# Review Worksheet 2

**1. Cruise Control**

Suppose that we're working with a more advanced version of the robot car we built in the lab. Its state at timestep $k$ is $n$ dimensional, captured in $\vec{x}[k] \in \mathbb{R}^n$. The control at each timestep $\vec{u}[k] \in \mathbb{R}^m$. The system evolves according to the discrete-time equation

$$\vec{x}[k+1] = A\vec{x}[k] + B\vec{u}[k]. \tag{1}$$

We know the values of the $n \times n$ matrix $A$ and the $n \times m$ matrix $B$ (say for example estimated through system identification). For all parts, the initial condition is $\vec{x}[0] = \vec{0}$.

(a) We want to transform our system to a nicer set of coordinates in the $S$ basis. $S$ is an $n \times n$ invertible matrix. Let us write the transformed state as $\vec{z}[k] = S^{-1}\vec{x}[k]$ for all $k$. **Show that eq. (1) can be written in the form**

$$\vec{z}[k+1] = \widetilde{A}\vec{z}[k] + \widetilde{B}\vec{u}[k]. \tag{2}$$

**with $\widetilde{A} = S^{-1}AS$ and $\widetilde{B} = S^{-1}B$. Show your work.**

(b) **Prove that the system in eq. (2) is controllable if and only if the system in eq. (1) is controllable. Show your work.**

*(HINT: Connect the controllability matrix of the system in eq. (2) to the controllability matrix of the system in eq. (1).)*

(c) Suppose (just for this problem subpart) that the system in eq. (1) is controllable, and define its controllability matrix as $C \in \mathbb{R}^{n \times mn}$. We want to reach a goal state $\vec{g} \in \mathbb{R}^n$ in exactly $n$ timesteps; that is, we want $\vec{x}[n] = \vec{g}$. Recall $\vec{x}[0] = \vec{0}$. We define the sequence of minimum energy controls as $\vec{u}^{\star} = \begin{bmatrix} \vec{u}^{\star}[n-1] \\ \vdots \\ \vec{u}^{\star}[0] \end{bmatrix}$ where

$$\vec{u}^{\star} = \underset{\vec{u}}{\operatorname{argmin}} \|\vec{u}\|^2 \tag{3}$$

$$\text{s.t. } C\vec{u} = \vec{g}. \tag{4}$$

**Prove that $\vec{u}^{*}$ is orthogonal to the nullspace of $C$.**

*(HINT: Consider a solution of $C\vec{u} = \vec{g}$ as $\vec{u}_{\text{sol}} = \vec{u}_{\text{null}} + \vec{u}_{\text{other}}$, where $\vec{u}_{\text{null}}$ is the component of $\vec{u}_{\text{sol}}$ in the nullspace of $C$, (i.e. $\vec{u}_{\text{null}}$ the projection of $\vec{u}_{\text{sol}}$ onto the nullspace of $C$). In this case, we would have that $\vec{u}_{\text{null}} \perp \vec{u}_{\text{other}}$)*

(d) Now let us work in the standard basis, with the system in eq. (1). Suppose $n = 3$ and $m = 1$ (so that $A \in \mathbb{R}^{3\times3}$, $B \in \mathbb{R}^3$, $\vec{x}[k] \in \mathbb{R}^3$, and $u[k] \in \mathbb{R}$). The SVD of the controllability matrix $C$ is given as

$$C = \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \end{bmatrix} \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{v}_1^\top \\ \vec{v}_2^\top \\ \vec{v}_3^\top \end{bmatrix}, \tag{5}$$

with $\alpha > \beta > 0$.

**Is the system controllable?** Justify your answer.

**If the system *is* controllable**, find a sequence of inputs $\vec{u} = \begin{bmatrix} u[2] & u[1] & u[0] \end{bmatrix}^\top$, such that $\vec{x}[3] = \vec{g}$, for a specific $\vec{g} \in \mathbb{R}^3$. (Here $\vec{u}$ should be a function of $\vec{g}$).

**If the system *is not* controllable, find a $\vec{g} \in \mathbb{R}^3$ that is unreachable by the system,** i.e. find $\vec{g}$ such that there is *no* sequence of inputs $\vec{u}$ that makes $\vec{x}[3] = \vec{g}$.

**All answers for this problem part should be in terms of $\vec{w}_i$, $\vec{v}_i$, $\alpha$, and $\beta$.**

*(HINT: Remember how the SVD is connected to the column space and null space of the matrix and that $\vec{x}[0] = \vec{0}$.)*

  

(e) We continue the setup of the previous part, repeated here. We work in the standard basis, with the system in eq. (1). The SVD of the controllability matrix $C$ is given as in (5), with $\alpha > \beta > 0$.

Let $H \subseteq \mathbb{R}^3$ be the vector subspace of inputs $\vec{u} = \begin{bmatrix} u[2] & u[1] & u[0] \end{bmatrix}^\top$ which set $\vec{x}[3] = \vec{0}$. **Give a basis for** $H$**.** Justify your answer.

**All answers for this problem part should be in terms of $\vec{w}_i$, $\vec{v}_i$, $\alpha$, and $\beta$. Show your work.**

*(HINT: Remember that $\vec{x}[0] = \vec{0}$ and $\vec{x}[3] = C\vec{u}$.)*

**2. Stability**

Consider the complex plane below, which is broken into non-overlapping regions A through H. The circle drawn on the figure is the unit circle $|\lambda| = 1$.

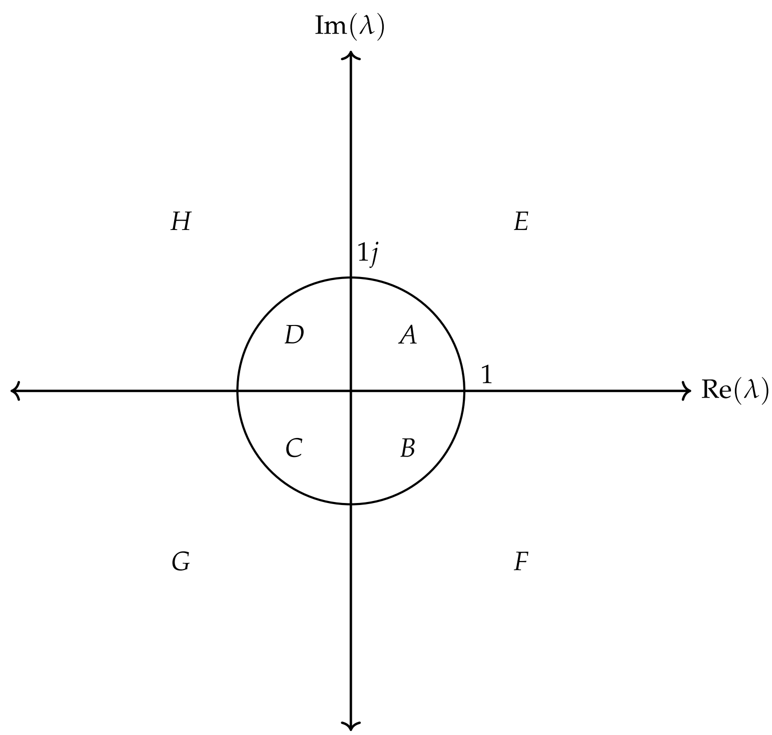


**Figure 1:** Complex plane divided into regions.

(a) Consider the continuous-time system $\frac{d}{dt}x(t) = \lambda x(t) + v(t)$ and the discrete-time system $y(t+1) = \lambda y(t) + w(t)$.

**In which regions can the eigenvalue $\lambda$ be for a *stable* system? Fill out the table below to indicate *stable* regions.** Assume that the eigenvalue $\lambda$ does not fall directly on the boundary between two regions.

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| **Continuous Time System** $x(t)$ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ |
| **Discrete Time System** $y(t)$ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ |

(b) Consider the continuous time system

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + u(t) \tag{6}$$

where $\lambda$ is real and $\lambda < 0$. Assume that $x(0) = 0$ and that $|u(t)| < \epsilon$ for all $t \geq 0$. **Prove that the solution $x(t)$ will be bounded (i.e. $\exists k$ so that $|x(t)| \leq k\epsilon$ for all time $t \geq 0$).**
*(HINT: Recall that the solution to such a first-order scalar differential equation is:*

$$x(t) = x(0)\mathrm{e}^{\lambda t} + \int_0^t u(\tau)\mathrm{e}^{\lambda(t-\tau)}\,\mathrm{d}\tau \tag{7}$$

*You may use this fact without proof.)*

**3. I bet Cal will win this year**

As huge fans of the Big Game, you and your friend want to bet on whether Cal or Stanford will win this year. You want to predict this year's result by analyzing historical records. Therefore, you decide to model this as a binary classification problem and do PCA for dimension reduction on the data you collected. The "+1" class represents victories of Cal and "−1" represents victories of Stanford.

After some research, you obtained a data matrix $A \in \mathbb{R}^{d \times n}$,

$$A = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix} \tag{8}$$

where each of the $n$ columns $\vec{x}_i$ denotes a game and each of the $d$ rows of $A$ contains information of a possibly relevant factor of the games (weather, location, date, air quality, etc).

(a) Let the full SVD of $A = U\Sigma V^\top$, where $A$ is given in eq. (8). You project your data along $\vec{u}_1$ and $\vec{u}_2$ (the first two principal components), and for comparison you also project your data along two randomly chosen directions $\vec{w}_1$ and $\vec{w}_2$ as well. You get the two pictures in Figure 2, but you forgot to label the axes. Of the two figures below, which one is the projection onto the principal components and which one is the projection onto the random directions? **Match axes (i), (ii), (iii), (iv) to $\vec{w}_1, \vec{w}_2, \vec{u}_1$, and $\vec{u}_2$, and justify your answer.**

*Note that there may be multiple correct matchings; you only need to find and justify one of them.*

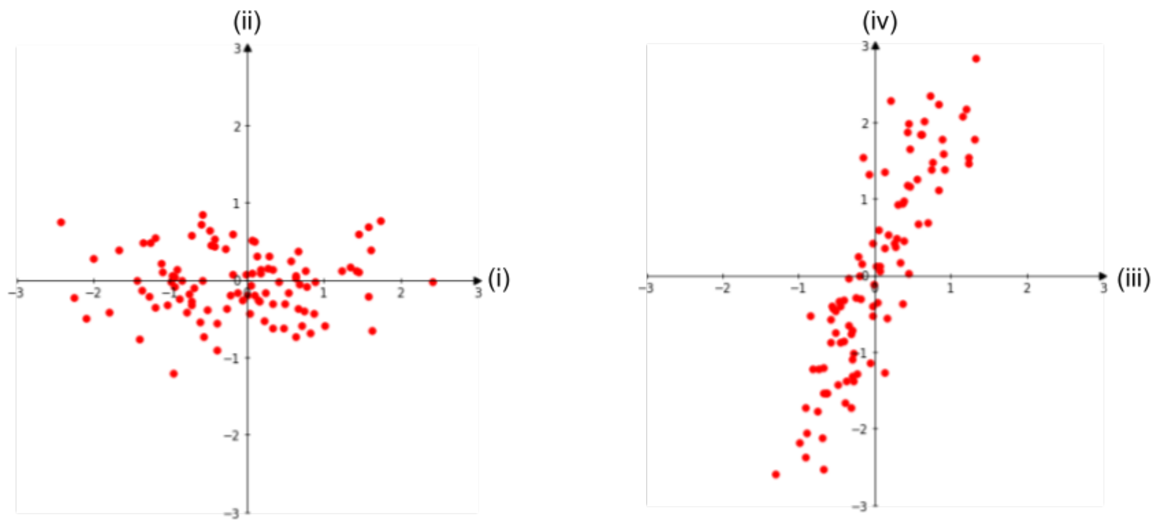


**Figure 2:** Projected datasets.

(b) In order to reduce the dimension of the data, we would like to project the data onto the first $k$ principal components of $A$, where $k$ is less than the original data dimension $d$. **Show how to find**

 

**the new vector $\vec{z}_i \in \mathbb{R}^k$ which is the $k$-dimesional, compressed version of $\vec{x}_i$.** You may use the SVD of $A$.

(c) Given a new set of projection coefficients denoted $\vec{z}_{\text{new}} \in \mathbb{R}^k$, we can define a classifier that will predict $+1$ (i.e., that Cal wins) if $\vec{w}_\star^\top \vec{z}_{\text{new}} > 0$ and $-1$ (i.e., that Stanford wins) otherwise.

Assume $d = 6$, $k = 4$, and $\vec{w}_\star = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^\top$. Let $A = U\Sigma V^\top$ for $A$ defined in eq. (8), and you find that $U$ is given by the identity matrix, i.e. $U = I_d$. Now suppose the data point for this year's big game $\vec{x}_{2021} = \begin{bmatrix} 3 & 6 & 4 & 1 & 9 & 6 \end{bmatrix}^\top$. **Would you bet on Cal or Stanford to win? Justify your answer.**
*(HINT: Don't forget to project your data onto the principal components.)*

  