EECS 16B Designing Information Devices and Systems II
Spring 2021 UC Berkeley **Homework 13**

This homework is due on Friday, April 23, 2021, at 11:00PM. Self-grades and HW Resubmission are due on Tuesday, April 27, 2021, at 11:00PM.

1. **Reading Lecture Notes**

   Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: Note 15

   (a) We know that a scalar function $f(x)$ can be linearly approximated around a particular point $x = x_*$ using Taylor's series expansion as follows:

   $$f(x) \approx f(x_*) + \frac{df}{dx}\Big|_{x=x_*} \cdot (x - x_*)$$

   What is the equivalent linear approximation of a multivariate scalar function $f(x, u)$ around a particular expansion point (x_*, u_*)?

   Solution: The linear approximation of $f(x, u)$ around the expansion point (x_*, u_*) is given by

   $$f(x,u) \approx f(x_*, u_*) + \frac{\partial f}{\partial x}\Big|_{x=x_*, u=u_*} \cdot (x - x_*) + \frac{\partial f}{\partial u}\Big|_{x=x_*, u=u_*} \cdot (u - u_*)$$

   (b) Now assume we have a vector valued function given by

   $$\vec{f}(\vec{x}, \vec{u}) = \begin{bmatrix} f_1(\vec{x}, \vec{u}) \\ f_2(\vec{x}, \vec{u}) \\ \vdots \\ f_n(\vec{x}, \vec{u}) \end{bmatrix}$$

   Let the state $\vec{x}$ be n dimensional, and control $\vec{u}$ be k dimensional. What is the linear approximation of the function $\vec{f}(\vec{x}, \vec{u})$ around a particular expansion point $(\vec{x}_*, \vec{u}_*)$?

   Solution: The linear approximation of $\vec{f}(\vec{x}, \vec{u})$ around the expansion point $(\vec{x}_*, \vec{u}_*)$ is given by

   $$\vec{f}(\vec{x}, \vec{u}) \approx \vec{f}(\vec{x}_*, \vec{u}_*) + \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\Bigg|_{\vec{x}=\vec{x}_*, \vec{u}=\vec{u}_*} (\vec{x} - \vec{x}_*) + \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_k} \end{bmatrix}\Bigg|_{\vec{x}=\vec{x}_*, \vec{u}=\vec{u}_*} (\vec{u} - \vec{u}_*).$$

2. Single-dimensional linearization

This is an exercise around linearization of a scalar system. The scalar nonlinear differential equation we have is

$$\frac{d}{dt}x(t) = \sin\big(x(t)\big) + u(t). \tag{1}$$

(a) The first thing we want to do is find equilibria (DC operating points) that this system can support. Suppose we want to investigate potential expansion points (x^*, u^*) with $u^* = 0$. **Sketch** $\sin(x^*)$ **for** $-4\pi \leq x^* \leq 4\pi$ **and intersect it with the horizontal line at** 0**.** This will show us the equilibria points, where $\sin(x^*) + u^* = 0$.

Solution:

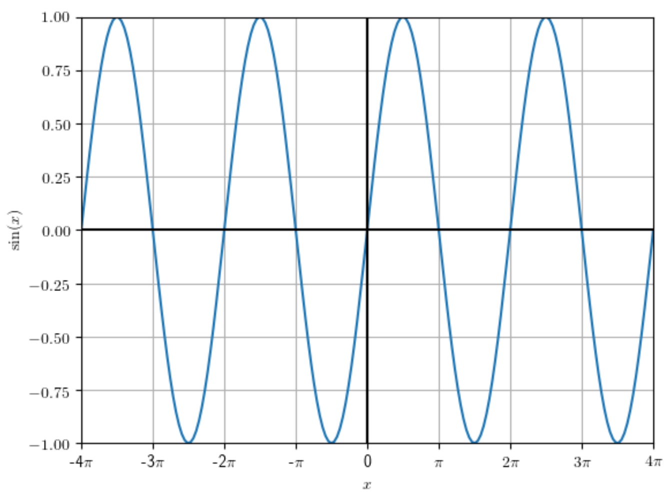


Figure 1: Plot of $\sin(x)$

We can see that all the multiples of π are where the line intersects the sine wave.

(b) **Show that all** $x(t) = x_m^* = m\pi$ **satisfy** (1) **together with** $u^* = 0$**, i.e.** $u(t) = u^* = 0\ \forall t$**.**

Solution: In figure 1, we plot $\sin(x^*)$ in from -4π to 4π. It is clear from the plot that $\sin(x^*) = 0$ at every integer multiple of π. Hence, the equilibrium points are

$$x_m^* = m\pi \quad \forall m \in \mathbb{Z}.$$

Now, for $x(t) = x_m^*$, we have:

$$\text{LHS: } \frac{d}{dt}x_m^* = 0$$
$$\text{RHS: } \sin\big(x_m^*\big) + u^* = 0$$

Hence, LHS = RHS, and (1) is satisfied.

Let us zoom in on two choices: $x_{-1}^* = -\pi$ and $x_0^* = 0$. Looking at the sketch we made, these seem like representative points.

(c) Linearize the system (1) around the equilibrium $(x_0^*, u^*) = (0,0)$. **What is the resulting linearized scalar differential equation for $x_\ell(t) = x(t) - x_0^* = x(t) - 0$, involving $u_\ell(t) = u(t) - u^* = u(t) - 0$?** Remember we are ignoring the higher order terms during linearization so we have to account for those using some $w(t)$ that can be thought of as noise.

Solution: First substituting for x and u in (1), then substituting the linearization for $\sin(x)$, we get,

$$\begin{aligned}\frac{dx}{dt} &= \sin\big(x(t)\big) + u(t)\\ \frac{dx_\ell}{dt} &= \sin\big(x_\ell(t)\big) + u_\ell(t)\\ &\approx \sin\big(x_0^*\big) + x_\ell(t)\frac{d}{dx}\sin(x)\big|_{x=x_0^*} + u_\ell(t)\\ &= 0 + x_\ell(t)\cos(0) + u_\ell(t) + w(t)\\ &= x_\ell(t) + u_\ell(t) + w(t)\end{aligned}$$

Notice that this approximate equality has been made an exact equality by calling the approximation error a disturbance $w(t)$.

(d) For the linearized approximate system model that you found in the previous part, what happens if we try to discretize time to intervals of duration Δ? Assume now we use a piecewise constant control input $u_\ell(t) = u_\ell[n]$ in the time interval $t \in [n\Delta, (n+1)\Delta)$, where Δ is small relative to the ranges of controls applied, and that we sample the state x every Δ (that is, at every $t = n\Delta$, where n is an integer) as well. **Write out the resulting scalar discrete-time control system model, i.e. what is $x_\ell[n+1]$ in terms of $x_\ell[n]$ and $u_\ell[n]$?** This model is an approximation of what will happen if we actually applied a piecewise constant control input to the original nonlinear differential equation at the operating point (x^*, u^*). Here you can ignore the $w(t)$ term.

Solution:

Here, we do a full derivation from scratch as review. But since you have seen this before in lecture, discussion and earlier homework, you didn't have to do this for full credit.

In the time interval $t \in [n\Delta, (n+1)\Delta)$, we can rewrite the differential equation as

$$\frac{dx_\ell}{dt} = x_\ell(t) + u_\ell[n]. \tag{2}$$

Applying the substitution $\widehat{x}_\ell(t) = x_\ell(t) + u_\ell[n]$, and solving the equation we get

$$\begin{aligned}\frac{dx_\ell}{dt} = \frac{d\widehat{x}_\ell}{dt} &= \widehat{x}_\ell(t)\\ \widehat{x}_\ell(t) &= ce^t \;\forall t \in [n\Delta, (n+1)\Delta)\end{aligned}$$

Next, use the initial condition at $t = n\Delta$ to get $\widehat{x}_\ell(n\Delta) = x_\ell[n] + u_\ell[n] = ce^{n\Delta}$. Hence, we get $c = \frac{x_\ell[n]+u_\ell[n]}{e^{n\Delta}}$. Substituting for c,

$$\begin{aligned}\widehat{x}_\ell(t) &= (x_\ell[n] + u_\ell[n])e^{t-n\Delta}\\ x_\ell(t) + u_\ell[n] &= (x_\ell[n] + u_\ell[n])e^{t-n\Delta}\\ x_\ell(t) &= x_\ell[n]e^{t-n\Delta} + u_\ell[n]\left(e^{t-n\Delta} - 1\right)\end{aligned}$$

Finally, we plug in $t = (n+1)\Delta$ in the above equation to find $x_\ell[n+1]$

$$x_\ell[n+1] = e^{\Delta}x_\ell[n] + u_\ell[n](e^{\Delta} - 1)$$

(e) **Is the (approximate) discrete-time system you found in the previous part stable or unstable?**

Solution: For a scalar discrete time recurrence relation, stability is determined by the coefficient of the system's variable, in this case x_ℓ. We know that if the magnitude of this coefficient is between -1 and 1, our system is stable. But $e^{\Delta} > 1$ for all positive Δ (and Δ by definition has to be positive). Hence, our system is *not* stable.

This makes sense. The differential equation for $x_\ell(t)$ had transient solutions that exponentially explode off to infinity. It makes sense that the discretization would also be unstable.

(f) Now linearize the system (1) around the equilibrium $(x^*_{-1}, u^*) = (-\pi, 0)$. **What is the resulting scalar differential equation for** $x_\ell(t) = x(t) - (-\pi)$ **involving** $u_\ell(t) = u(t) - 0$**?**

Again, don't forget to include the $w(t)$ term to capture the approximation error.

Solution: As before, substituting for $x(t)$ and $u(t)$ in (1), we get

$$\begin{aligned}\frac{d}{dt}(x_\ell(t) + \pi) &= \sin\big(x_\ell(t) + \pi\big) + u_\ell(t) \\ \frac{d}{dt}x_\ell(t) &= -\sin\big(x_\ell(t)\big) + u_\ell(t)\end{aligned}$$

Next, we can linearize the system around $x^*_{-1} = -\pi$ to obtain

$$\begin{aligned}\frac{dx_\ell}{dt} &\approx -\sin(0) + x_\ell(t)\frac{d}{dx_\ell}\left(-\sin(x_\ell)\big|_{x=x^*_{-1}}\right) + u_\ell(t) \\ &= -x_\ell(t)\cos(x_\ell)\big|_{x_\ell=0} + u_\ell(t) + w(t) \\ &= -x_\ell(t) + u_\ell(t) + w(t)\end{aligned}$$

(g) For the linearized approximate system model that you found in the previous part, what happens if we try to discretize time to intervals of duration Δ? Assume now we use a piecewise constant control input $u_\ell(t) = u_\ell[n]$ in the time interval $t \in [n\Delta, (n+1)\Delta)$, where Δ is small relative to the ranges of controls applied, and that we sample the state x every Δ (that is, at every $t = n\Delta$, where n is an integer) as well. **Write out the resulting scalar discrete-time control system model, i.e. what is** $x_\ell[n+1]$ **in terms of** $x_\ell[n]$ **and** $u_\ell[n]$**?** This model is an approximation of what will happen if we actually applied a piecewise constant control input to the original nonlinear differential equation at the operating point (x^*, u^*). Here you can ignore the $w(t)$ term.

Solution: Following a similar route to part (d), we can write the differential equation in the time interval $t \in [n\Delta, (n+1)\Delta)$ as

$$\frac{dx_\ell}{dt} = -x_\ell(t) + u_\ell[n]. \tag{3}$$

Applying the substitution $\widehat{x}_\ell(t) = -x_\ell(t) + u_\ell[n]$, and solving the equation we get

$$\begin{aligned}\frac{dx_\ell}{dt} &= -\frac{d\widehat{x}_\ell}{dt} = \widehat{x}_\ell(t) \\ \widehat{x}_\ell(t) &= ce^{-t}\end{aligned}$$

Next, use the initial condition at $t = n\Delta$ to get $\widehat{x}_\ell(n\Delta) = -x_\ell[n] + u_\ell[n] = ce^{-n\Delta}$. Hence, we get $c = \frac{-x_\ell[n]+u_\ell[n]}{e^{-n\Delta}} = (-x_\ell[n] + u_\ell[n])e^{n\Delta}$. Substituting for c,

$$\begin{aligned}
\widehat{x}_\ell(t) &= (-x_\ell[n] + u_\ell[n])e^{-(t-n\Delta)} \\
-x_\ell(t) + u_\ell[n] &= (-x_\ell[n] + u_\ell[n])e^{-(t-n\Delta)} \\
x_\ell(t) &= x_\ell[n]e^{-(t-n\Delta)} - u_\ell[n]\left(e^{-(t-n\Delta)} - 1\right)
\end{aligned}$$

Finally, we plug in $t = (n+1)\Delta$ in the above equation to find $x_\ell[n+1]$

$$x_\ell[n+1] = e^{-\Delta}x_\ell[n] + u_\ell[n](1 - e^{-\Delta}).$$

(h) **Is the (approximate) discrete-time system you found in the previous part stable or unstable?**

Solution: In this case, $0 < e^{-\Delta} < 1$ for all positive Δ, hence our system is stable. Again, this makes sense since the continuous-time linearization had transient solutions that decayed exponentially to zero.

(i) Suppose for the two *linearized discrete-time systems* derived in parts (d) and (g), we chose to apply a feedback law

$$u_\ell[n] = -k(x_\ell[n] - x^*).$$

For what range of k values, would the resulting linearized discrete-time systems be stable? Your answer will depend on Δ.

(HINT: Your solution to part (d) should be

$$x_\ell[n+1] = e^{\Delta}x_\ell[n] + u_\ell[n](e^{\Delta} - 1)$$

and solution to part (g) should be

$$x_\ell[n+1] = e^{-\Delta}x_\ell[n] + u_\ell[n](1 - e^{-\Delta})$$

)

Solution: Let's begin with the first case, $x^* = 0$. Based on our defintion of x_ℓ, we have, $u_\ell[n] = -kx_\ell[n]$. Substituting and grouping the terms, we get

$$x_\ell[n+1] = x_\ell[n]\left(e^{\Delta} - k(e^{\Delta} - 1)\right)$$

Hence, we want the above coefficient to between -1 and 1.

$$\begin{aligned}
-1 < e^{\Delta} - k(e^{\Delta} - 1) &< 1 \\
-(1 + e^{\Delta}) < -k(e^{\Delta} - 1) &< 1 - e^{\Delta} \\
\implies 1 < k &< \frac{e^{\Delta} + 1}{e^{\Delta} - 1}
\end{aligned}$$

Looking at the second case, with $x^* = -\pi$, we get

$$x_\ell[n+1] = e^{-\Delta}x_\ell[n] + kx_\ell[n](e^{-\Delta} - 1) + k\pi(e^{-\Delta} - 1).$$

Grouping terms, and further simplifying

$$x_\ell[n+1] = x_\ell[n]\left(e^{-\Delta} + k(e^{-\Delta} - 1)\right) + k\pi(e^{-\Delta} - 1).$$

As before, we want this coefficient to be between -1 and 1, hence

$$
\begin{aligned}
-1 < e^{-\Delta} + k(e^{-\Delta} - 1) < 1 \\
-(1 + e^{-\Delta}) < k(e^{-\Delta} - 1) < 1 - e^{-\Delta} \\
\implies -1 < k < \frac{1 + e^{-\Delta}}{1 - e^{-\Delta}}
\end{aligned}
$$

3. Linearizing for understanding amplification

Linearization isn't just something that is important for control, robotics, machine learning, and optimization — it is one of the standard tools used across different areas, including thinking about circuits.

The circuit below is a voltage amplifier, where the element inside the box is a bipolar junction transistor (BJT). You do not need to know what a BJT is to do this question.

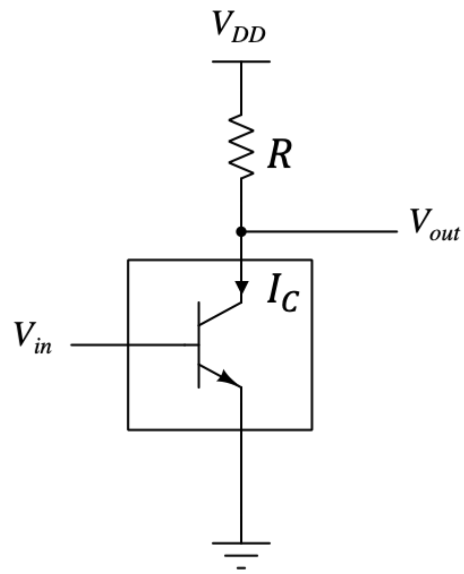


Figure 2: Voltage amplifier circuit using a BJT

The BJT in the circuit can be modeled quite accurately as a nonlinear, voltage-controlled current source, where the collector current I_C is given by:

$$I_C(V_{in}) = I_S \cdot e^{\frac{V_{in}}{V_{TH}}}, \tag{4}$$

where V_{TH} is the thermal voltage. We can assume $V_{TH} = 26\text{mV}$ at room temperature. I_S is a constant whose exact value we are not giving you because we want you to find ways of eliminating it in favor of other quantities whenever possible.

Let's consider the 2N3904 model of a BJT, where the above expression for $I_C(V_{in})$ holds as long as $0.2\text{V} < V_{out} < 40\text{V}$, and $0.1\text{mA} < I_C < 10\text{mA}$. (Note that the 2N3904 is a cheap transistor that people often use in personal projects. You can get them for 3 cents each if you buy in bulk.)

The goal of this circuit is to pick a particular point (V_{in}^*, V_{out}^*) so that any small variation δV_{in} in the input voltage V_{in} can be amplified to a relatively larger variation δV_{out} in the output voltage V_{out}. In other words, if $V_{in} = V_{in}^* + \delta V_{in}$ and $V_{out} = V_{out}^* + \delta V_{out}$, then we want the magnitude of the 'amplification gain' given by $\left|\frac{\delta V_{out}}{\delta V_{in}}\right|$ to be large. We're going to investigate this amplification using linearization.

(a) **Write a symbolic expression for V_{out} as a function of I_C, V_{DD} and R in Fig 2.**

Solution:

$$V_{out} = V_{DD} - RI_C \tag{5}$$

since we have a voltage drop of $I_C R$ across the resistor and the top voltage is V_{DD}.

(b) Now let's linearize I_C in the neighborhood of an input voltage V_{in}^* and a specific I_C^*. Assume that you have a found a particular pair of input voltage V_{in}^* and current I_C^* that satisfy the current equation (4). We can look at nearby input voltages and see how much the current changes. We can write the linearized expression for the collector current around this point as:

$$I_C(V_{in}) = I_C(V_{in}^*) + m(V_{in} - V_{in}^*) = I_C^* + m\,\delta V_{in} \tag{6}$$

where $\delta V_{in} = V_{in} - V_{in}^*$ is the change in input voltage.

What is m here as a function of I_C^* and V_{TH}?

(If you take EE105, you will learn that this m is called the transconductance, which is usually written g_m, and is the single most important parameter in most analog circuit designs.)

(HINT: First just find m by taking the appropriate derivative and using the chain rule as needed. Then leverage the Taylor's series expansion of the exponential function to express it in terms of the desired quantities.)

Solution:

We start out by writing out the linearization form that we are looking for.

$$I_C(V_{in}) = I_C^* + m\,\delta V_{in}$$

Now, using the linearization approximation of Taylor series.

$$m = \left.\frac{dI_C}{dV_{in}}\right|_{V_{in}^*} \tag{7}$$

$$= \left.\frac{1}{V_{TH}} I_S e^{\frac{V_{in}}{V_{TH}}}\right|_{V_{in}^*} \tag{8}$$

$$= \frac{I_C^*}{V_{TH}} \tag{9}$$

where in the last line, we recognize that the expression in the exponential with the I_S before it is just I_C^* itself. This is why the I_S constant didn't need to be told to you.

We can use these equations to linearize I_C at certain chosen values of V_{in}, such as values $V_{in}^* = 0.65\text{V}$ and $V_{in}^* = 0.7\text{V}$ given in parts (d) and (e) below. We plot these linearizations here to help visualize our results.

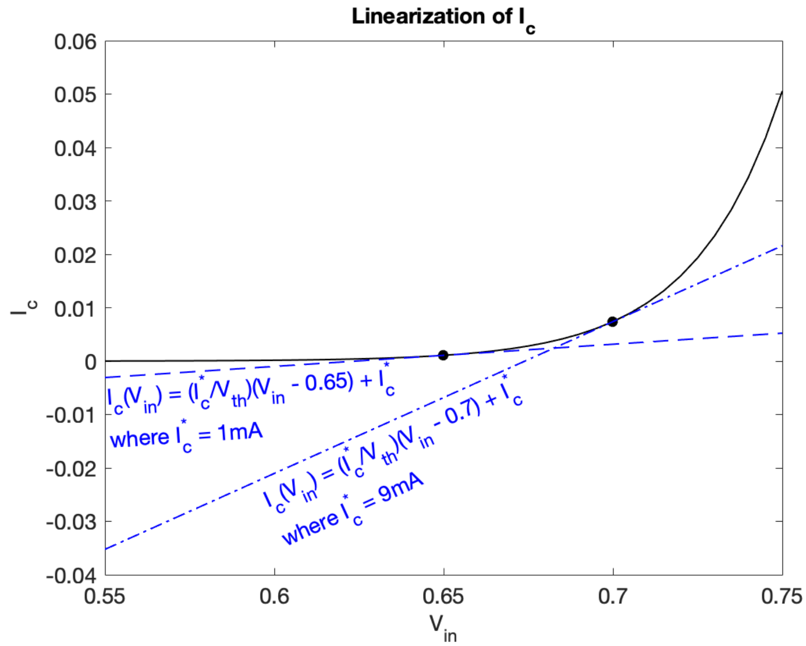


Figure 3: Linearization of the non linear I_C (black curve)

(c) We now have a linear relationship between small changes in current and voltage, $\delta I_C = m\ \delta V_{in}$ around a known solution (I_C^*, V_{in}^*). This is called a "bias point" in circuits terminology. (This is also why related things in neural nets are called bias terms — their job is to get the nonlinearity to behave the way we want it to.)

As a reminder, **the goal of this problem is to pick a particular point (V_{in}^*, V_{out}^*) so that any small variation δV_{in} in the input voltage V_{in} can be amplified to a relatively larger variation δV_{out} in the output voltage V_{out}. In other words, if $V_{in} = V_{in}^* + \delta V_{in}$ and $V_{out} = V_{out}^* + \delta V_{out}$, then we want the magnitude of the 'amplification gain' given by $\left|\frac{\delta V_{out}}{\delta V_{in}}\right|$ to be large.**

Plug in your linearized equation for I_C in the answer from part (a). Define

$$V_{out}^* = V_{DD} - RI_C^*$$

so that it makes sense to view $V_{out} = V_{out}^* + \delta V_{out}$ when we have $V_{in} = V_{in}^* + \delta V_{in}$, and **find the approximate linear relationship between δV_{out} and δV_{in}.**

The ratio $\frac{\delta V_{out}}{\delta V_{in}}$ is called the small-signal voltage gain of this amplifier around this bias point.

Solution: Expanding out and remembering the equation for V_{out} from part (a):

$$V_{out} = V_{out}^* + \delta V_{out} = V_{DD} - R(I_C^* + m\ \delta V_{in})$$

And so we have $V_{out}^* = V_{DD} - RI_C^*$ and then have

$$\delta V_{out} = -R\ m\ \delta V_{in}$$

where the m is as it was in part (b). Namely

$$\frac{\delta V_{out}}{\delta V_{in}} = -\frac{I_C^* R}{V_{TH}} = -\frac{V_{DD} - V_{out}^*}{V_{TH}}. \tag{10}$$

You don't have to simplify it to this point, but this form is useful because it shows you that the gap between the operating point V_{out}^* to the supply rail V_{DD} matters to understand the small-signal gain.

We want as much current as possible to make the gain magnitude big, but there is a limit to how big the current can get.

We can use these equations to linearize V_{out} at certain chosen values of V_{in}, such as values $V_{in}^* = 0.65\text{V}$ and $V_{in}^* = 0.7\text{V}$ given in parts (d) and (e) below. We plot these linearizations here to help visualize our results. The slope of these lines are the small signal voltage gain $\frac{\delta V_{out}}{\delta V_{in}} = -\frac{I_C^* R}{V_{th}}$.

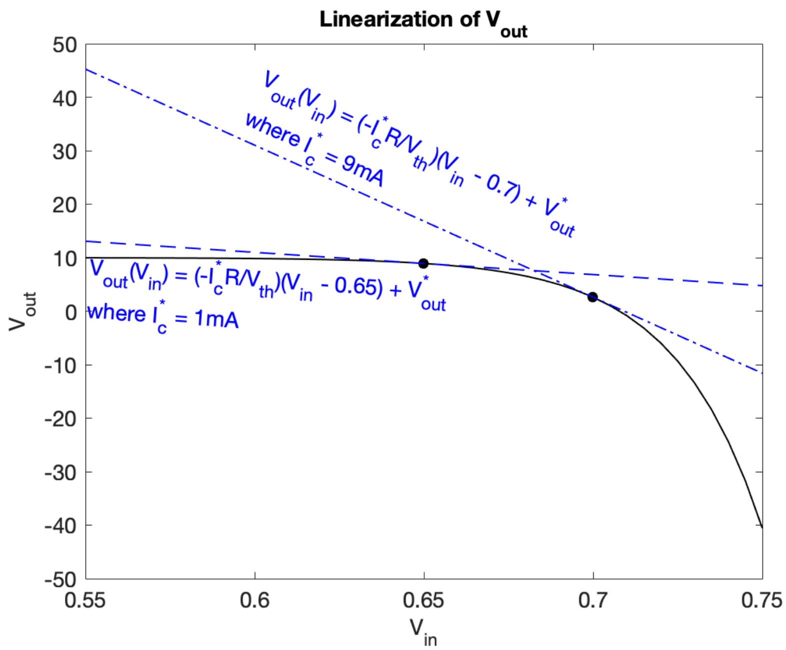


Figure 4: Linearization of the non-linear V_{out} (black curve)

(d) Assuming that $V_{DD} = 10\text{V}$, $R = 1\text{k}\Omega$, and $I_C^* = 1\text{mA}$ when $V_{in}^* = 0.65\text{V}$, **verify that the magnitude of the small-signal voltage gain** $\left|\frac{\delta V_{out}}{\delta V_{in}}\right|$ **between the input and the output around this bias point is approximately** 38**.**

(HINT: Remember $V_{TH} = 26\text{mV}$*)*

Solution: Just plugging in to (10)

$$\left|\frac{\delta V_{out}}{\delta V_{in}}\right| = \frac{I_C^* R}{V_{TH}} = \left(\frac{1\text{mA} \times 1\text{k}\Omega}{26\text{mV}}\right) = \frac{1\text{V}}{26\text{mV}} \approx 38.$$

(e) If $I_C^* = 9\text{mA}$ when $V_{in}^* = 0.7\text{V}$ with all other parameters remaining fixed, **verify that the magnitude of the small-signal voltage gain** $\left|\frac{\delta V_{out}}{\delta V_{in}}\right|$ **between the input and the output around this bias point is approximately** 350**.**

Solution:

$$\left|\frac{\delta V_{out}}{\delta V_{in}}\right| = \frac{I_C^* R}{V_{TH}} = \left(\frac{9\text{mA} \times 1\text{k}\Omega}{26\text{mV}}\right) = \frac{9\text{V}}{26\text{mV}} \approx 350.$$

Notice here that V_{out}^* has already been pulled down to around 1V. So, this is close to as big as this gain can get. It is not obvious that it is actually V_{DD} and V_{TH} that provide the fundamental limit on the small-signal gain for such circuits, but the simple linearization analysis in part (c) reveals this. Before doing this analysis, it would be tempting to believe that it is the size of the resistance that matters a lot.

These circuit insights and more are developed further in courses like 105 and then in 140 where ideas of feedback control and circuit design come together in interesting ways.

(f) If you wished to make an amplifier with as large of a small signal gain as possible, which operating (bias) point would you choose among $V_{in}^* = 0.65\text{V}$ (part d) and $V_{in}^* = 0.7\text{V}$ (part e)?

Solution: We would choose $V_{in}^* = 0.7\text{V}$ since the magnitude of the small signal gain in this bias point is much higher than that at $V_{in}^* = 0.65\text{V}$.

Note that since I_C^* is related to V_{in}^* by (4), and V_{out}^* is related to I_C^* by (5), just choosing V_{in}^* fixes the small signal gain of the circuit in Fig 2.

This shows you how by appropriately biasing (choosing an operating point), we can adjust what our gain is for small signals. Although here, we just wanted to show you this as a simple application of linearization, these ideas are developed a lot further in 105, 140, and other courses to create things like op-amps and other analog information-processing systems.

4. Inverse Kinematics

Inverse Kinematics is critical in robotics, control, and computer graphics applications.

We need to be able to go backward from what we want to have happen in the real (or virtual) world to how to set parameters.

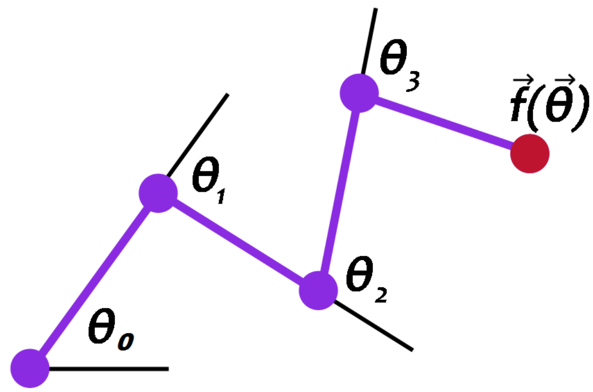


Figure 5: An example of an arm parameterized by θ_0, θ_1, θ_2, and θ_3 with the end effector at point $\vec{f}(\vec{\theta})$.

Suppose you have a robotic arm composed of several rotating joints.

The lengths r_i of the arm are fixed, but you can control the arm by specifying the amount of rotation θ_i for each joint. If we have an arm with four joints, it can be parameterized by:

$$\vec{\theta} = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}. \tag{11}$$

Suppose further that we have some target $\vec{t} \in \mathbb{R}^2$, which represents a point in the 2D space, and we would like for the end of the arm, called the end effector, to reach for the target. We have some function $\vec{f}(\vec{\theta})$ that rotates each joint of the arm according to the input and returns the position of the end effector. Figure 5 shows a visualization of an arm rotated by $\vec{\theta}$. To make the arm reach for the target $\vec{t}$, we want to find where the function $\vec{g}$ defined as

$$\vec{\theta} \in \mathbb{R}^4 \mapsto \vec{g}(\vec{\theta}) = \vec{f}(\vec{\theta}) - \vec{t} \tag{12}$$

is equal to $\vec{0}$. To accomplish this, we use the spirit of Newton's method for solving potentially nonlinear equations. (This is related to the spirit of the earlier problem on using iterative ways of solving least-squares problems.)

You might have seen Newton's method in your calculus course in the 1-D case. In this case, you have a real function g of a single parameter θ and we want to find a $\hat{\theta}$ so that $g(\hat{\theta}) = 0$. The method is the following. Step i of Newton's method does the following:

(a) Linearize g around $\theta^{(i)}$, the current estimate of $\hat{\theta}$: $\hat{g}^{(i)}(\theta) = g(\theta^{(i)}) + g'(\theta^{(i)})(\theta - \theta^{(i)})$

(b) $\theta^{(i+1)}$ solves $\hat{g}^{(i)}(\theta) = 0$. So we have

$$0 = g(\theta^{(i)}) + g'(\theta^{(i)})(\theta - \theta^{(i)}) \tag{13}$$

$$\theta = -\frac{g(\theta^{(i)})}{g'(\theta^{(i)})} + \theta^{(i)} \tag{14}$$

Therefore $\theta^{(i+1)} = -\frac{g(\theta^{(i)})}{g'(\theta^{(i)})} + \theta^{(i)}$. Note also that doing this is solving: $g'(\theta^{(i)})(\theta - \theta^{(i)}) = g(\theta^{(i)})$, which reduces to "inverting" the linear operator $z \mapsto g'(\theta^{(i)})z$.

We will iterate this until $g(\theta)$ is close enough to 0 for our application. In practice, instead of solving exactly for $\hat{g}^{(i)} = 0$, in the second step of iteration i, we may chose to move $\theta^{(i)}$ by a fixed step-size η in the direction that the first-order approximation to the function suggests, but not all the way. This is done because the derivative $g'(\theta)$ where $\hat{g}(\theta) = 0$ might be very different from $g'(\theta^{(i+1)})$ where $\theta^{(i+1)}$. After all, linearization is only valid in a local neighborhood. (This is the spirit of gradient descent as well.)

While you might have seen Newton's method as described above in your calculus courses, you might not have seen the vector-generalization of it. It follows exactly the same spirit. The first-order approximation to the vector valued function $\vec{g}(\vec{\theta})$ at $\vec{\theta}^{(i)}$ is now $\vec{g}(\vec{\theta}^{(i)}) + J_{\vec{g}}(\vec{\theta}^{(i)})(\vec{\theta} - \vec{\theta}^{(i)})$ where $J_{\vec{g}}(\vec{\theta})$ is the Jacobian matrix of the function $\vec{g}(\vec{\theta})$. For this problem, we will be using a robotic arm with 4 joints in a 2-dimensional space. Therefore, the Jacobian of $\vec{g}(\vec{\theta})$ will be a 2x4 matrix, and it is computed by calculating the partial derivatives of $\vec{g}(\vec{\theta})$:

$$J_{\vec{g}} = \begin{bmatrix} \frac{\partial g_x(\vec{\theta})}{\partial \theta_1} & \frac{\partial g_x(\vec{\theta})}{\partial \theta_2} & \frac{\partial g_x(\vec{\theta})}{\partial \theta_3} & \frac{\partial g_x(\vec{\theta})}{\partial \theta_4} \\ \frac{\partial g_y(\vec{\theta})}{\partial \theta_1} & \frac{\partial g_y(\vec{\theta})}{\partial \theta_2} & \frac{\partial g_y(\vec{\theta})}{\partial \theta_3} & \frac{\partial g_y(\vec{\theta})}{\partial \theta_4} \end{bmatrix}. \tag{15}$$

In this notation, we use $\vec{g}(\vec{\theta}) = [g_x(\vec{\theta}) \quad g_y(\vec{\theta})]^T$ where $g_x(\vec{\theta})$ is the x coordinate of the end effector and $g_y(\vec{\theta})$ is the y coordinate in our 2D space. There is nothing mysterious about this, if you think about it, this matrix of partial derivatives (a partial derivative is just a regular derivative with respect to a particular variable, treating all the other variables as constants) is the natural n-d candidate to replace g'.

The Newton algorithm in this case is an iterative method that gives us successively better estimates for our vector $\hat{\vec{\theta}}$. If we start with some guess $\vec{\theta}^{(i)}$, then the next guess is given by

$$\vec{\theta}^{(i+1)} = \vec{\theta}^{(i)} - \eta J^{\dagger}_{\vec{g}}(\vec{\theta}^{(i)})\vec{g}(\vec{\theta}^{(i)}) \tag{16}$$

where η is adjusted to determine how large of a step we make between $\vec{\theta}^{(i)}$ and $\vec{\theta}^{(i+1)}$. Notice that we need to invert the Jacobian matrix of first-partial-derivatives, and this matrix is not square. It is in fact a wide matrix. Fortunately, we know how to "invert" wide matrices, using the Moore Penrose pseudo-inverse that you saw in a previous homework. The minimality property of the Moore-Penrose pseudoinverse that we studied then is useful here because we would rather take small steps than big ones. And when tracking a moving reference, we'd like to have the joint angles change in a minimal way rather than in some very convoluted fashion.

The following problem will guide you step-by-step through the implementation of the pseudoinverse. The three steps of the psuedoinverse algorithm are:

- First, compute the compact-form SVD of the input matrix.
- Next, we compute Σ^{-1} by inverting each singular value σ_i.
- Finally, we compute the pseudoinverse by multiplying the matrices together in the right order.

The next three parts guide you through the code to be written for the pseudoinverse.

(a) In the "pseudoinverse" function, **compute** the SVD of the input matrix A by using the appropriate NumPy function.

(HINT: It is useful to read the documentation for the numpy functions involved so that you call them with the right arguments. For example, for this problem you need to call `svd(A, full_matrices=False)`,

What is the default? What do you want the function to return? What exactly does the SVD function return in numpy?)

Solution: See the IPython notebook for the solution.

The challenge here was basically in understanding the numpy documentation about what the SVD function actually does. The fact that it has an optional argument that defaults to something other than what you want (i.e. it defaults to the full SVD as opposed to the compact form that you prefer for this problem, and basically would prefer anytime the nullspace is not interesting to you) adds complexity, but this is something that you always have to watch out for when using libraries.

(b) To save memory space, the NumPy algorithm returns the matrix Σ as a one-dimensional array of the singular values.

Use this vector to **compute the diagonal entries of** Σ^{-1}. Be careful of numerical issues: first threshold the singular values, and only invert the singular values above a certain value ϵ, considering smaller ones are 0.

The reason is that you don't want to have very big entries in the pseudoinverse because that will defeat the point of you using a small step-size η to stay within the rough neighborhood that your linear approximation is valid. So you need to stop that from happening. That is what considering small singular values as being 0 does.

Solution: See the IPython notebook for the solution.

Numerical errors can arise by dividing by numbers that are too small. Threshold them first, i.e. set the inverse of A along the directions corresponding to very small singular vectors to 0.

You might have wondered how you can systematically pick the value for the threshold ϵ and the answer here was that you had no such systematic approach. You had to guess something. The hint told you a conservative approach. The principled reason you need to pick a threshold is to prevent yourself from moving too far away.

Now, our provided code was actually somewhat intelligent (i.e. adaptive) in picking this update's step-size η. It would shrink the step-size whenever it saw something odd. (This is a cheap variant of something called "line-search" — it is possible to do something more sophisticated, but it won't help in a big way for this problem.) And so, the real issue here was in avoiding a bad interaction with our adaptive η code.

(c) We now have all of the parts to compute the relevant pseudoinverse of A. **Add this computation to the function.**

(HINT: `np.diag` *can be a very useful tool in converting a vector into a square diagonal matrix. Also remember that numpy knows how to multiply matrices and using* `.T` *compute transposes.)*

Solution: See the IPython notebook for the solution.

In the real world, so-called "hyperparameters" like ϵ are set by more sophisticated variants of "trial and error." Basically, you try a value and see if it works for some of the cases that you want to make sure that it works for. If it doesn't, you adjust it. You keep trying. Once you've settled on something as being good enough, you just need to check to see whether you were deluded. To test that, you use another fresh and different set of test cases to make sure that things still work. In this problem, we thought of the animation as being like that final set of test cases. But ideally, we would have had a second animated path that did something else.

(d) There are three test cases that you can use to determine if your pseudoinverse function works correctly. In the first case, the arm is able to reach the target, and the end of the arm will be touching the target. In the second case, the arm should be pointing in a straight line towards the blue circle. The last case is the same as the second with the addition that a singular value will be very close to zero to test your pseudoinverse function's ability to handle small singular values. There is also an animated test case that will move the target in and out of the reach of the arm. Verify that the arm follows the target correctly and points towards the target when it is out of reach.

Solution: You should observe the arm getting as close to the blue dot as possible in a smooth manner.

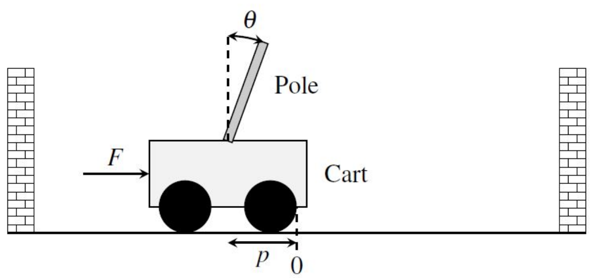
5. Segway Tours

A segway is a stand on two wheels, and can be thought of as an inverted pendulum. The segway works by applying a force (through the spinning wheels) to the base of the segway, This controls both the position on the segway and the angle of the stand. As the driver pushes on the stand, the segway tries to bring itself back to the upright position, and it can only do this by moving the base.

You may recall analyzing a problem related to segway in 16A homework. You were given a linear discrete time representation of the segway dynamics, and were asked to analyze if it's possible to make the segway reach some desired states. Now, we will see how to derive the linear discrete time system from the equations of motion and do some further detailed analysis.

Note that this problem is completely independent from the 16A version and does not require any prior knowledge or results of the previous problem.

Is it possible for the segway to be brought upright and to a stop from any initial configuration? There is only one input (force) used to control two outputs (position and angle). Let's model the segway as a cart-pole system and analyze.



A cart-pole system can be fully described by its position p, velocity $\frac{dp}{dt}$, angle θ, and angular velocity $\frac{d\theta}{dt}$. We can write this as the continuous time state vector $\vec{x}$ as follows:

$$\vec{x} = \begin{bmatrix} p \\ \frac{dp}{dt} \\ \theta \\ \frac{d\theta}{dt} \end{bmatrix}$$

The input to this system is a scalar quantity $u(t)$ at time t, which is the force F applied to the cart (or base of the segway). Let the co-efficient of friction be k.

The equations of motion for this system are as follows:

$$\begin{aligned} \frac{d^2p}{dt^2} &= \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{u}{m} + \left(\frac{d\theta}{dt}\right)^2 l\sin\theta - g\sin\theta\cos\theta - \frac{k}{m}\frac{dp}{dt}\right) \\ \frac{d^2\theta}{dt^2} &= \frac{1}{l\left(\frac{M}{m} + \sin^2\theta\right)}\left(-\frac{u}{m}\cos\theta - \left(\frac{d\theta}{dt}\right)^2 l\cos\theta\sin\theta + \frac{M+m}{m}g\sin\theta + \frac{k}{m}\frac{dp}{dt}\cos\theta\right) \end{aligned} \tag{17}$$

The derivation of these equations is a mechanics problem and not in 16B syllabus, but interested students can look up the details online.

(a) First let us linearize the system of equations in (17) about the upright position at rest, i.e. $\theta_* = 0$ and $\frac{d\theta}{dt}_* = 0$. **Show that the linearized system of equations is given by the following state space form:**

$$\frac{d\vec{x}(t)}{dt} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{k}{M} & -\frac{m}{M}g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{k}{Ml} & \frac{M+m}{Ml}g & 0 \end{bmatrix}}_{A} \vec{x}(t) + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{Ml} \end{bmatrix}}_{\vec{b}} u(t) \tag{18}$$

(HINT: Since we are linearizing around $\theta_* = 0$ and $\frac{d\theta}{dt}_* = 0$, you can use the following approximations for small values of θ:

$$\begin{aligned} \sin\theta &\approx \theta \\ \sin^2\theta &\approx 0 \\ \cos\theta &\approx 1 \\ \left(\frac{d\theta}{dt}\right)^2 &\approx 0. \end{aligned}$$

You do not have to do the full linearization using Taylor series, you can just substitute the values above. You will get the same answer as doing the linear Taylor series approximation.)

Solution: Using the approximations from the hint, (17) is linearized as follows:

$$\begin{aligned} \frac{d^2p}{dt^2} &= -\frac{k}{M}\frac{dp}{dt} - \frac{m}{M}g\theta + \frac{1}{M}u \\ \frac{d^2\theta}{dt^2} &= \frac{k}{Ml}\frac{dp}{dt} + \frac{M+m}{Ml}g\theta - \frac{1}{Ml}u \end{aligned} \tag{19}$$

Now (19) can be represented in state space form as

$$\frac{d}{dt}\begin{bmatrix} p \\ \frac{dp}{dt} \\ \theta \\ \frac{d\theta}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{k}{M} & -\frac{m}{M}g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{k}{Ml} & \frac{M+m}{Ml}g & 0 \end{bmatrix}}_{A} \begin{bmatrix} p \\ \frac{dp}{dt} \\ \theta \\ \frac{d\theta}{dt} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{Ml} \end{bmatrix}}_{\vec{b}} u(t)$$

(b) For all subsequent parts, assume that $m = 1$, $M = 10$, $g = 10$, $l = 1$ and $k = 0.1$. Let's consider the discrete time representation of the state space (18) at time $t = n\Delta$. For simplicity, assume $\Delta = 1$. The discrete time state $\vec{x}_d$ follows the following linear model:

$$\vec{x}_d[n+1] = A_d\vec{x}_d[n] + \vec{b}_d u_d[n] \tag{20}$$

where $A_d \in \mathbb{R}^{4\times 4}$ and $\vec{b}_d \in \mathbb{R}^{4\times 1}$. **Find A_d and $\vec{b}_d$.** Use the Jupyter notebook **segway.ipynb** for numerical calculations, and approximate the results to 2 or 3 significant figures.

(HINT: Recall that the continuous time scalar differential equation

$$\frac{dz(t)}{dt} = \lambda z(t) + cw(t)$$

can be represented in discrete time ($n\Delta = t$) as follows:

$$z_d[n+1] = \begin{cases} (e^{\lambda\Delta}) \cdot z_d[n] + \left(\frac{e^{\lambda\Delta}-1}{\lambda}\right) \cdot cw_d[n] & \textit{if } \lambda \neq 0 \\ (1) \cdot z_d[n] + (\Delta) \cdot cw_d[n] & \textit{if } \lambda = 0 \end{cases}$$

Use the eigendecompostion of $A = V\Lambda V^{-1}$ to do change of basis variables, and you should finally reach

$$\vec{x}_d[n+1] = \underbrace{V\Lambda_d V^{-1}}_{A_d}\vec{x}_d[n] + \underbrace{VM_dV^{-1}\vec{b}}_{\vec{b}_d}u_d[n]$$

What are the elements of Λ_d and M_d in terms of the elements of Λ?)

Solution: Using the eigendecompostion of $A = V\Lambda V^{-1}$ and a change of variable $\vec{y}(t) = V^{-1}\vec{x}(t)$, we can transform (18) into a system of decoupled equations

$$\frac{d\vec{y}(t)}{dt} = \Lambda\vec{y}(t) + V^{-1}\vec{b}u(t) \tag{21}$$

This can be represented in discrete time as follows:

$$\vec{y}_d[n+1] = \Lambda_d\vec{y}_d[n] + M_dV^{-1}\vec{b}u_d[n] \tag{22}$$

where Λ_d is a diagonal matrix given by

$$\Lambda_{dii} = \begin{cases} e^{\Lambda_{ii}\Delta} & \text{if } \Lambda_{ii} \neq 0 \\ 1 & \text{if } \Lambda_{ii} = 0 \end{cases}$$

and M_d is a diagonal matrix given by

$$M_{dii} = \begin{cases} \frac{e^{\Lambda_{ii}\Delta}-1}{\Lambda_{ii}} & \text{if } \Lambda_{ii} \neq 0 \\ \Delta & \text{if } \Lambda_{ii} = 0 \end{cases}$$

Changing variables back to $\vec{x}_d[n] = V\vec{y}_d[n]$, we get

$$\vec{x}_d[n+1] = \underbrace{V\Lambda_d V^{-1}}_{A_d}\vec{x}_d[n] + \underbrace{VM_dV^{-1}\vec{b}}_{\vec{b}_d}u_d[n]$$

Plugging in the values of $m = 1$, $M = 10$, $g = 10$, $l = 1$, $k = 0.1$, $\Delta = 1$ in the Jupyter notebook, we get

$$\begin{aligned} A_d &\approx \begin{bmatrix} 1 & 0.994 & -1.161 & -0.286 \\ 0 & 0.987 & -4.138 & -1.161 \\ 0 & 0.012 & 13.797 & 4.150 \\ 0 & 0.041 & 45.634 & 13.797 \end{bmatrix} \\ \vec{b}_d &\approx \begin{bmatrix} 0.056 \\ 0.128 \\ -0.116 \\ -0.414 \end{bmatrix} \end{aligned} \tag{23}$$

(c) **Show that the discrete time system in (20) is controllable by using the appropriate matrix in the Jupyter notebook.**

(HINT: Is the controllability matrix full rank? You have to use numerical values of A_d and $\vec{b}_d$ from the previous part. Use the Jupyter notebook for all numerical calculations.)

Solution: Since $A_d \in \mathbb{R}^{4\times 4}$ and $\vec{b}_d \in \mathbb{R}^{4\times 1}$, the controllability matrix is given by

$$\mathcal{C} = \begin{bmatrix} \vec{b}_d & A_d\vec{b}_d & A_d^2\vec{b}_d & A_d^3\vec{b}_d \end{bmatrix}$$

Using the Jupyter notebook, we can see that $\mathcal{C}$ has rank $= 4$, hence the discrete time system is controllable.

(d) Since the discrete time system is controllable, it is possible to reach any final state $\vec{x}_{d,f}$ starting from initial state $\vec{x}_{d,i}$ using an appropriate sequence of inputs in exactly 4 steps, provided that the deviations are small enough so that the linearization approximation is valid. **Set up a set of linear equations to solve for the $u_d[0]$, $u_d[1]$, $u_d[2]$, $u_d[3]$ given the initial and final states. Find the input sequence to reach the upright position $\vec{x}_{d,f} = \vec{x}_d[4] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ starting from an initial state $\vec{x}_{d,i} = \vec{x}_d[0] = \begin{bmatrix} -2 \\ 3.1 \\ 0.3 \\ -0.6 \end{bmatrix}$. Use the Jupyter notebook for all numerical calculations and simulation. Explain qualitatively what you observe from the segway simulation.**

(HINT: Use (20) and loop unrolling to express $\vec{x}_d[4]$ as a linear combination of $\vec{x}_d[0]$, $u_d[3]$, $u_d[2]$, $u_d[1]$, $u_d[0]$.)

Solution: In 4 time steps, the discrete time system in (20) reaches

$$\vec{x}_d[4] = A_d^4\vec{x}_d[0] + \vec{b}_d u_d[3] + A_d\vec{b}_d u_d[2] + A_d^2\vec{b}_d u_d[1] + A_d^3\vec{b}_d u_d[0]$$

$$\implies \vec{x}_d[4] - A_d^4\vec{x}_d[0] = \begin{bmatrix} \vec{b}_d & A_d\vec{b}_d & A_d^2\vec{b}_d & A_d^3\vec{b}_d \end{bmatrix} \begin{bmatrix} u_d[3] \\ u_d[2] \\ u_d[1] \\ u_d[0] \end{bmatrix}$$

$$= \mathcal{C} \begin{bmatrix} u_d[3] \\ u_d[2] \\ u_d[1] \\ u_d[0] \end{bmatrix}$$

$$\implies \begin{bmatrix} u_d[3] \\ u_d[2] \\ u_d[1] \\ u_d[0] \end{bmatrix} = \mathcal{C}^{-1}\left(\vec{x}_d[4] - A_d^4\vec{x}_d[0]\right)$$

Using the notebook, we can calculate $\begin{bmatrix} u_d[3] \\ u_d[2] \\ u_d[1] \\ u_d[0] \end{bmatrix} \approx \begin{bmatrix} -1.636 \\ 48.650 \\ -97.747 \\ 17.433 \end{bmatrix}$.

The simulation shows the inverted pendulum stabilizing to the upright position at rest from the initial position.

(e) Now suppose we try to use an initial state $\vec{x}_{d,i} = \vec{x}_d[0] = \begin{bmatrix} -2 \\ 3.1 \\ 3.3 \\ -0.6 \end{bmatrix}$ that does not satisfy the linearization constraint, since $\theta_i = 3.3$ is very far from the linearization point $\theta_* = 0$. **Using the equations derived in the previous part, use the Jupyter notebook to determine the input sequence to reach the same final upright position. Explain qualitatively what you observe from the segway simulation.** Use the Jupyter notebook for all numerical calculations and simulation.

Solution: Using the notebook, we can calculate $\begin{bmatrix} u_d[3] \\ u_d[2] \\ u_d[1] \\ u_d[0] \end{bmatrix} \approx \begin{bmatrix} -15.049 \\ 445.384 \\ -851.258 \\ 387.623 \end{bmatrix}$.

The simulation shows that the inverted pendulum again stabilizes to the upright position at rest from the initial position, but does some weird unexpected rotations before reaching there.

Compare the simulation results in parts (d) and (e). In both cases, the segway finally stabilizes to an upright position at rest. However, in part (d) the behavior of the segway looks more realistic whereas in part (e) it is doing some wild unexpected rotations.

This is because the linearization approximation is valid with the small initial values of θ and $\frac{d\theta}{dt}$ in part (d). So this discrete time linear model is a good representation of the original continuous time non-linear system. Hence the trajectory taken by the segway from the initial to the final position is similar to what we may expect from real life physics.

However in part (e), the linearization approximation is violated. The model still converges to the final upright position because (20) is controllable as we proved in part (c). However, since the approximation is not valid anymore, this discrete time linear model is **not** a good representation of the original continuous time non-linear system. Hence the trajectory is extremely weird with the segway undergoing a few full rotations, and does not match what we would expect from the real system.

We can still analyze the system in continuous time by directly solving the set of non-linear differential equations in (17) (out of 16B scope) or in discrete time using a discretized version of (17). Note that there are two independent distinctions we are making, i.e. continuous vs discrete, and linear vs non-linear. Part (e) failed because it's beyond the scope of the linear model, not because we are using a discrete time system. A non-linear discrete time analysis would also give the correct solution.

(f) We have attached two videos: **cart_pole_non_linearized.mp4** and **cart_pole_linear.mp4**. These videos show the behavior of the continuous time segway system before (17) and after (18) we linearize it, when no control input is applied. More specifically, we use a starting state of

$$\vec{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0.1\text{rad} \\ 0 \end{bmatrix}$$

and we set the input $u(t) = 0$ for the entire simulation. **Contrasting the non-linear and linearized continuous time simulations in the two videos, do you notice any differences in the trajectory when the angle θ gets large? Why is this the case?**

Solution: We notice that the non-linear system is a much more accurate representation of what a real segway looks like. The linearized system starts off looking realistic, but then behaves oddly when the angle gets large. This is because we have approximated a non-linear system with a linear one, so there will be a discrepancy when we have a large deviation from the operating point.

6. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) **What sources (if any) did you use as you worked through the homework?**

(b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

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