Discussion 4A

The following notes are useful for this discussion: Note 4, Note 5.

1. Phasor Analysis

Any sinusoidal time-varying function $x(t)$, representing a voltage or a current, can be expressed in the form

$$x(t) = \widetilde{X}e^{j\omega t} + \overline{\widetilde{X}}e^{-j\omega t}, \tag{1}$$

where $\widetilde{X}$ is a time-independent (possibly) complex quantity called the **phasor** representation of $x(t)$ (recall that $\overline{z}$ denotes the complex conjugate of z. The complex conjugate of a complex number $z = a + jb$ is $\overline{z} = a - jb$). Note that:

(a) $\widetilde{X}$ and $\overline{\widetilde{X}}$ are complex conjugates of each other.

(b) $e^{j\omega t}$ and $e^{-j\omega t}$ are complex conjugates of each other.

(c) $\widetilde{X}\,e^{j\omega t}$ and $\overline{\widetilde{X}}\,e^{-j\omega t}$ are also complex conjugates of each other.

NOTE: We define the phasor corresponding to $x(t)$ as the coefficient of $e^{j\omega t}$ in eq. (1). Other resources may define it differently. Some reasons for competing definitions are discussed in Note 5. Be careful to use the definition above and in other course materials this semester.

The phasor analysis method consists of five steps. The steps below are phrased in terms of any general circuit, but our goal is to apply these steps to the circuit we're given. Specifically, consider the RC circuit in fig. 1.

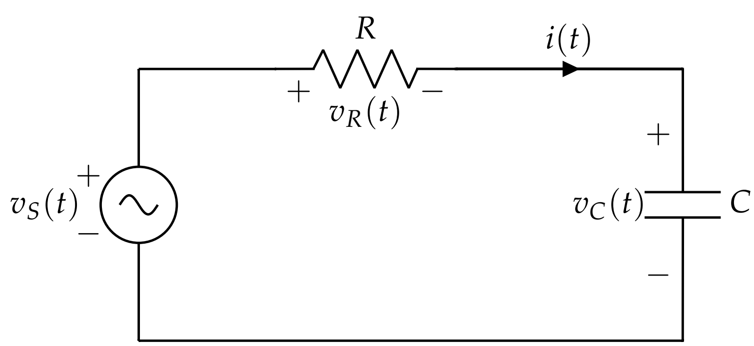

Figure 1

The voltage source is given by the sinusoid

$$v_S(t) = 12\sin\left(\omega t - \frac{\pi}{4}\right), \tag{2}$$

with $\omega = 1 \times 10^3\ \frac{\text{rad}}{\text{s}}$, $R = \sqrt{3}\,\text{k}\Omega$, and $C = 1\,\mu\text{F}$.

We seek to obtain a solution for $i(t)$ with the sinusoidal voltage source[1] $v_S(t)$.

[1]The voltage source symbol here has a squiggly, not $+/-$. This is the symbol denoting a time-dependent, *sinusoidal* source; we have previously had input voltages dependent on time but in a piecewise-constant way (turns on at some time t). These do *not* have the mini-sine inside the source symbol.

(a) **Step 1: Write sources as exponentials:** $\widetilde{X}e^{j\omega t} + \overline{\widetilde{X}}e^{-j\omega t}$**.**
All voltages and currents with known sinusoidal functions should be expressed in the standard exponential format. **Convert** $v_S(t)$ **into a exponential and write down its phasor representation** $\widetilde{V}_S$**.** Note that $v_S(t)$ is given in terms of a sine wave, not a cosine wave.

(b) **Step 2: Transform circuits to phasor domain.** The voltage source $v_S(t)$ is represented by its phasor $\widetilde{V}_S$. Similarly, $v_R(t)$ has phasor $\widetilde{V}_R$, and $v_C(t)$ has phasor $\widetilde{V}_C$.
The current $i(t)$ is related to its phasor counterpart $\widetilde{I}$ by

$$i(t) = \widetilde{I}e^{j\omega t} + \overline{\widetilde{I}}e^{-j\omega t}. \tag{3}$$

We redraw the circuit in phasor domain as in fig. 2. Recall that the impedances of the resistor,

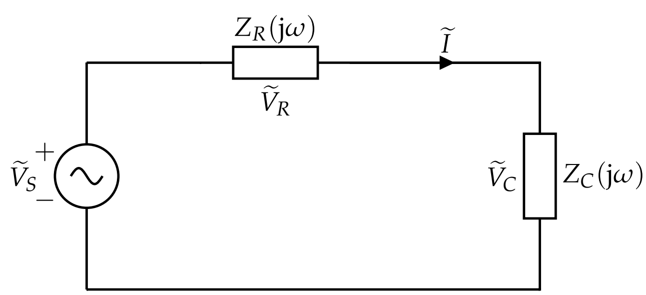

Figure 2: Circuit in “phasor domain”

$Z_R(j\omega)$, and capacitor, $Z_C(j\omega)$, are as given below. We sometimes also refer to this as the "phasor representation" of R and C.

$$Z_R(j\omega) = R \tag{4}$$

$$Z_C(j\omega) = \frac{1}{j\omega C} \tag{5}$$

Using the numbers given in the problem statement ($\omega = 1 \times 10^3 \, \frac{\text{rad}}{\text{s}}$, $R = \sqrt{3}\,\text{k}\Omega$, and $C = 1\,\mu\text{F}$), **compute the numerical values of these impedances.**

(c) **(PRACTICE)** As an intermediate step to use in the next subpart, **show that the fact that the first equation holds for all t implies the second equation:**

$$v_S(t) = v_R(t) + v_C(t) \tag{6}$$
$$\widetilde{V}_S = \widetilde{V}_R + \widetilde{V}_C \tag{7}$$

(d) **Step 3: Cast the branch and element equations in the phasor domain.**

The previous subpart gave us a concrete relation we can use in the phasor domain to relate the voltages of the circuit elements. Specifically, **we know that $\widetilde{V}_S = \widetilde{V}_R + \widetilde{V}_C$.**

Now, we must **substitute in the voltage phasors corresponding to these terms, using the element impedances given in Step 2.** At this point, feel free to leave the terms symbolic; in the next part, we will substitute in numbers.

(e) **Step 4: Solve for unknown variables**

Solve the equation you derived in Step 3 for $\widetilde{I}$ and $\widetilde{V}_C$. What is the polar form of $\widetilde{I}$ and $\widetilde{V}_C$? The polar form is given by $A\mathrm{e}^{\mathrm{j}\theta}$, where A is a positive real number.

Hints:

- $\frac{\sqrt{3}}{2} - \frac{j}{2} = \mathrm{e}^{-\mathrm{j}\frac{\pi}{6}}$

(f) **Step 5: Transform solutions back to time domain**
To return to time domain, we apply the relation between a sinusoidal function and its phasor counterpart. **What is $i(t)$ and $v_C(t)$? What is the phase difference between $i(t)$ and $v_C(t)$?**

(g) **(PRACTICE)** Now, suppose that instead of wherever we analyzed the phasor as $\widetilde{X}$ (the coefficient associated with the $e^{j\omega t}$ term), we had instead selected to work with $\overline{\widetilde{X}}$, or we solved using both $\widetilde{X}$ and $\overline{\widetilde{X}}$. **How would our answer or problem-solving procedure have changed?**

2. (PRACTICE) ***Note 5 Companion Problem:*** **Inductor Impedance**

Given the voltage-current relationship of an inductor $v(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t}$, we want to show that its complex impedance is $Z_L(j\omega) = \mathrm{j}\omega L$. We will perform this analysis in steps.

A sample inductor circuit is in fig. 3.

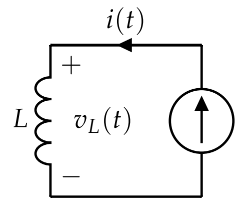

Figure 3: A simple inductor circuit.

(a) Suppose that the input current source in fig. 3 has value $i(t) = I_0 \mathrm{e}^{st}$, where I_0 is some (not necessarily real) constant. **What is the corresponding s-phasor for the current?**

(b) Now, using the governing voltage-current equation for an inductor, **derive the time-domain inductor voltage using the current expression** and **solve for the corresponding voltage s-phasor.**

(c) Using the voltage and current s-phasors, **solve for the s-impedance of the inductor $Z_L(s)$.** (This is the ratio between these phasor quantities).

(d) Now, suppose that our current source value was instead $i(t) = I_0 \cos(\omega t + \phi)$, where ω is the frequency of the cosine wave and ϕ is the phase-offset. $\phi = 0$ corresponds to the standard cosine centered at $t = 0$.

Using Euler's formula, represent $i(t)$ as the sum of two complex exponentials. Using this, **Find the new phasor $\widetilde{I}$ associated with the complex exponential $\mathrm{e}^{\mathrm{j}\omega t}$.**

(e) Same as before, **use $i(t)$ to derive** $v(t)$ and **find the new phasor $\widetilde{V}$ associated with the complex exponential $\mathrm{e}^{\mathrm{j}\omega t}$.**

(f) Once again, using the voltage and current phasors, **solve for the impedance of the inductor $Z_L(s)$.** Is this the same quantity that we found in the earlier subpart, as expected?

Now, let's see how we could have used the first result (for a single complex exponential) and taken a shortcut for the generic sinusoid using superposition. By pattern-matching the expansion of $i(t) = I_0 \frac{1}{2}\left(e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}\right)$ to the single s-exponential at the very start, we find that there are 2 components:

i. Component 1: $i_1(t) = \left(\frac{I_0}{2} e^{j\phi}\right) e^{j\omega t}$

ii. Component 2: $i_2(t) = \left(\frac{I_0}{2} e^{-j\phi}\right) e^{-j\omega t}$

(g) Now, **evaluate your expression for $Z_L(s)$ (from the single exponential case) at $s = j\omega$, and $s = -j\omega$.** What do you notice?

(h) Using the current components given above, **solve for the voltage phasors $\widetilde{V}_1$ and $\widetilde{V}_2$ as the product of the associated current phasors $\widetilde{I}_1$ and $\widetilde{I}_2$, and the corresponding impedances**. What do you notice about the current phasors? What do you notice about the voltage phasors? How can we explain the relationships between these results?

Contributors:

- Yen-Sheng Ho.
- Regina Eckert.
- Jaijeet Roychowdhury.
- Sidney Buchbinder.
- Druv Pai.
- Neelesh Ramachandran.