0.02\times 10^6 + j(2\times 10^6))t} \\ -j(0.01)e^{(-0.02\times 10^6 - j(2\times 10^6))t} \end{bmatrix} \tag{74}$$

$$\begin{aligned}
x_1(t) &= j(0.01)e^{(-0.02\times 10^6 + j(2\times 10^6))t} - j(0.01)e^{(-0.02\times 10^6 - j(2\times 10^6))t} && (75)\\
&= j(0.01)e^{-(0.02\times 10^6)t}e^{j(2\times 10^6)t} - j(0.01)e^{-(0.02\times 10^6)t}e^{-j(2\times 10^6)t} && (76)\\
&= e^{-(0.02\times 10^6)t}\left(j(0.01)e^{j(2\times 10^6)t} - j(0.01)e^{-j(2\times 10^6)t}\right) && (77)\\
&= -0.02e^{-(0.02\times 10^6)t}\sin\left((2\times 10^6)t\right) && (78)\\
x_2(t) &= (0.5 - j(0.005))e^{(-0.02\times 10^6 + j(2\times 10^6))t} + (0.5 + j(0.005))e^{(-0.02\times 10^6 - j(2\times 10^6))t} && (79)\\
&= (0.5 - j(0.005))e^{-(0.02\times 10^6)t}e^{j(2\times 10^6)t} + (0.5 + j(0.005))e^{-(0.02\times 10^6)t}e^{-j(2\times 10^6)t} && (80)\\
&= e^{-(0.02\times 10^6)t}\left((0.5 - j(0.005))e^{j(2\times 10^6)t} + (0.5 + j(0.005))e^{-j(2\times 10^6)t}\right) && (81)\\
&= e^{-(0.02\times 10^6)t}\cos\left((2\times 10^6)t\right) + 0.01\cdot e^{-(0.02\times 10^6)t}\sin\left((2\times 10^6)t\right). && (82)
\end{aligned}$$

(c) In the `RLCSliders.ipynb` Jupyter notebook, move the sliders to approximately $R = 1\Omega$ and $C = 10nF$. **Comment on the graph of $V_C(t)$ and the location of the eigenvalues on the complex plane. Do the waveforms for $x_1(t)$ and $x_2(t)$ decay to 0?**

Note: Because the resistance is so small, this is called the "underdamped" case. It is good to reflect upon these waveforms to see why engineers consider such behavior to be reflective of systems that don't have enough damping.

Solution: Yes, the waveforms decay to 0. They appear to be sinusoids that are decaying exponentially. The eigenvalues should be located at coordinates $(-0.02\times 10^6, 2\times 10^6)$ and $(-0.02\times 10^6, -2\times 10^6)$.

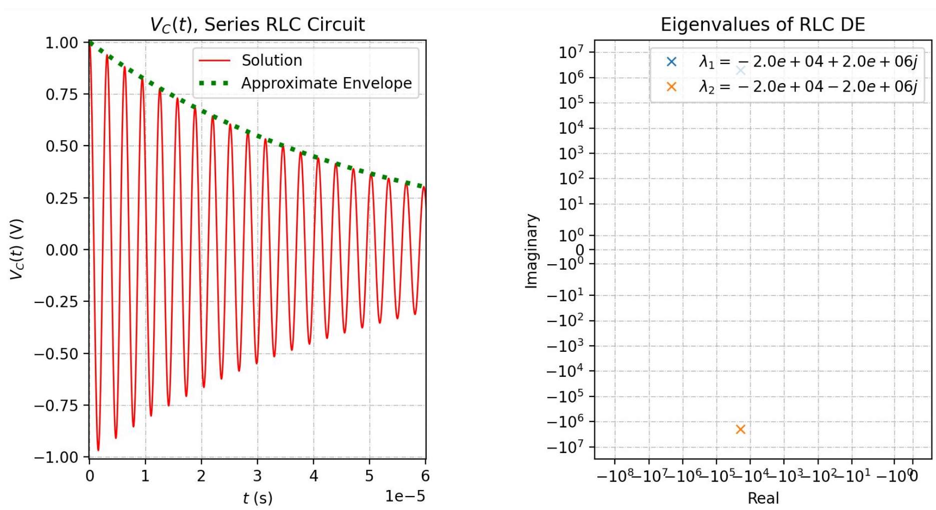


Figure 4: $V_C(t)$ and eigenvalues for underdamped case

(d) Notice that you got answers in terms of complex exponentials. **Why did the final voltage and current waveforms end up being purely real?**

Solution: In this case, it's because of the complex conjugacy of the quantities in the problem. The eivenvalues and their associated eigenvectors were complex conjugates, as were the transformed solutions $\widetilde{x}_1(t)$ and $\widetilde{x}_2(t)$. When we applied the inverse transformation to $\widetilde{x}_1(t)$ and $\widetilde{x}_2(t)$, we added together many complex conjugate terms, and the imaginary parts cancelled out.

Now, was this just a fluke that just happened to line up perfectly? Is there some A matrix out there with real-valued entries that will result in a complex solution? Or is something more profound going on?

It turns to be no fluke. If the entries in the A matrix are real, and the initial condition $\vec{x}_0$ is real, then the solution to the differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = A\vec{x}$ with $\vec{x}(0) = \vec{x}_0$ will also be real, *regardless of whether the eigenvalues of A are real, imaginary, or complex.* If a matrix $A \in \mathbb{R}^{n\times n}$ has some complex eigenvalues, then those eigenvalues will always arise in complex conjugate pairs. Furthermore, the eigenvectors associated to those eigenvalues arise on complex conjugate pairs. This will lead to the kind of cancellation that you saw in here, every single time.

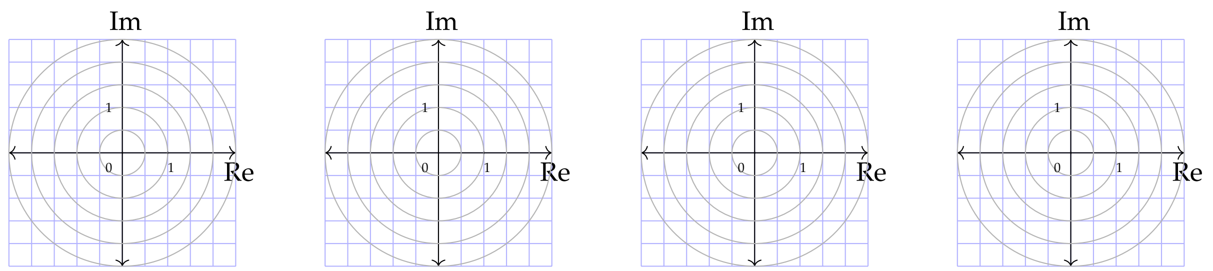
After all, the quantities that we observe in the world are always purely real, so we would expect that the solutions to our models would also be real-valued.

6. Phasors

(a) Three components in series. **(b)** Vector sum of $\vec{u}$ and $\vec{v}$.

Figure 5: Relevant problem figures.

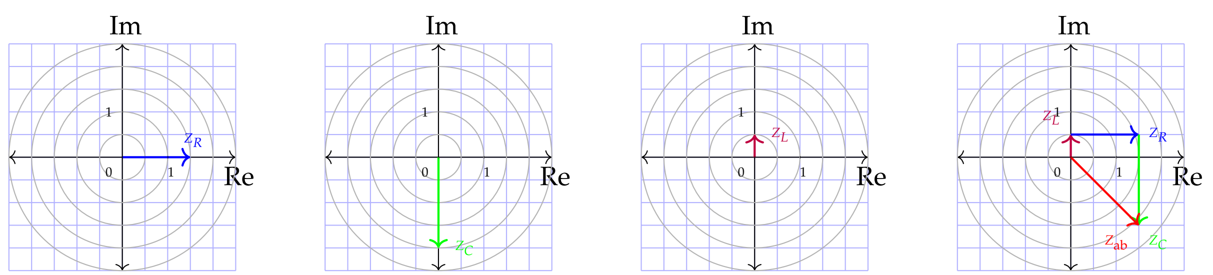
(a) **For the component values given in figure 5a, evaluate the impedances Z_R, Z_C, Z_L and the series equivalent impedance Z_{ab} for the case $\omega = \frac{1}{2}\frac{\text{rad}}{\text{s}}$. Draw the individual impedances as "vectors" on the complex plane. On the last plot draw Z_{ab} as a vector sum (as shown in figure 5b) of Z_R, Z_C, and Z_L on the complex plane. Then give the magnitude and phase of Z_{ab}.**



(a) $Z_R(@\omega = 0.5\,\frac{\text{rad}}{\text{s}})$ **(b)** $Z_C(@\omega = 0.5\,\frac{\text{rad}}{\text{s}})$ **(c)** $Z_L(@\omega = 0.5\,\frac{\text{rad}}{\text{s}})$ **(d)** $Z_{\text{ab}}(@\omega = 0.5\,\frac{\text{rad}}{\text{s}})$

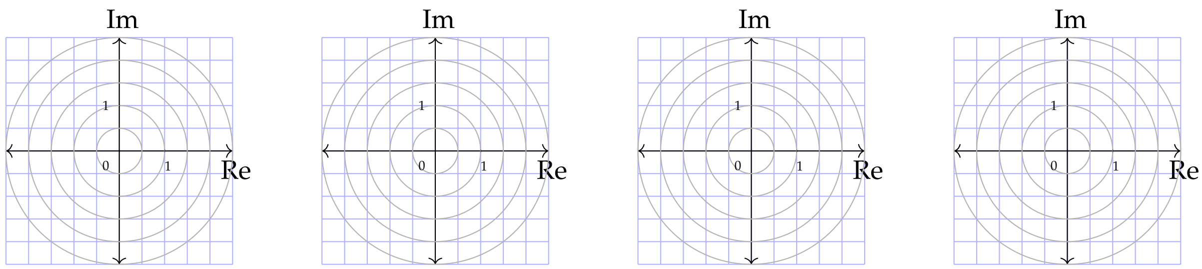
Solution:

Substituting for $\omega = \frac{1}{2}\frac{\text{rad}}{\text{s}}$ in the impedance formulas, we get: $Z_R = 1.5\,\Omega$, $Z_C = -\text{j}2\Omega$ and $Z_L = \text{j}0.5\Omega$. Since the elements are in series, $Z_{\text{ab}} = Z_L + Z_C + Z_R = (1.5 - \text{j}1.5)\,\Omega$. Z_{ab} has magnitude $\sqrt{(1.5\,\Omega)^2 + (-1.5\,\Omega)^2} = 1.5\sqrt{1+1}\Omega = 1.5\sqrt{2}\ \Omega$ and phase $\text{atan2}(-1.5, 1.5) = -\frac{\pi}{4}$ rad or $-45°$. Following are the plots:



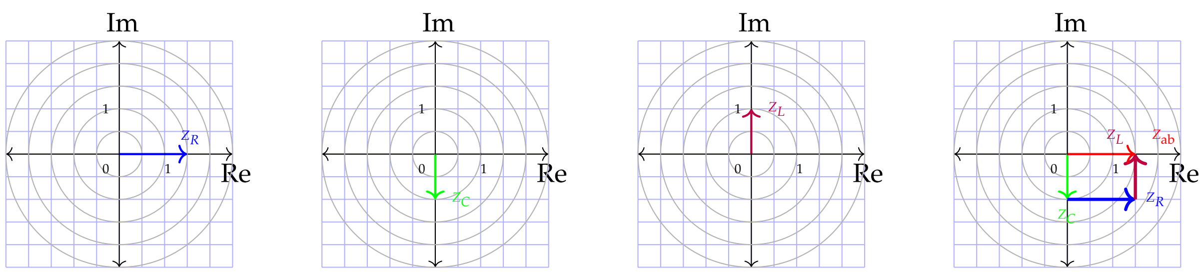
(a) $Z_R(@\omega = 0.5\,\frac{\text{rad}}{\text{s}})$ **(b)** $Z_C(@\omega = 0.5\,\frac{\text{rad}}{\text{s}})$ **(c)** $Z_L(@\omega = 0.5\,\frac{\text{rad}}{\text{s}})$ **(d)** $Z_{\text{ab}}(@\omega = 0.5\,\frac{\text{rad}}{\text{s}})$

(b) **For the component values given in figure 5a, evaluate the impedances Z_R, Z_C, Z_L and the series equivalent impedance Z_{ab} for the case $\omega = 1\,\frac{\text{rad}}{\text{s}}$. Draw the individual impedances as "vectors" on the complex plane. On the last plot draw Z_{ab} as a vector sum (as shown in figure 5b) of Z_R, Z_C, and Z_L on the complex plane. Then give the magnitude and phase of Z_{ab}.**



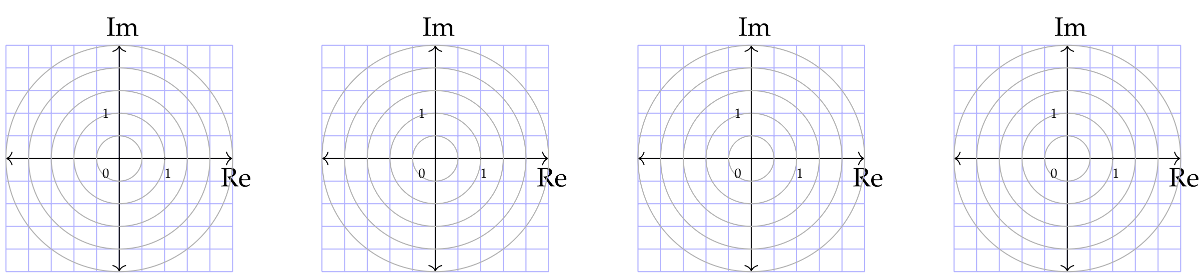
(a) $Z_R(@\omega = 1\,\frac{\text{rad}}{\text{s}})$ **(b)** $Z_C(@\omega = 1\,\frac{\text{rad}}{\text{s}})$ **(c)** $Z_L(@\omega = 1\,\frac{\text{rad}}{\text{s}})$ **(d)** $Z_{\text{ab}}(@\omega = 1\,\frac{\text{rad}}{\text{s}})$

Solution:
Following the same method as last time, with $\omega = 1\,\frac{\text{rad}}{\text{s}}$, we can compute $Z_R = 1.5\,\Omega$, $Z_C = -\text{j}\Omega$, $Z_L = \text{j}\Omega$, and $Z_{\text{ab}} = 1.5\,\Omega$. Z_{ab} has magnitude $1.5\,\Omega$ and phase 0 rad or $0°$.



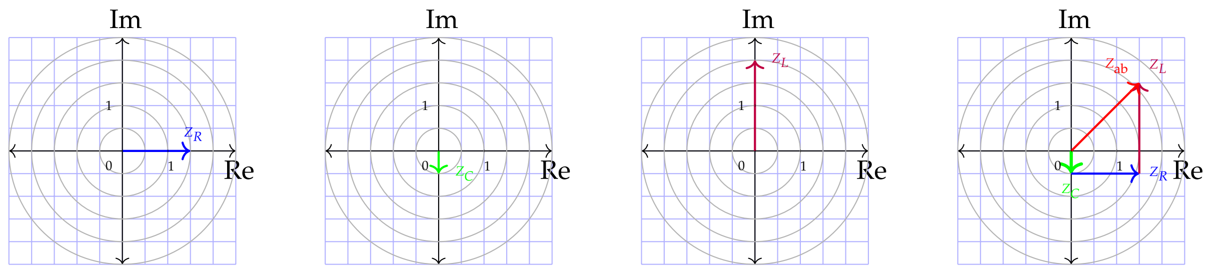
(a) $Z_R(@\omega = 1\,\frac{\text{rad}}{\text{s}})$ **(b)** $Z_C(@\omega = 1\,\frac{\text{rad}}{\text{s}})$ **(c)** $Z_L(@\omega = 1\,\frac{\text{rad}}{\text{s}})$ **(d)** $Z_{\text{ab}}(@\omega = 1\,\frac{\text{rad}}{\text{s}})$

(c) **For the component values given in figure 5a, evaluate the impedances Z_R, Z_C, Z_L and the series equivalent impedance Z_{ab} for the case $\omega = 2\,\frac{\text{rad}}{\text{s}}$. Draw the individual impedances as "vectors" on the complex plane. On the last plot draw Z_{ab} as a vector sum (as shown in figure 5b) of Z_R, Z_C, and Z_L on the complex plane. Then give the magnitude and phase of Z_{ab}.**



(a) $Z_R(@\omega = 2\,\frac{\text{rad}}{\text{s}})$ **(b)** $Z_C(@\omega = 2\,\frac{\text{rad}}{\text{s}})$ **(c)** $Z_L(@\omega = 2\,\frac{\text{rad}}{\text{s}})$ **(d)** $Z_{\text{ab}}(@\omega = 2\,\frac{\text{rad}}{\text{s}})$

Solution: Again, following the same method as last time, with $\omega = 2\,\frac{\text{rad}}{\text{s}}$, we can compute $Z_R = 1.5\,\Omega$, $Z_C = -\text{j}0.5\Omega$, $Z_L = \text{j}2\Omega$, and $Z_{\text{ab}} = (1.5 + \text{j}1.5)\ \Omega$. Z_{ab} magnitude $1.5\sqrt{2}\ \Omega$ and phase $+\frac{\pi}{4}$ rad or $45°$.



(a) $Z_R(@\omega = 2\,\frac{\text{rad}}{\text{s}})$ **(b)** $Z_C(@\omega = 2\,\frac{\text{rad}}{\text{s}})$ **(c)** $Z_L(@\omega = 2\,\frac{\text{rad}}{\text{s}})$ **(d)** $Z_{\text{ab}}(@\omega = 2\,\frac{\text{rad}}{\text{s}})$

(d) The "natural frequency" ω_n is defined as the frequency ω_n where the net impendance is purely real. **For the series combination of RLC elements, Z_{ab}, appearing in figure 5a, what is the "natural frequency" ω_n?**

Fact: We call this the "natural frequency" since it is the frequency at which the magnitude of the impedance is the smallest. It turns out to be the case that such a circuit will oscillate at this frequency if it was underdamped (if R was small enough) and we set it up in a problem like that of the underdamped problem on this HW set.

Solution:

From our above answers, the natural frequency, $\omega_n = 1\,\frac{\text{rad}}{\text{s}}$. This is where the imaginary parts of the impedance cancel each other.

7. Low-pass Filter

You have a 1 kΩ resistor and a 1 µF capacitor wired up as a low-pass filter.

(a) **Draw the filter circuit, labeling the input node, output node, and ground.**

Solution:

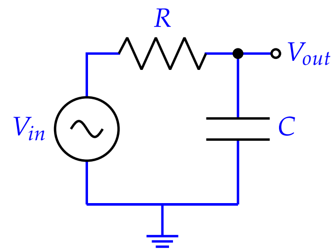


Figure 12: A simple RC circuit

(b) **Write down the transfer function of the filter, $H(j\omega)$ that relates the output voltage phasor to the input voltage phasor.** Be sure to use the given values for the components.

Solution: First, we convert everything into the phasor domain. We have,

$$Z_R = R = 1 \times 10^3 \Omega \tag{83}$$
$$Z_C = \frac{1}{j\omega C} = \frac{1}{j\omega \times 10^{-6}} \mathrm{F} \tag{84}$$

In phasor domain, we can treat these impedances essentially like we treat resistors and recognize the voltage divider. Hence,

$$\widetilde{V}_{out} = \frac{Z_C}{Z_C + Z_R} \widetilde{V}_{in} \tag{85}$$
$$\frac{\widetilde{V}_{out}}{\widetilde{V}_{in}} = H(j\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \tag{86}$$
$$= \frac{1}{1 + j\omega RC} \tag{87}$$
$$= \frac{1}{1 + j\omega / \frac{1}{RC}} \tag{88}$$
$$= \frac{1}{1 + j\omega \times 10^{-3}} \tag{89}$$
$$\tag{90}$$

For any system, a corner/cutoff frequency is the point in the system's frequency response after which the input begins to be attenuated. More concretely, suppose that for a particular system the corner frequency is ω_C. Then this means that for input signals with frequency $\omega_{\text{in}} << \omega_C$ the gain of the system is approximately 1 and for input signals with frequency $\omega_{\text{in}} >> \omega_C$ the gain tends to 0. For our particular circuit, the corner frequency $\omega_C = \frac{1}{RC} = \frac{1}{10^3 \cdot 10^{-6}} = 10^3$ rad/sec.

(c) **Write an exact expression for the *magnitude* of $H(j\omega = j10^6)$, and give an approximate numerical answer.**

Solution: From the previous subpart

$$H(j\omega) = \frac{1}{1 + j\omega/\omega_C}.$$

Then

$$|H(j\omega)| = \frac{|1|}{|1 + j\omega/\omega_C|}.$$

Since $|1 + j\omega/\omega_C| = \sqrt{1 + \omega^2/\omega_c^2}$,

$$|H(j\omega)| = \frac{1}{\sqrt{1 + \omega^2/\omega_c^2}}.$$

Plugging in for $\omega = 10^6$:

$$|H(j\omega = 10^6)| = \frac{1}{\sqrt{1 + 10^6}}.$$

Approximately:

$$|H(j\omega = 10^6)| \approx \frac{1}{\sqrt{10^6}} = \frac{1}{10^3} = 10^{-3}$$

$$|H(j\omega = 10^6)| \approx 10^{-3}$$

(d) **Write an exact expression for the *phase* of $H(j\omega = j1)$, and give an approximate numerical answer.**

Solution: As before

$$H(j\omega) = \frac{1}{1 + j\omega/\omega_C}.$$

Then

$$\angle H(j\omega) = \angle 1 - \angle(1 + j\omega/\omega_C) = -\angle(1 + j\omega/\omega_C) = \angle(1 - j\omega/\omega_C).$$

Thus the expression for the transfer function's phase is given by:

$$\angle H(j\omega) = \text{atan2}(-\frac{\omega}{\omega_c}, 1)$$

Plugging in for $\omega = 1$:

$$\angle H(j\omega = 1) = \text{atan2}(-\frac{10^0}{10^3}, 1) = \tan^{-1}(-10^{-3})$$

By the small angle approximation, this is:

$$\angle H(j\omega = 1) \approx -10^{-3}\text{rad}$$

(e) **Write down an expression for the time-domain output waveform $V_{out}(t)$ of this filter if the input voltage is $V(t) = 1\sin(1000t)$ V.** You can assume that any transients have died out — we are interested in the steady-state waveform.

Solution: We can find the magnitude and phase of the transfer function at this point:

$$|H(j\omega = 10^3)| = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

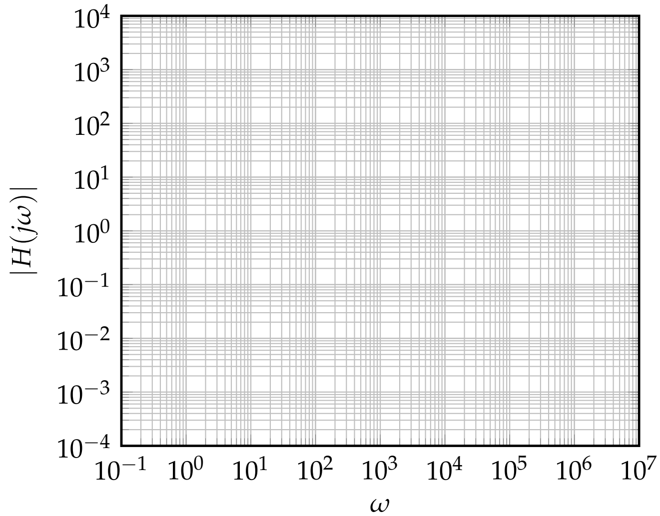
$$\angle H(j\omega = 10^3) = \text{atan2}(-1, 1) = -45^\circ$$

Therefore the output will be:

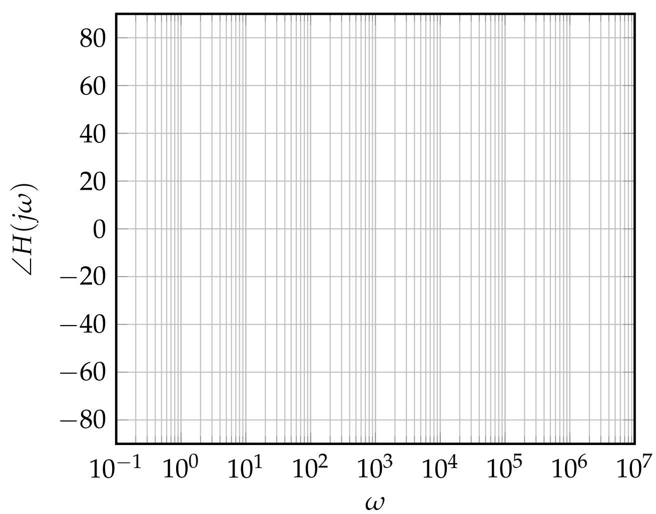
$$V_{out}(t) = \frac{1}{\sqrt{2}}\sin(1000t - 45^\circ)$$

(f) **Sketch (by hand) the Bode plot (both magnitude and phase) of the filter on the graph paper below.**

Log-log plot of transfer function magnitude

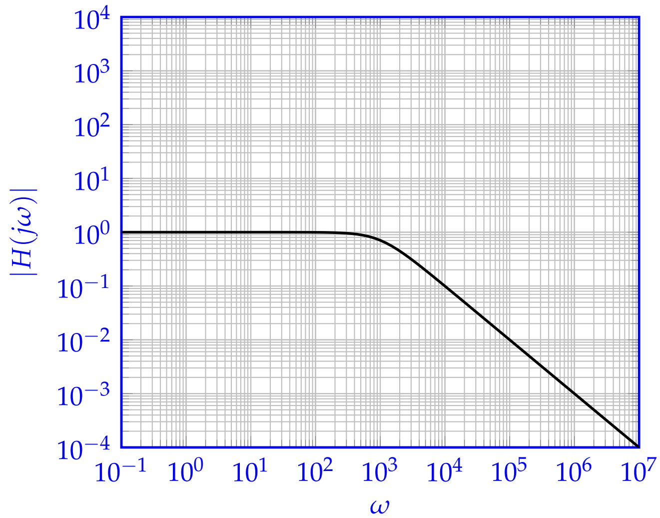


Semi-log plot of transfer function phase

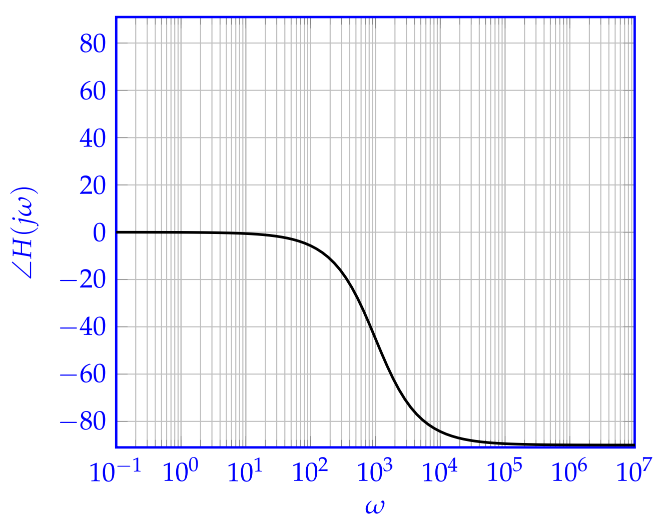


Solution:

Log-log plot of transfer function magnitude



Semi-log plot of transfer function phase



8. (OPTIONAL) Make Your Own Problem.

Write your own problem about content covered in the course thus far, and provide a thorough solution to it.

NOTE: This can be a totally new problem, a modification on an existing problem, or a Jupyter part for a problem that previously didn't have one. Please cite all sources for anything (including course material) that you used as inspiration.

NOTE: High-quality problems may be used as inspiration for the problems we choose to put on future homeworks or exams.

9. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!

We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) **What sources (if any) did you use as you worked through the homework?**

(b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

(c) **Roughly how many total hours did you work on this homework? Write it down here where you'll need to remember it for the self-grade form.**

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- Nathan Lambert.
- Moses Won.
- Kris Pister.
- Tanmay Gautam.