Discussion 7A

The following notes are useful for this discussion: Note 10 and Note 11.

1. Changing behavior through feedback

In this question, we discuss how *feedback control* can be used to change the effective behavior of a system.

(a) Consider the scalar system:

$$x[i+1] = 0.9x[i] + u[i] + w[i] \tag{1}$$

where $u[i]$ is the control input we get to apply based on the current state and $w[i]$ is the external disturbance, each at time i.

Is the system stable? If $|w[i]| \leq \epsilon$, what can you say about $|x[i]|$ at all times i if you further assume that $u[i] = 0$ and the initial condition $x[0] = 0$? How big can $|x[i]|$ get?

(b) Suppose that we decide to choose a control law $u[i] = f x[i]$ to apply in feedback. **Given a specific λ, you want the system to behave like:**

$$x[i+1] = \lambda x[i] + w[i]? \tag{2}$$

To do so, how would you pick f?

NOTE: In this case, $w[i]$ can be thought of like another input to the system, except we can't control it.

(c) **For the previous part, which f would you choose to minimize how big $|x[i]|$ can get?**

(d) **What if instead of a 0.9, we had a 3 in the original eq. (1). Would system stability change? Would our ability to control λ change?**

(e) Now suppose that we have a vector-valued system with a vector-valued control:

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i] \tag{3}$$

where we further assume that B is an invertible square matrix. Futher, suppose we decide to apply linear feedback control using a square matrix F so we choose $\vec{u}[i] = F\vec{x}[i]$.
Given a specific A_{CL} we want the system to behave like:

$$\vec{x}[i+1] = A_{\text{CL}}\vec{x}[i] + \vec{w}[i]? \tag{4}$$

How would you pick F given knowledge of A, B and the desired goal dynamics A_{CL}? Will this work for any desired A_{CL}?

2. Controlling states by designing sequences of inputs

Consider the following matrix, with a simple structure (what does it do when it acts on a vector?):

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{5}$$

Let's assume we have a *discrete-time* system defined as follows:

$$\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]. \tag{6}$$

(a) We are given the initial condition $\vec{x}[0] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Let's say we want to achieve $\vec{x}[\ell] = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ for some specific $\ell \geq 0$ (Note that ℓ denotes total sequence length.) The key is that we want to be in this specific state at this specific timestep, ℓ. **What is the smallest ℓ such that this is possible? What is our choice of sequence of inputs $u[i]$?**

(b) **What if we started from $\vec{x}[0] = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$? What is the smallest ℓ and what is our choice of $u[i]$?**

(c) **If we start from $\vec{x}[0] = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, what is smallest ℓ such that $\vec{x}[\ell] = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, what is corresponding $u[i]$?**

(d) **If you would like to make sure that at time ℓ we are at $\vec{x}[\ell] = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ for the state, what controls could you use to get there? How big does ℓ have to be for this strategy to work?**

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