Homework 4

This homework is due on Friday, February 18, 2022 at 11:59PM. Self-grades and HW Resubmissions are due the following Friday, February 25, 2022 at 11:59PM.

1. **Reading Lecture Notes**

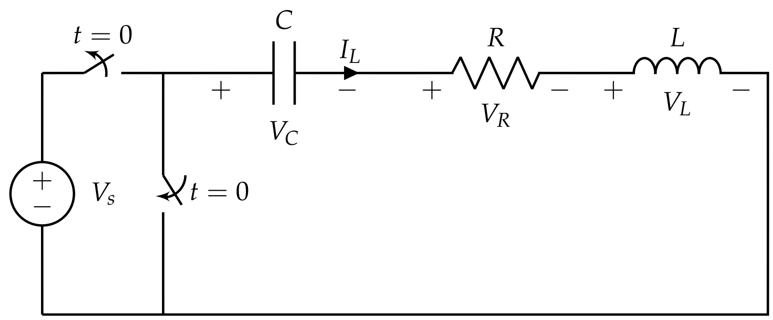
   Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: Note 5 and Note 6.

   (a) What is the equation that governs the relationship that an inductor enforces on the voltage across it and the current through it? What is the behavior of an inductor under DC current (i.e. constant current)?

   (b) What is the definition of impedance? What quantity is impedance analogous to in the static DC circuit analysis you learned in 16A?

2. RLC Responses: Initial Part

Consider the following circuit:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

The sequence of problems 2 - 6 combined will try to show you the various RLC system responses and how they relate to changing circuit properties.

(a) We first need to construct our state space system. Our state variables are the current through the inductor $x_1(t) = I_L(t)$ and the voltage across the capacitor $x_2(t) = V_C(t)$ since these are the quantities whose derivatives show up in the system of equations governing our circuit. Now, **show that the system of differential equations in terms of our state variables that describes this circuit for $t \geq 0$ is**

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = -\frac{R}{L}x_1(t) - \frac{1}{L}x_2(t) \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = \frac{1}{C}x_1(t). \tag{2}$$

(b) **Write the system of equations in vector/matrix form with the vector state variable $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} I_L(t) \\ V_C(t) \end{bmatrix}$.** This should be in the form $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t)$ with a 2×2 matrix A.

(c) **Show that, for the 2×2 matrix A, the two eigenvalues of A are**

$$\lambda_1 = -\frac{1}{2}\frac{R}{L} + \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}} \tag{3}$$

$$\lambda_2 = -\frac{1}{2}\frac{R}{L} - \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}. \tag{4}$$

(*HINT: The quadratic formula will be involved.*)

(d) **Under what condition on the circuit parameters R, L, C will A have two distinct real eigenvalues?**

(e) **Under what condition on the circuit parameters R, L, C will A have two imaginary eigenvalues? What will the eigenvalues be in this case?**

(f) Assuming that the circuit parameters are such that there are a pair of (potentially complex) eigenvalues λ_1, λ_2 so that $\lambda_1 \neq \lambda_2$, **show that the corresponding eigenvectors $\vec{v}_{\lambda_1}, \vec{v}_{\lambda_2}$ are**

$$\vec{v}_{\lambda_1} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_1 C} \end{bmatrix} \quad \text{and} \quad \vec{v}_{\lambda_2} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_2 C} \end{bmatrix}. \tag{5}$$

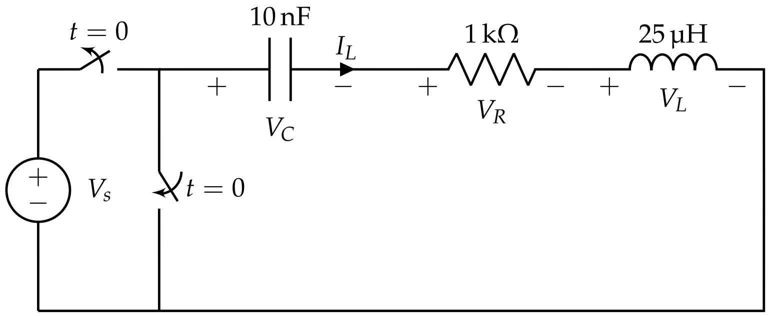
(g) Assuming circuit parameters such that the two eigenvalues of A are distinct, let $V = \begin{bmatrix} \vec{v}_{\lambda_1} & \vec{v}_{\lambda_2} \end{bmatrix}$ be a specific eigenbasis. Consider a coordinate system for which we can write $\vec{x}(t) = V\vec{\widetilde{x}}(t)$. **Show that the $\widetilde{A}$ so that $\frac{\mathrm{d}}{\mathrm{d}t}\vec{\widetilde{x}}(t) = \widetilde{A}\vec{\widetilde{x}}(t)$ is**

$$\widetilde{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \tag{6}$$

(*HINT: Write out the original differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t)$, then use the given change of coordinates to write everything in terms of $\vec{\widetilde{x}}(t)$.*)

3. RLC Responses: Overdamped Case

Building on the previous problem, consider the following circuit with specified component values:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2. You may use a calculator or the attached `RLC_Calc.ipynb` Jupyter Notebook for numerical calculations.

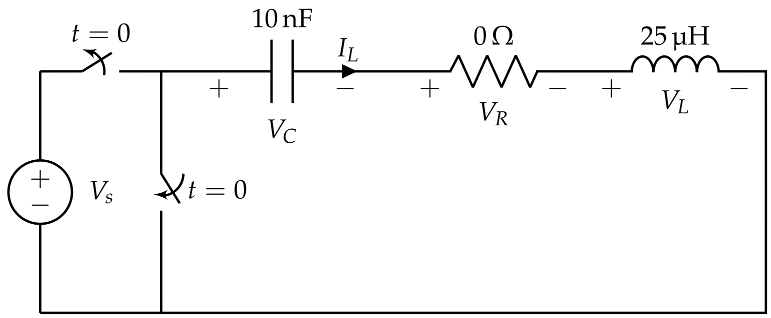
(a) Suppose $R = 1\,\text{k}\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1\,\text{V}$. **Find the initial conditions for $\vec{\tilde{x}}(0)$.** Recall that $\vec{\tilde{x}}$ are the eigenbasis coordinates from the first question.

(b) Using the diagonalized system from 2(g) and continuing the previous part, **find $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ for $t \geq 0$.**

(c) In the `RLCSliders.ipynb` Jupyter notebook, move the sliders to approximately $R = 1\,\text{k}\Omega$ and $C = 10\,\text{nF}$. **Comment on the graph of $V_C(t)$ and the location of the eigenvalues on the complex plane.**

4. RLC Responses: Undamped Case

Building on the previous problem, consider the following circuit with specified component values:



Assume that the capacitor is charged to V_s and there is no current in the inductor for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2. You may use a calculator or the attached `RLC_Calc.ipynb` Jupyter Notebook for numerical calculations.

(a) Suppose $R = 0\,\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1\,\text{V}$. **Find the initial conditions for $\vec{\tilde{x}}(0)$.** Recall that $\vec{\tilde{x}}$ is in the changed "nice" eigenbasis coordinates from the first problem.

(b) Using the diagonalized system from 2(g) and continuing the previous part, **find $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ for $t \geq 0$.** Remember that your final expressions for $x_1(t)$ and $x_2(t)$ should be real functions (no imaginary terms).

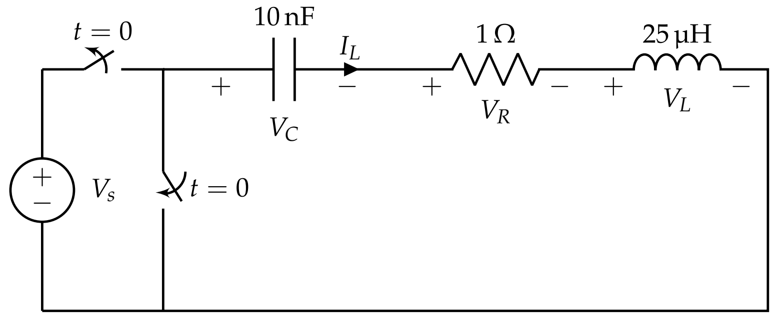
(HINT: Use Euler's formula.)

(c) In the `RLCSliders.ipynb` Jupyter notebook, move the sliders to approximately $R = 0\,\Omega$ and $C = 10\,\text{nF}$. **Comment on the graph of $V_C(t)$ and the location of the eigenvalues on the complex plane. Do the waveforms for $x_1(t)$ and $x_2(t)$ decay to 0?**

Note: Because there is no resistance, this is called the "undamped" case.

5. RLC Responses: Underdamped Case

Building on the previous problem, consider the following circuit with specified component values:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2. You may round numbers to make the algebra more simple. You may use a calculator or the attached `RLC_Calc.ipynb` Jupyter Notebook for numerical calculations.

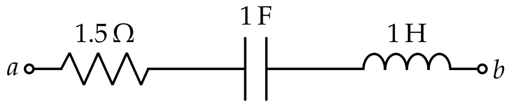
(a) Now suppose that $R = 1\,\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1\,\text{V}$. **Find the initial conditions for $\vec{\tilde{x}}(0)$.** Recall that $\vec{\tilde{x}}$ is in the changed "nice" eigenbasis coordinates from the first problem.

(b) Using the diagonalized system from 2(g) and continuing the previous part, **find $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ for $t \geq 0$.** Remember that your final expressions for $x_1(t)$ and $x_2(t)$ should be real functions (no imaginary terms).
(HINT: Remember that $\mathrm{e}^{a+\mathrm{j}b} = \mathrm{e}^{a}\mathrm{e}^{\mathrm{j}b}$. Use Euler's formula.)

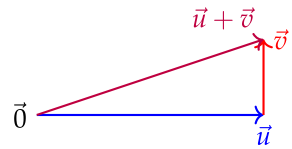
(c) In the `RLCSliders.ipynb` Jupyter notebook, move the sliders to approximately $R = 1\Omega$ and $C = 10nF$. **Comment on the graph of $V_C(t)$ and the location of the eigenvalues on the complex plane. Do the waveforms for $x_1(t)$ and $x_2(t)$ decay to 0?**
Note: Because the resistance is so small, this is called the "underdamped" case. It is good to reflect upon these waveforms to see why engineers consider such behavior to be reflective of systems that don't have enough damping.

(d) Notice that you got answers in terms of complex exponentials. **Why did the final voltage and current waveforms end up being purely real?**

6. Phasors



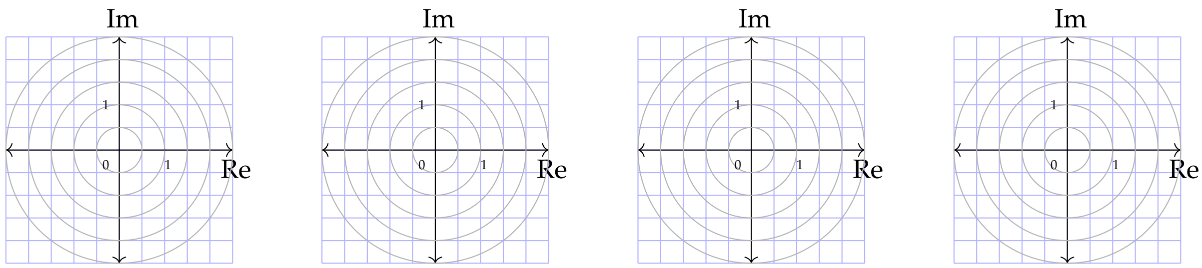
(a) Three components in series.



(b) Vector sum of $\vec{u}$ and $\vec{v}$.

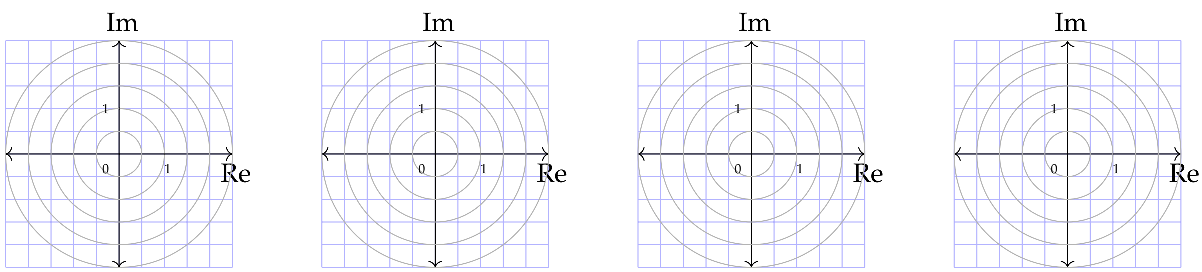
Figure 1: Relevant problem figures.

(a) **For the component values given in figure 1a, evaluate the impedances Z_R, Z_C, Z_L and the series equivalent impedance Z_{ab} for the case $\omega = \frac{1}{2}\,\frac{\text{rad}}{\text{s}}$. Draw the individual impedances as "vectors" on the complex plane. On the last plot draw Z_{ab} as a vector sum (as shown in figure 1b) of Z_R, Z_C, and Z_L on the complex plane. Then give the magnitude and phase of Z_{ab}.**



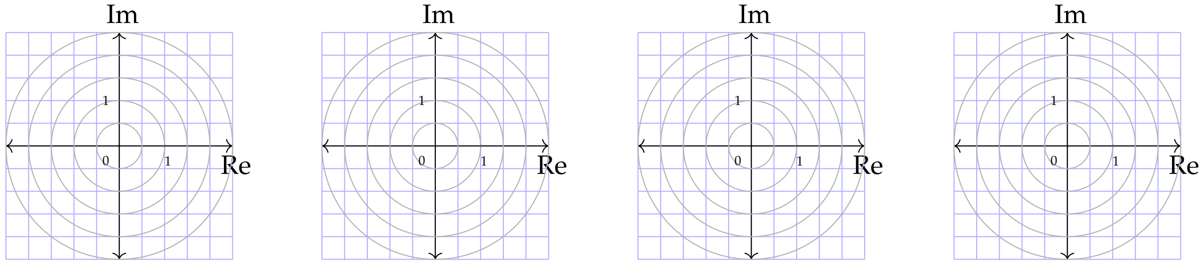
(a) $Z_R(@\omega = 0.5\,\frac{\text{rad}}{\text{s}})$ **(b)** $Z_C(@\omega = 0.5\,\frac{\text{rad}}{\text{s}})$ **(c)** $Z_L(@\omega = 0.5\,\frac{\text{rad}}{\text{s}})$ **(d)** $Z_{\text{ab}}(@\omega = 0.5\,\frac{\text{rad}}{\text{s}})$

(b) **For the component values given in figure 1a, evaluate the impedances Z_R, Z_C, Z_L and the series equivalent impedance Z_{ab} for the case $\omega = 1\,\frac{\text{rad}}{\text{s}}$. Draw the individual impedances as "vectors" on the complex plane. On the last plot draw Z_{ab} as a vector sum (as shown in figure 1b) of Z_R, Z_C, and Z_L on the complex plane. Then give the magnitude and phase of Z_{ab}.**



(a) $Z_R(@\omega = 1\,\frac{\text{rad}}{\text{s}})$ **(b)** $Z_C(@\omega = 1\,\frac{\text{rad}}{\text{s}})$ **(c)** $Z_L(@\omega = 1\,\frac{\text{rad}}{\text{s}})$ **(d)** $Z_{\text{ab}}(@\omega = 1\,\frac{\text{rad}}{\text{s}})$

(c) **For the component values given in figure 1a, evaluate the impedances Z_R, Z_C, Z_L and the series equivalent impedance Z_{ab} for the case $\omega = 2\,\frac{\text{rad}}{\text{s}}$. Draw the individual impedances as "vectors" on the complex plane. On the last plot draw Z_{ab} as a vector sum (as shown in figure 1b) of Z_R, Z_C, and Z_L on the complex plane. Then give the magnitude and phase of Z_{ab}.**



(a) $Z_R(@\omega = 2\,\frac{\text{rad}}{\text{s}})$ **(b)** $Z_C(@\omega = 2\,\frac{\text{rad}}{\text{s}})$ **(c)** $Z_L(@\omega = 2\,\frac{\text{rad}}{\text{s}})$ **(d)** $Z_{\text{ab}}(@\omega = 2\,\frac{\text{rad}}{\text{s}})$

(d) The "natural frequency" ω_n is defined as the frequency ω_n where the net impendance is purely real. **For the series combination of RLC elements, Z_{ab}, appearing in figure 1a, what is the "natural frequency" ω_n?**

Fact: We call this the "natural frequency" since it is the frequency at which the magnitude of the impedance is the smallest. It turns out to be the case that such a circuit will oscillate at this frequency if it was underdamped (if R was small enough) and we set it up in a problem like that of the underdamped problem on this HW set.

7. Low-pass Filter

You have a 1 kΩ resistor and a 1 μF capacitor wired up as a low-pass filter.

(a) **Draw the filter circuit, labeling the input node, output node, and ground.**

(b) **Write down the transfer function of the filter, $H(j\omega)$ that relates the output voltage phasor to the input voltage phasor.** Be sure to use the given values for the components.

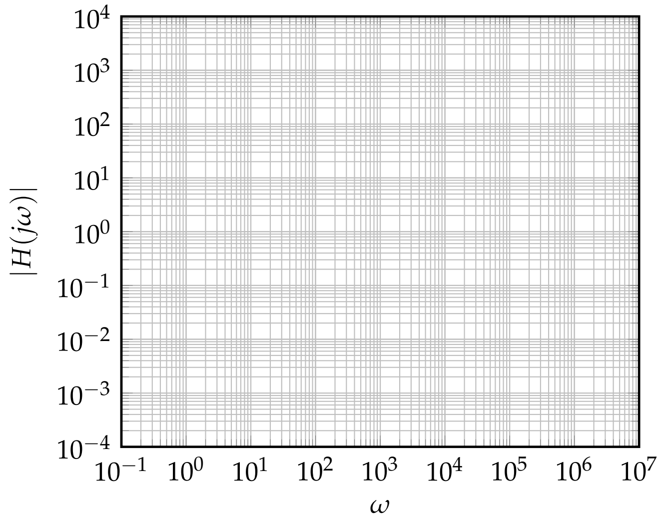
(c) **Write an exact expression for the *magnitude* of $H(j\omega = j10^6)$, and give an approximate numerical answer.**

(d) **Write an exact expression for the *phase* of $H(j\omega = j1)$, and give an approximate numerical answer.**

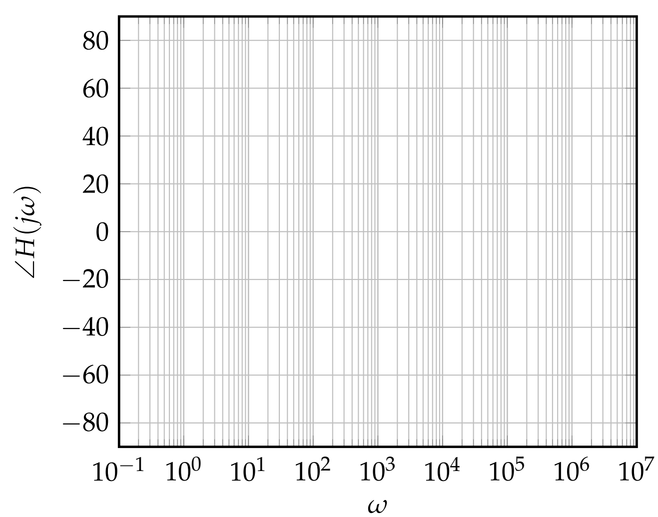
(e) **Write down an expression for the time-domain output waveform $V_{out}(t)$ of this filter if the input voltage is $V(t) = 1\sin(1000t)$ V.** You can assume that any transients have died out — we are interested in the steady-state waveform.

(f) **Sketch (by hand) the Bode plot (both magnitude and phase) of the filter on the graph paper below.**

Log-log plot of transfer function magnitude



Semi-log plot of transfer function phase



8. **(OPTIONAL) Make Your Own Problem.**

   Write your own problem about content covered in the course thus far, and provide a thorough solution to it.

   NOTE: This can be a totally new problem, a modification on an existing problem, or a Jupyter part for a problem that previously didn't have one. Please cite all sources for anything (including course material) that you used as inspiration.

   NOTE: High-quality problems may be used as inspiration for the problems we choose to put on future homeworks or exams.

9. **Homework Process and Study Group**

   Citing sources and collaborators are an important part of life, including being a student!

   We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

   (a) **What sources (if any) did you use as you worked through the homework?**

   (b) **If you worked with someone on this homework, who did you work with?**
   List names and student ID's. (In case of homework party, you can also just describe the group.)

   (c) **Roughly how many total hours did you work on this homework? Write it down here where you'll need to remember it for the self-grade form.**

Contributors:

- Anant Sahai.
- Jaijeet Roychowdhury.
- Sanjeet Batra.
- Aditya Arun.
- Alex Devonport.
- Regina Eckert.
- Kuan-Yun Lee.
- Druv Pai.
- Mike Danielczuk.
- Ayan Biswas.
- Sally Hui.
- Antroy Roy Chowdhury.
- Nathan Lambert.
- Moses Won.
- Kris Pister.
- Tanmay Gautam.