EECS 16B Spring 2021 — Designing Information Devices and Systems II, UC Berkeley

Homework 03

This homework is due on Friday, February 5, 2021, at 11:00PM. Self-grades and HW Resubmission are due on Tuesday, February 9, 2021, at 11:00PM.

1. Reading Lecture Notes

Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: Note 2 and Note 3A

(a) How do we deal with piecewise constant inputs as introduced in the notes?

Solution: For piecewise constant inputs, we can just treat them in the same way that we dealt with circuits with switches changing configuration. Make the state be instantaneously constant across the configuration change, and solve the differential equation with that initial condition.

(b) What conclusions do we get after approximating any function $u(t)$ as being piecewise constant over fixed interval widths Δ?

Solution: After taking the limit $\Delta \to 0$, we get the solution to the differential equation $\frac{d}{dt}V(t) = \lambda V(t) - \lambda u(t)$ to be $V(t) = v_0 e^{\lambda t} - \lambda \int_0^t u(\theta) e^{\lambda(t-\theta)} d\theta$.

2. Simple scalar differential equations driven by an input

In class, you learned that the solution for $t \geq 0$ to the simple scalar first-order differential equation

$$\frac{d}{dt}x(t) = \lambda x(t) \tag{1}$$

with initial condition

$$x(t=0) = x_0 \tag{2}$$

is given for $t \geq 0$ by

$$x(t) = x_0 e^{\lambda t}. \tag{3}$$

In an earlier homework, you proved that these solutions are *unique*– that is, that $x(t)$ of the form in (3) are the only possible solutions to the equation (1) with the specified initial condition (2).

In this question, we will analyze differential equations with inputs and prove that their solutions are unique.

In particular, we consider the scalar differential equation

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t) \tag{4}$$

where $u(t)$ is a known function of time from $t = 0$ onwards.

(a) Suppose that you are given an $x_g(t)$ that satisfies both (2) and (4) for $t \geq 0$.
Show that if $y(t)$ also satisfies (2) and (4) for $t \geq 0$, then it must be that $y(t) = x_g(t)$ for all $t \geq 0$.
(HINT: You already used ratios in an earlier HW to prove that two things were necessarily equal. This time, you might want to use differences. Be sure to leverage what you already proved earlier instead of having to redo all that work.)

Solution: We had already used ratios in the earlier homework, and so the natural thing to look at is the difference. Consider $z(t) = y(t) - x_g(t)$. We know that

$$z(0) = y(0) - x_g(0) \tag{5}$$
$$= x_0 - x_0 \tag{6}$$
$$= 0. \tag{7}$$

Furthermore,

$$\frac{d}{dt}z(t) = \frac{d}{dt}y(t) - \frac{d}{dt}x_g(t) \tag{8}$$
$$= \lambda y(t) + u(t) - (\lambda x_g(t) + u(t)) \tag{9}$$
$$= \lambda (y(t) - x_g(t)) + (u(t) - u(t)) \tag{10}$$
$$= \lambda z(t) \tag{11}$$

By (11) and (7), we know that $z(t)$ satisfies the exact conditions for which we proved uniqueness in an earlier homework. (This is why we cared so much about getting the zero initial condition case correct without handwaving.) So we know that $z(t) = 0e^{\lambda t} = 0$ for all $t \geq 0$.
This means that $y(t) - x_g(t) = 0$ and hence $y(t) = x_g(t)$. This successfully proves the uniqueness of solutions to simple scalar differential equations with inputs.

(b) Suppose that the given $u(t)$ starts at $t = 0$ (it is zero before that) and is a nicely integrable function (feel free to assume bounded and continuously differentiably with bounded derivative — whatever conditions you assumed in your calculus course when considering integration and the fundamental theorem of calculus). Let

$$x_c(t) = x_0 e^{\lambda t} + \int_0^t u(\tau) e^{\lambda(t-\tau)} d\tau \tag{12}$$

for $t \geq 0$.

Show that the $x_c(t)$ defined in (12) indeed satisfies (4) and (2).

Note: the τ here in (12) is just a dummy variable of integration. We could have used any letter for that local variable. We just used τ because it visually reminds us of t while also looking different. If you think they look too similar in your handwriting, feel free to change the dummy variable of integration to another symbol of your choice.

(HINT: Remember the fundamental theorem of calculus that you proved in your calculus class and manipulate the expression in (12) to get it into a form where you can apply it along with other basic calculus rules.)

Solution:

Checking (2) just involves plugging in $t = 0$ into (12).

$$x_c(0) = x_0 e^0 + \int_0^0 u(\tau) e^{\lambda(0-\tau)} d\tau = x_0 + 0 = x_0. \tag{13}$$

The real action is in checking that this satisfies the differential equation with the given input. To do this, there are many approaches. One would be to use the big hammer of the full Fundamental Theorem of Calculus in Leibniz form. However, in this case, we can do this in a simpler way by first simplifying the integral by pulling out factors that do not vary with the variable of integration. Notice that:

$$x_c(t) = x_0 e^{\lambda t} + \int_0^t u(\tau) e^{\lambda(t-\tau)} d\tau \tag{14}$$
$$= x_0 e^{\lambda t} + e^{\lambda t} \int_0^t u(\tau) e^{-\lambda \tau} d\tau. \tag{15}$$

Taking a derivative we have:

$$\frac{d}{dt}x_c(t) = \lambda x_0 e^{\lambda t} + \lambda e^{\lambda t}\int_0^t u(\tau)e^{-\lambda\tau}d\tau + e^{\lambda t}\frac{d}{dt}\int_0^t u(\tau)e^{-\lambda\tau}d\tau \tag{16}$$

$$= \lambda x_c(t) + e^{\lambda t}\frac{d}{dt}\int_0^t u(\tau)e^{-\lambda\tau}d\tau \tag{17}$$

$$= \lambda x_c(t) + e^{\lambda t}u(t)e^{-\lambda t} \tag{18}$$

$$= \lambda x_c(t) + u(t) \tag{19}$$

where we used the product rule in (16) and the basic fundamental theorem of calculus in (18).

Because we have checked that the solution satisfies the conditions of the differential equation, and we have shown uniqueness in the previous part, we know that this must be the only solution.

The discussion and the supported course notes show how an argument building from approximating by piecewise constant functions and then Riemann integration lets you naturally guess this form of the solution. But we thought that this HW was already too long to include redoing that part here. It is important to prove that it works however, so that is what you did.

(c) **Use the previous part to get an explicit expression for $x_c(t)$ for $t \geq 0$ when $u(t) = e^{st}$ for some constant s, when $s \neq \lambda$ and $t \geq 0$.**

Solution: In this part we are given input $u(t) = e^{st}$. We have to solve a differential equation of the form:

$$\frac{d}{dt}x(t) = \lambda x(t) + e^{st} \tag{20}$$

In the previous part we were given equation (12) to solve for $x_c(t)$ for $t \geq 0$, for a nonhomogeneous ODE of the form (4) with any input $u(t)$.

Plugging into equation (12) we get:

$$x_c(t) = x_0 e^{\lambda t} + \int_0^t e^{s\tau}e^{\lambda(t-\tau)}d\tau \tag{21}$$

$$= x_0 e^{\lambda t} + e^{\lambda t}\int_0^t e^{\tau(s-\lambda)}d\tau \tag{22}$$

$$= x_0 e^{\lambda t} + e^{\lambda t}\left[\frac{1}{s-\lambda}e^{(s-\lambda)\tau}\right]_{\tau=0}^{\tau=t} \tag{23}$$

$$= x_0 e^{\lambda t} + e^{\lambda t}\left[\frac{1}{s-\lambda}e^{(s-\lambda)t} - \frac{1}{s-\lambda}e^{(s-\lambda)0}\right] \tag{24}$$

$$= x_0 e^{\lambda t} + e^{\lambda t}\left[\frac{1}{s-\lambda}e^{(s-\lambda)t} - \frac{1}{s-\lambda}\right] \tag{25}$$

$$= x_0 e^{\lambda t} + e^{\lambda t}\frac{1}{s-\lambda}e^{(s-\lambda)t} - e^{\lambda t}\frac{1}{s-\lambda} \tag{26}$$

$$= x_0 e^{\lambda t} + \frac{1}{s-\lambda}e^{st} - \frac{1}{s-\lambda}e^{\lambda t} \tag{27}$$

$$= (x_0 - \frac{1}{s-\lambda})e^{\lambda t} + \frac{e^{st}}{s-\lambda} \tag{28}$$

Thus we get $x_c(t) = (x_0 - \frac{1}{s-\lambda})e^{\lambda t} + \frac{e^{st}}{s-\lambda}$.

(d) **Similarly, what is $x_c(t)$ for $t \geq 0$ when $u(t) = e^{\lambda t}$ for $t \geq 0$.**

(HINT: Don't worry if this seems too easy.)

Solution:

Similar to the previous part here we are given another exponential e^{st} as an input. However here $s = \lambda$. Just like the previous part we will proceed by plugging our new input into (4):

$$
\begin{aligned}
x_c(t) &= x_0 e^{\lambda t} + \int_0^t e^{\lambda \tau} e^{\lambda (t-\tau)} d\tau && (29)\\
&= x_0 e^{\lambda t} + e^{\lambda t} \int_0^t e^{\tau(\lambda-\lambda)} d\tau && (30)\\
&= x_0 e^{\lambda t} + e^{\lambda t} \int_0^t 1 d\tau && (31)\\
&= x_0 e^{\lambda t} + e^{\lambda t} \Big[\tau\Big]_{\tau=0}^{\tau=t} && (32)\\
&= x_0 e^{\lambda t} + t e^{\lambda t} && (33)
\end{aligned}
$$

Thus we get $x_c(t) = x_0 e^{\lambda t} + t e^{\lambda t}$.
This basic pattern can be continued. We can plug in $t e^{\lambda t}$ as an input and get further polynomials in t multiplying the same exponential.

3. Uniqueness Counterexample

This problem explores an example of a differential equation that does not have a unique solution. The purpose is to show that uniqueness cannot always be assumed. There is a reason we are making you prove uniqueness to trust solutions.

Along the way, this problem will also show you a heuristic way to guess the solutions to differential equations that is often called "separation of variables." The advantage of the separation of variables technique is that it can often be helpful in systematically coming up with guesses for nonlinear differential equations. However, as with any technique for guessing, it is not a proof and the guess definitely needs to be checked and uniqueness verified before proceeding.

The idea of separation-of-variables is to pretend that $\frac{d}{dt}x(t) = \frac{dx}{dt}$ is a ratio of quantities rather than what it is — a shorthand for taking the derivative of the function $x(\cdot)$ with respect to its single argument, and then writing the result in terms of the free variable "t" for that argument. This little bit of make-believe (sometimes euphemistically called "an abuse of notation") allows one the freedom to do calculations.

To demonstrate, let's do this for a case where we know the correct solution: $\frac{d}{dt}x(t) = \lambda x(t)$. This is how a separation-of-variables approach would try to get a guess:

$$\frac{d}{dt}x(t) = \lambda x(t) \tag{34}$$

$$\frac{dx}{dt} = \lambda x \tag{35}$$

$$\frac{dx}{x} = \lambda dt \text{ separating variables to sides} \tag{36}$$

$$\int \frac{dx}{x} = \int \lambda dt \text{ integrating both sides} \tag{37}$$

$$\ln x + C_1 = \lambda t + C_2 \tag{38}$$

$$x(t) = Ke^{\lambda t} \text{ exponentiating both sides and folding constants} \tag{39}$$

With the above guess obtained, $x(t) = Ke^{\lambda t}$ can be plugged in and seen to solve the original differential equation because the steps above are vaguely reversible, if a bit hallucinatory in nature. Then of course, a uniqueness proof is required, but you did that in the previous homework.

To see why this technique is a bit fraught, we will consider the following nonlinear differential equation involving a third root. (If we had more time in making this problem, we would have showed you how this sort of differential equation can arise from a toy physical setting of a inverted pyramidal container that had $x(t)$ liters of water in it, where the rate of water being poured in is proportional to the height of the water $x^{\frac{1}{3}}$. This fractional power arises since volume is a cubic quantity while the water is being poured in at a rate governed by a one-dimensional quantity of length. Similar equations can arise in microfluidic dynamics.)

Anyway, consider the differential equation

$$\frac{d}{dt}x(t) = \alpha x^{\frac{1}{3}}(t) \tag{40}$$

with the initial condition

$$x(0) = 0.$$

Let's take the "separation-of-variables pill" and see what trip it takes us on:

$$\frac{d}{dt}x(t) = \alpha x^{\frac{1}{3}}(t) \tag{41}$$
$$\frac{dx}{dt} = \alpha x^{\frac{1}{3}} \tag{42}$$

Due to the "abuse of notation," we can write it as follows

$$x^{-\frac{1}{3}}dx = \alpha dt \tag{43}$$
$$\int x^{-\frac{1}{3}}dx = \int \alpha dt \tag{44}$$
$$\frac{3}{2}x^{\frac{2}{3}} + C_1 = \alpha t + C_2 \tag{45}$$
$$x(t) = (\frac{2}{3}\alpha t + C_3)^{\frac{3}{2}} \tag{46}$$

That didn't seem like too wild a ride. Let's see what rabbit hole we have actually landed in.

(a) Given our separation-of-variables based calculation, let us guess a solution of the form

$$x(t) = \left(\frac{2}{3}\alpha t + c\right)^{\frac{3}{2}}$$

.

Show that this is a solution to the differential equation (40), and find the c that satisfies the initial condition.

Solution: We check that this is a solution by differentiating the guess for $x(t)$, and showing that it satisfies (40).

$$\begin{aligned}
\frac{d}{dt}x(t) &= \frac{d}{dt}\left(\left(\frac{2}{3}\alpha t + c\right)^{\frac{3}{2}}\right) \\
&= \frac{3}{2}\cdot\left(\frac{2}{3}\alpha t + c\right)^{\frac{1}{2}}\cdot\frac{2}{3}\alpha \\
&= \alpha\left(\frac{2}{3}\alpha t + c\right)^{\frac{1}{2}} \\
&= \alpha\left(\left(\frac{2}{3}\alpha t + c\right)^{\frac{3}{2}}\right)^{\frac{1}{3}} \\
&= \alpha x^{\frac{1}{3}}(t)
\end{aligned}$$

We thus satisfy (40).

To solve for the initial condition, we evaluate

$$\begin{aligned}
0 = x(0) &= \left(\frac{2}{3}\alpha\cdot 0 + c\right)^{\frac{3}{2}} \\
&= c^{\frac{3}{2}} \\
&\longrightarrow c = 0
\end{aligned}$$

This is the only solution to the found form that satisfies the initial condition.

The equation $x(t) = \left(\frac{2}{3}\alpha t\right)^{\frac{3}{2}}$ thus satisfies the differential equation and the initial value condition.

(b) Let us guess a second solution:

$$x(t) = 0 \tag{47}$$

Show that this new guess also satisfies (40), and the initial condition ($x(0) = 0$).

Solution:

$$\frac{d}{dt}x(t) = \frac{d}{dt}0 = 0 = \alpha \cdot 0 = \alpha \cdot x(t) \tag{48}$$

Furthermore, $x(0) = 0$.

So this second solution also satisfies the differential equation and the initial value condition.

(c) **Show that any solution of the form**

$$x(t) = \begin{cases} 0, & \text{if } t < t_0 \\ \left(\frac{2}{3}\alpha\left(t - t_0\right)\right)^{\frac{3}{2}}, & t \geq t_0 \end{cases} \tag{49}$$

also satisfies (40) and the initial condition $x(0) = 0$, for any $t_0 > 0$.

So this actually has *infinitely* many solutions.

Solution:

To see if this is a solution, we see if it satisfies (40). We handle the two cases separately:

For $t < t_0$:

$$\frac{d}{dt}x(t) = \frac{d}{dt}0 = 0 = \alpha \cdot 0 = \alpha \cdot 0^{\frac{1}{3}} = \alpha \cdot x^{\frac{1}{3}}(t)$$

For $t \geq t_0$:

$$\begin{aligned} \frac{d}{dt}x(t) &= \frac{d}{dt}\left(\left(\frac{2}{3}\alpha\left(t - t_0\right)\right)^{\frac{3}{2}}\right) \\ &= \frac{3}{2} \cdot \left(\frac{2}{3}\alpha\left(t - t_0\right)\right)^{\frac{1}{2}} \cdot \frac{2}{3}\alpha \\ &= \alpha\left(\left(\frac{2}{3}\alpha\left(t - t_0\right)\right)^{\frac{3}{2}}\right)^{\frac{1}{3}} \\ &= \alpha x^{\frac{1}{3}}(t) \end{aligned}$$

At the boundary $t = t_0$, we see that

$$x(t_0) = \left(\frac{2}{3}\alpha\left(t_0 - t_0\right)\right)^{\frac{3}{2}} = \frac{2}{3}(0)^{\frac{3}{2}} = 0$$

Thus, $x(t)$ is a continuous function.

We also can verify that $\frac{d}{dt}x(t)$ is continuous at the boundary $t = t_0$:

$$\begin{aligned} \frac{d}{dt}x(t)|_{t=t_0} &= \alpha\left(\alpha(t - t_0)\right)^{\frac{1}{2}}|_{t=t_0} \\ &= \alpha\left(\alpha(t_0 - t_0)\right)^{\frac{3}{2}} \\ &= 0 \end{aligned}$$

The derivative also continuous.

The initial condition holds, as $t_0 > 0$, so $x(t=0)$ will always be 0.

(d) One of the actual sufficient conditions for the uniqueness of solutions to differential equations of the form $\frac{d}{dt}x(t) = f(x(t))$ is that the function $f(x)$ be continuously differentiable with a bounded derivative at the initial condition $x(0)$ and everywhere that the solution purports to go. (You will understand the importance of this condition and where it comes from better when we are in Module 2 of 16B.)

Does this differential equation problem satisfy this condition that would let us trust guessing and checking?

Solution: No. To see this, consider our differential equation (40) at $t = 0$. In this case, $f(x) = \alpha x^{\frac{1}{3}}$. We see that $\frac{d}{dx}f(x) = \frac{d}{dx}\left(\alpha x^{\frac{1}{3}}\right) = \frac{\alpha}{3}x^{-\frac{2}{3}} = \frac{\alpha}{3x^{\frac{2}{3}}}$.

Note that at $x(t=0) = 0$, our initial condition, the derivative becomes discontinuous in an unbounded fashion. It blows up. Thus this differential equation does not satisfy the uniqueness condition that we have provided.

(e) In this particular case, there is a little bit of a warning in the separation-of-variables trip (you can consider it a dream sequence if you'd like — it has a logic to it, just not a logic that you can rely upon in the real world). **Explain why** (43) **might be a bit problematic in this case.**

Solution: The problem with (43) is that there is $x^{-\frac{1}{3}}$ on the left hand side. If $x = 0$, this results in division by 0. As $x(0) = 0$ is the initial condition, this unbounded point becomes a problem for the separation-of-variables technique.

(f) To see that separation of variables can actually give you reasonable guesses even for equations with interesting solutions, try to use it to solve the following differential equation $\frac{d}{dt}x(t) = \frac{1}{2}x^2$ for initial condition $x(0) = 1$. **Sketch your solution. Describe anything interesting that happens. Does this problem satisfy the uniqueness conditions for** $t \in [0,1]$**?**

Solution: Applying separation of variables:

$$\begin{aligned}
\frac{d}{dt}x(t) &= \frac{1}{2}x^2 \\
\frac{dx}{dt} &= \frac{1}{2}x^2 \\
x^{-2}dx &= \frac{1}{2}dt \\
-x^{-1} &= \frac{t}{2} + C_1 \\
x(t) &= \frac{-1}{\frac{t}{2} + C_1} \\
&= \frac{-2}{t + C_2}
\end{aligned}$$

To satisfy the initial condition $x(0) = 1$: $1 = x(0) = \frac{-2}{0+C_2} \longrightarrow C_2 = -2$ Thus

$$x(t) = \frac{-2}{t-2}$$

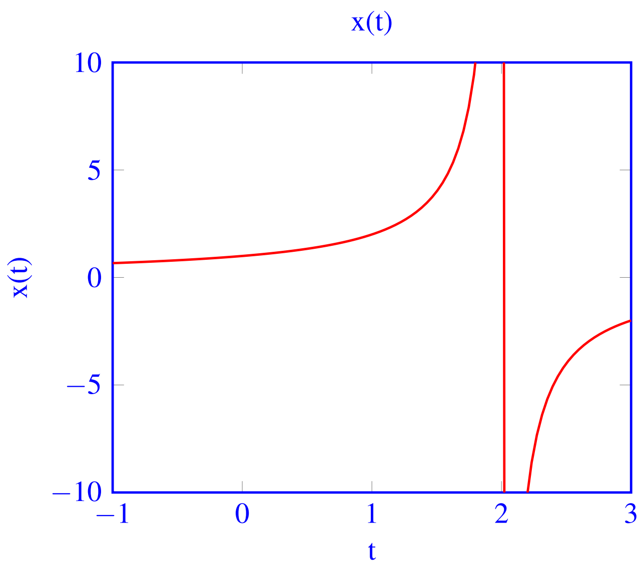


Figure 1: Notice that this function escapes to infinity at a finite time $t = 2$. While a computer will happily plot it after that point, such a plot is meaningless there since there is no reason to come down from $+\infty$.

To check whether the uniqueness condition holds over $t \in [0,1]$, we need to check that $f(x) = \frac{1}{2}x^2$ is continuously differentiable with a bounded derivative over $t \in [0,1]$.

First, note that $f'(x) = \frac{d}{dx}f(x) = x$ exists.

For $t \in [0,1]$, we see that the solution moves smoothly from $x(t = 0) = 1$ to $x(t = 1) = 2$. Thus we must check that $f'(x)$ is continuous and bounded for $x \in [1,2]$.

Indeed this is the case, as $f'(x) = x$ is a smooth function that varies from 1 to 2 for $x \in [1,2]$.

Thus, the solution is unique.

Note that at $t = 2$, the solution becomes unbounded. Beyond that point, our differential equation ceases to have meaning.

However, as $t = 2$ is outside the range over which we are trying to find a solution, it does not impact the uniqueness criteria question we asked. For any interval that is on the left side of $t = 2$, the uniqueness criteria are satisfied so this solution indeed must run off to infinity in finite time.

In practice, such finite-time-escape behavior is an indication that whatever physical model we are using is breaking down.

4. Op-Amp Stability

In this question we will revisit the basic op-amp model that was introduced in EECS 16A and we will add a capacitance C_{out} to make the model more realistic (refer to figure 2). Now that we have the tools to do so, we will study the behavior of the op-amp in positive and negative feedback (refer to figure 3). Furthermore, we will begin looking at the integrator circuit (refer to figure 4) to see how a capacitor in the negative feedback can behave. In the next homework, you will see why it ends up being close to an integrator.

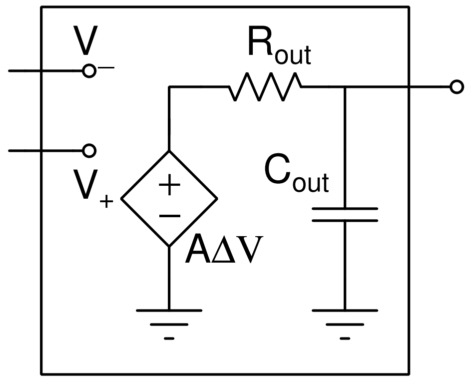
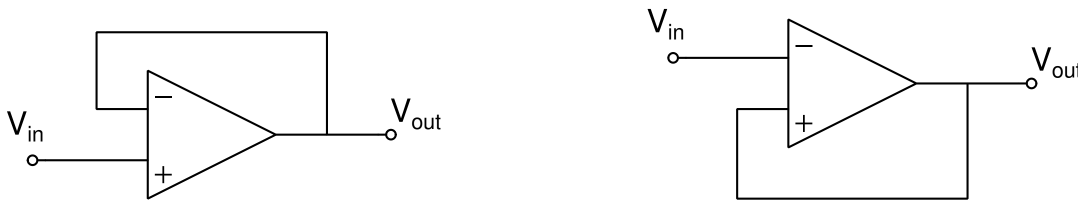


Figure 2: Op-amp model: $\Delta V = V_+ - V_-$



(a) Buffer in negative feedback

(b) Op-amp in positive feedback

Figure 3: Op-amp in positive and negative feedback

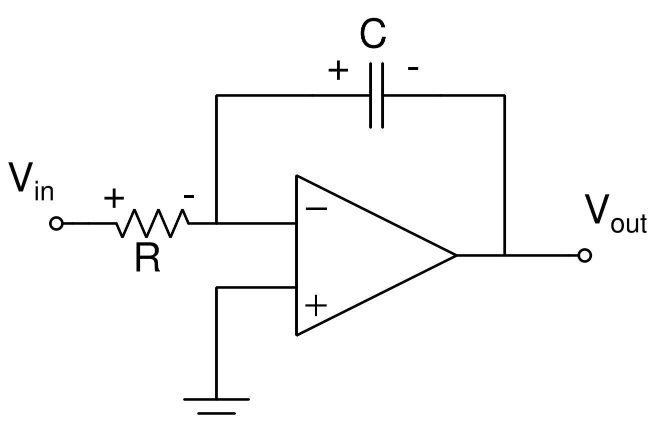


Figure 4: Integrator circuit

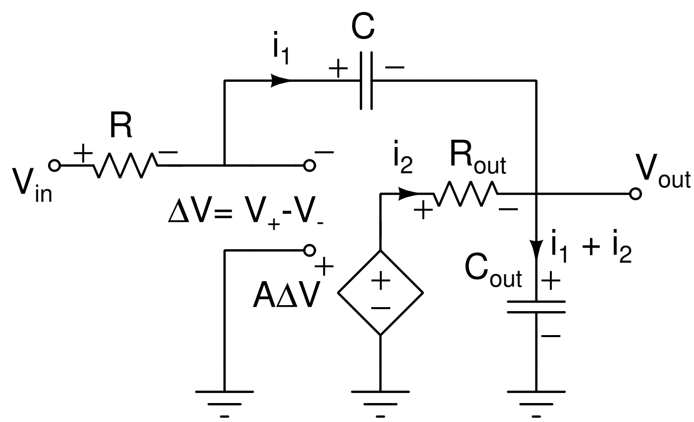


Figure 5: Integrator circuit with Op-amp model

(a) Using the op-amp model in figure 2 and the buffer in negative-feedback configuration in figure 3(a), **draw a combined circuit**. Remember that $\Delta V = V_+ - V_-$, the voltage difference between the positive and negative labeled input terminals of the op-amp.

(HINT: Look at figure 5 to see how this was done for the integrator. That might help.)

Note: here, we have used the equivalent model for the op-amp gain. In more advanced analog circuits courses, it is traditional to use a controlled current source with a resistor in parallel instead.

Solution: Please refer to Figure 6 for the completed circuit.

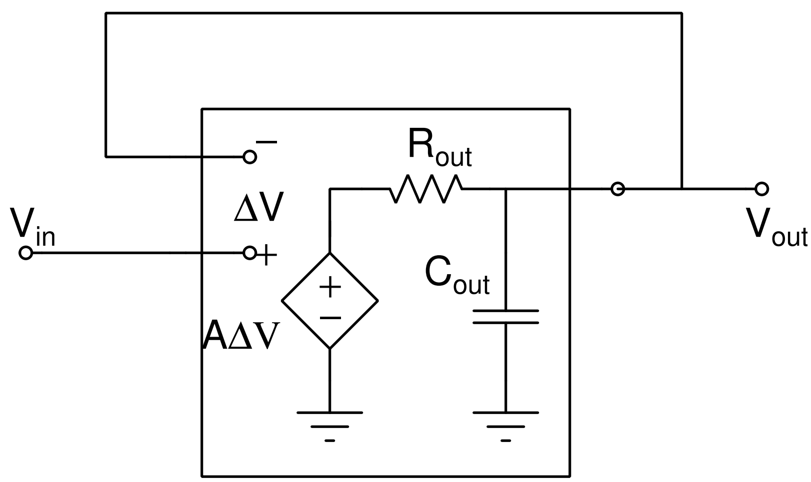


Figure 6: Negative-feedback buffer configuration using given op-amp model

(b) Let's look at the op-amp in negative feedback. From our discussions in EECS 16A, we know that the buffer in figure 3(a) should work with $V_{out} \approx V_{in}$ by the golden rules. **Write a differential-equation for V_{out} by replacing the op-amp with the given model and show what the solution will be as a function of time for a static V_{in}. What does it converge to as $t \to \infty$?** Note: We assume the gain $A > 1$ for all parts of the question.

Solution: We have $\Delta V = V_{in} - V_{out}$. Next, we can write the following branch equations:

$$\begin{aligned} i &= C_{out}\frac{d}{dt}V_{out} \\ A\Delta V &= V_R + V_{out} \\ A(V_{in} - V_{out}) &= RC\frac{d}{dt}V_{out} + V_{out} \end{aligned}$$

Simplifying the last line, we get:

$$\begin{aligned} AV_{in} &= RC\frac{d}{dt}V_{out} + (1+A)V_{out} \\ \frac{d}{dt}V_{out} &+ \frac{1+A}{RC}V_{out} - \frac{A}{RC}V_{in} = 0 \end{aligned}$$

Solving the above differential equation, with the substitution $\widetilde{V}_{out} = V_{out} - \frac{A}{A+1}V_{in}$, we get

$$\widetilde{V}_{out}(t) = ke^{-\frac{A+1}{RC}t}.$$

Substituting for the initial condition $V_{out}(0) = 0$, we get:

$$V_{out} = \frac{AV_{in}}{A+1}\left(1 - e^{-\frac{A+1}{RC}t}\right).$$

Since $A > 1$, the exponent is negative, hence as $t \to \infty$, the solution will converge to $V_{out} \to \frac{A}{A+1}V_{in}$.

(c) Next, let's look at the op-amp in positive feedback. We know that the configuration given in figure 3(b) is unstable and V_{out} will just rail. **Again, using the op-amp model in figure 2, show that V_{out} does not converge and hence the output will rail. For positive DC input $V_{in} > 0$, will V_{out} rail to the positive or negative side? Explain.**

Solution: This time, we have $\Delta V = V_{out} - V_{in}$. Next, we can write the following branch equations:

$$\begin{aligned} i &= C_{out}\frac{d}{dt}V_{out} \\ A\Delta V &= V_R + V_{out} \\ A(V_{out} - V_{in}) &= RC\frac{d}{dt}V_{out} + V_{out} \end{aligned}$$

Simplifying the last line, we get:

$$\begin{aligned} -AV_{in} &= RC\frac{d}{dt}V_{out} + (1-A)V_{out} \\ \frac{d}{dt}V_{out} &+ \frac{1-A}{RC}V_{out} + \frac{A}{RC}V_{in} = 0 \end{aligned}$$

Solving the above differential equation, with the substitution $\widetilde{V}_{out} = V_{out} - \frac{A}{A-1}V_{in}$, we get:

$$\begin{aligned} \widetilde{V}_{out} &= ke^{-\frac{1-A}{RC}t} \\ &= ke^{\frac{A-1}{RC}t} \end{aligned}$$

Substituting for the hypothetical initial condition $V_{out}(0) = 0$, we get:

$$V_{out} = -\frac{AV_{in}}{A-1}\left(e^{\frac{A-1}{RC}t} - 1\right)$$

Since $A > 1$, the exponent is positive, hence as $t \to \infty$, the solution will be unbounded, and $V_{out} \to -\infty$. Of course, it can't grow to negative infinity, and so we can conclude that V_{out} will rail to the negative side if V_{out} had started at zero. The same exact story would hold if V_{out} started anywhere below V_{in}.

The case of V_{out} starting out significantly greater than V_{in} deserves some mention, although not necessary for full credit on this question. In this case, the initial condition for $\widetilde{V}_{out}$ would be positive, and would proceed to unstably attempt to run away to positive infinity. It would be stopped at the positive rail, where it would stay.

Essentially, all that matters is the initial condition of $\widetilde{V}_{out}$ — start out positive, then we rail to a positive rail for V_{out}. Start out negative, then we rail to the negative rail for V_{out}. An op-amp in positive feedback retains its comparator-like character.

(d) For an ideal op-amp, we can assume that it has an infinite gain, *i.e.*, $A \to \infty$. **Under these assumptions, show that the op-amp in negative feedback behaves as an ideal buffer, *i.e.*, $V_{out} = V_{in}$.**

Solution: Taking the limit of our solution in part (a),

$$\begin{aligned} V_{out} &= \lim_{A\to\infty} \frac{AV_{in}}{A+1}\left(1 - e^{-\frac{A+1}{RC}t}\right) \\ &= V_{in} \end{aligned}$$

The coeffecient of the exponent goes to 1 whereas the exponent itself goes to $-\infty$, and hence $1 - e^{-\infty} \to 1$.

(e) Let's extend our analysis to the integrator circuit shown in figure 4. Simplifying all the equations, we get a system of differential equations in two variables V_C and V_{out}, where V_C and V_{out} are the voltage drops across the capacitors C and C_{out}. **Fill in the missing term in the following matrix differential equation.**

$$\frac{d}{dt}\begin{bmatrix} V_{out}(t) \\ V_C(t) \end{bmatrix} = \begin{bmatrix} -\left(\frac{A+1}{R_{out}C_{out}} + \frac{1}{RC_{out}}\right) & -\left(\frac{1}{RC_{out}} + \frac{A}{R_{out}C_{out}}\right) \\ -\frac{1}{RC} & ? \end{bmatrix}\begin{bmatrix} V_{out}(t) \\ V_C(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{RC_{out}} \\ \frac{1}{RC} \end{bmatrix} V_{in}(t) \tag{50}$$

(HINT: We picked an easier term to hide. You don't have to write out all the equations and do a lot of algebra to figure out what the missing term is.)

Solution: We can define a convenient variable as follows:

$$\Delta V = 0 - (V_{in} - i_1 R) \tag{51}$$

$$i_1 = C\frac{d}{dt}V_C = \frac{1}{R}(V_{in} - V_C - V_{out}) \tag{52}$$

$$i_2 = \frac{1}{R_{out}}(A\Delta V - V_{out}) \tag{53}$$

$$= \frac{AR}{R_{out}}i_1 - \frac{A}{R_{out}}V_{in} - \frac{1}{R_{out}}V_{out} \tag{54}$$

Next, we can write the branch equations as follows:

$$\begin{aligned} V_{out} &= V_{in} - i_1 R - V_C \\ C_{out}\frac{d}{dt}V_{out} &= i_1 + i_2 \end{aligned}$$

Substituting from equations (51), (52) and (54)

$$\begin{aligned}
\frac{d}{dt}V_C &= -\frac{1}{RC}V_{out} - \frac{1}{RC}V_C + \frac{1}{RC}V_{in} \\
C_{out}\frac{d}{dt}V_{out} &= \frac{1}{R}(V_{in} - V_C - V_{out}) + \frac{A}{R_{out}}(V_{in} - V_C - V_{out}) - \frac{A}{R_{out}}V_{in} - \frac{1}{R_{out}}V_{out} \\
&= \frac{1}{R}V_{in} - \left(\frac{1}{R} + \frac{A}{R_{out}}\right)V_C - \left(\frac{1}{R_{out}} + \frac{A}{R_{out}} + \frac{1}{R}\right)V_{out}
\end{aligned}$$

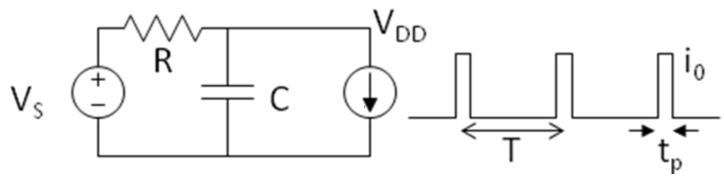
Hence, we can write the matrix differential equation as follows:

$$\frac{d}{dt}\begin{bmatrix} V_{out} \\ V_C \end{bmatrix} = \begin{bmatrix} -\left(\frac{A+1}{R_{out}C_{out}} + \frac{1}{RC_{out}}\right) & -\left(\frac{1}{RC_{out}} + \frac{A}{R_{out}C_{out}}\right) \\ -\frac{1}{RC} & -\frac{1}{RC} \end{bmatrix}\begin{bmatrix} V_{out} \\ V_C \end{bmatrix} + \begin{bmatrix} \frac{1}{RC_{out}} \\ \frac{1}{RC} \end{bmatrix} V_{in}$$

5. IC Power Supply

Digital integrated circuits (ICs) often have very non-uniform current requirements which can cause voltage noise on the supply lines. If one IC is adding a lot of noise to the supply line, it can affect the performance of other ICs that use the same power supply, which can hinder performance of the entire device. For this reason, it is important to take measures to mitigate, or "smooth out", the power supply noise that each IC creates. A common way of doing this is to add a "supply capacitor" between each IC and the power supply. (If you look at a circuit board, and the supply capacitor is the small capacitor next to each IC.)

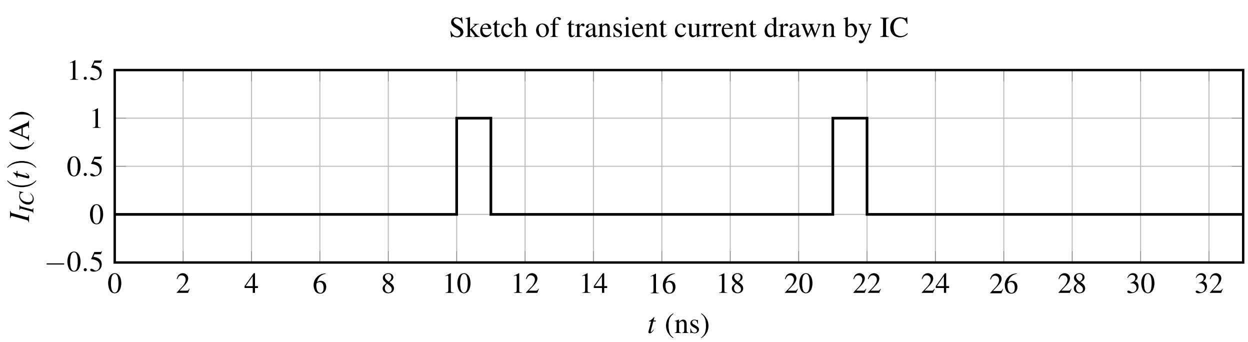
Here's a simple model for a power supply and digital circuit:



The current source is modeling the "spiky," non-uniform nature of digital circuit current consumption. 'The resistor represents the sum of the source resistance of the supply and any wiring resistance between the supply and the load.

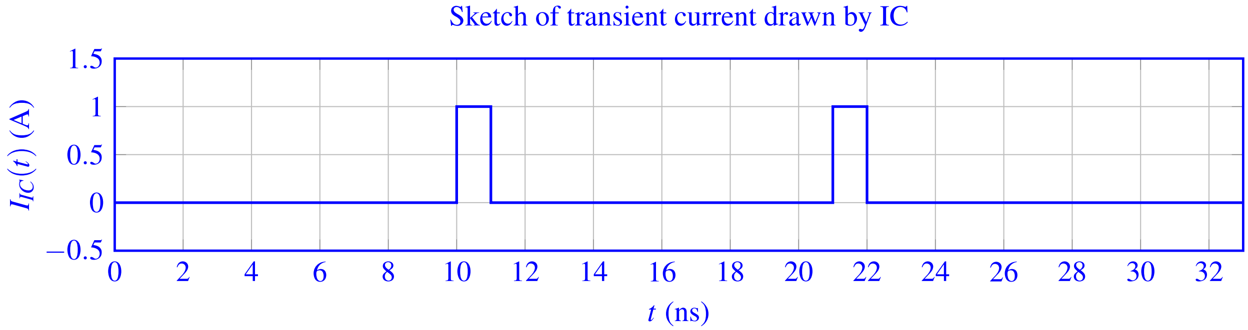
The capacitor is added to try to minimize the noise on V_{DD}. Assuming that $V_s = 3\text{V}$, $R = 1\Omega$, $i_0 = 1\text{A}$, $T = 10\text{ns}$, and $t_p = 1\text{ns}$.

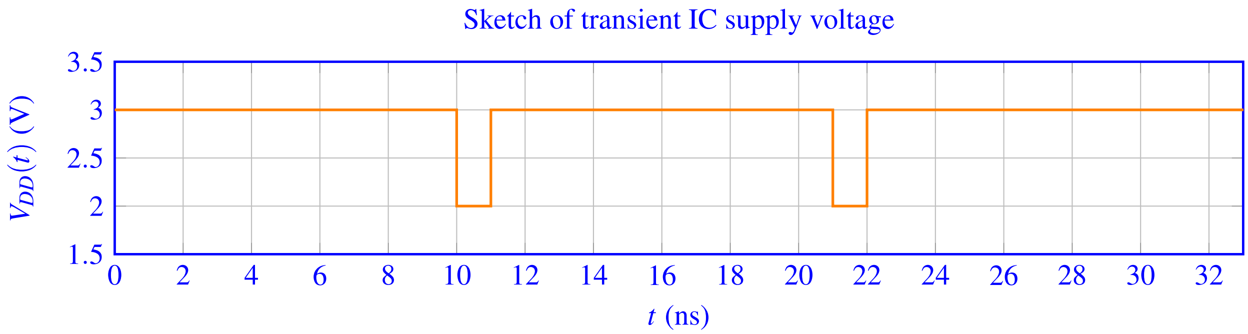
(a) Sketch the voltage V_{DD} vs. time for two periods assuming that $C = 0$. Assume the transient current drawn by IC behaves as follows.



Solution:

If $C = 0$, then this circuit will respond instantaneously to changes in the current; thus we may break this down into two segments, where in the current source $I_{IC}(t)$ equals 0 (and thus $V_{DD} = V_s$), and where the current source $I_{IC}(t)$ equals i_0 (and thus $V_{DD} = V_s - i_0 R = 2\text{V}$). These will follow the current source's flips precisely. With that in mind, your sketch should look something like this:





In the above sketch, we have the first current spike at $t = 10$ ns. Yours doesn't have to align with that: for example, if you had the first current spike at $t = 0$, that's okay. However, what *does* matter is that you have the timing between the current spikes drawn correctly.

(b) Give expressions for and sketch the voltage V_{DD} vs. time for two periods for each of three different capacitor values for C: 1pF , 1nF, 1μF. (1pF $= 10^{-12}$F, 1nF $= 10^{-9}$F, 1μF $= 10^{-6}$F). You may assume $I_C = 0$ for a long time before $t = 0$. Feel free to use the script in the iPython notebook for plotting.

Solution:

Since the current through the source is a series of pulses, it will be easiest if we solve for $V_{DD}(t)$ assuming a piecewise constant I_C. Starting with KVL:

$$V_S = V_R + V_{DD} \tag{55}$$

$$V_S = (I_C + C\frac{d}{dt}V_{DD})R + V_{DD} \tag{56}$$

$$\frac{d}{dt}V_{DD} = \frac{1}{RC}(V_S - RI_C - V_{DD}) \tag{57}$$

Where I_C is the piecewise constant value of the current flowing through the digital circuit. At this point we can use substitution with $\widetilde{V} = (V_S - RI_C - V_{DD})$.

$$\frac{d}{dt}\widetilde{V} = -\frac{d}{dt}V_{DD} \tag{58}$$

$$\frac{d}{dt}V_{DD} = \frac{\widetilde{V}}{RC} \tag{59}$$

$$\frac{d}{dt}\widetilde{V} = -\frac{\widetilde{V}}{RC} \tag{60}$$

$$\widetilde{V}(t) = Ae^{-\frac{t}{RC}} \tag{61}$$

Substituting back to solve for $V_{DD}(t)$ we get the following general expression for the voltage during any one piecewise constant time slice:

$$V_{DD}(t) = V_S - RI_C - Ae^{-\frac{t}{RC}} \tag{62}$$

From here, based on the value of the piecewise current I_C and the initial conditions imposed by previous time segments we can simplify the V_{DD} expression and solve for A. At the start, we will assume a convenient initial condition which corresponds with the behavior of V_{DD} if $I_C = 0$ for a long time before $t = 0$. In this case, $V_{DD} = V_S$ and there is zero current flowing through the circuit. Things get exciting once the first current pulse starts such that $I_C = i_0$. The initial condition for this piecewise section is the final voltage V_{DD} from the previous section, V_S.

$$V_S = V_S - Ri_0 - Ae^0 \tag{63}$$

$$A = -Ri_0 \tag{64}$$

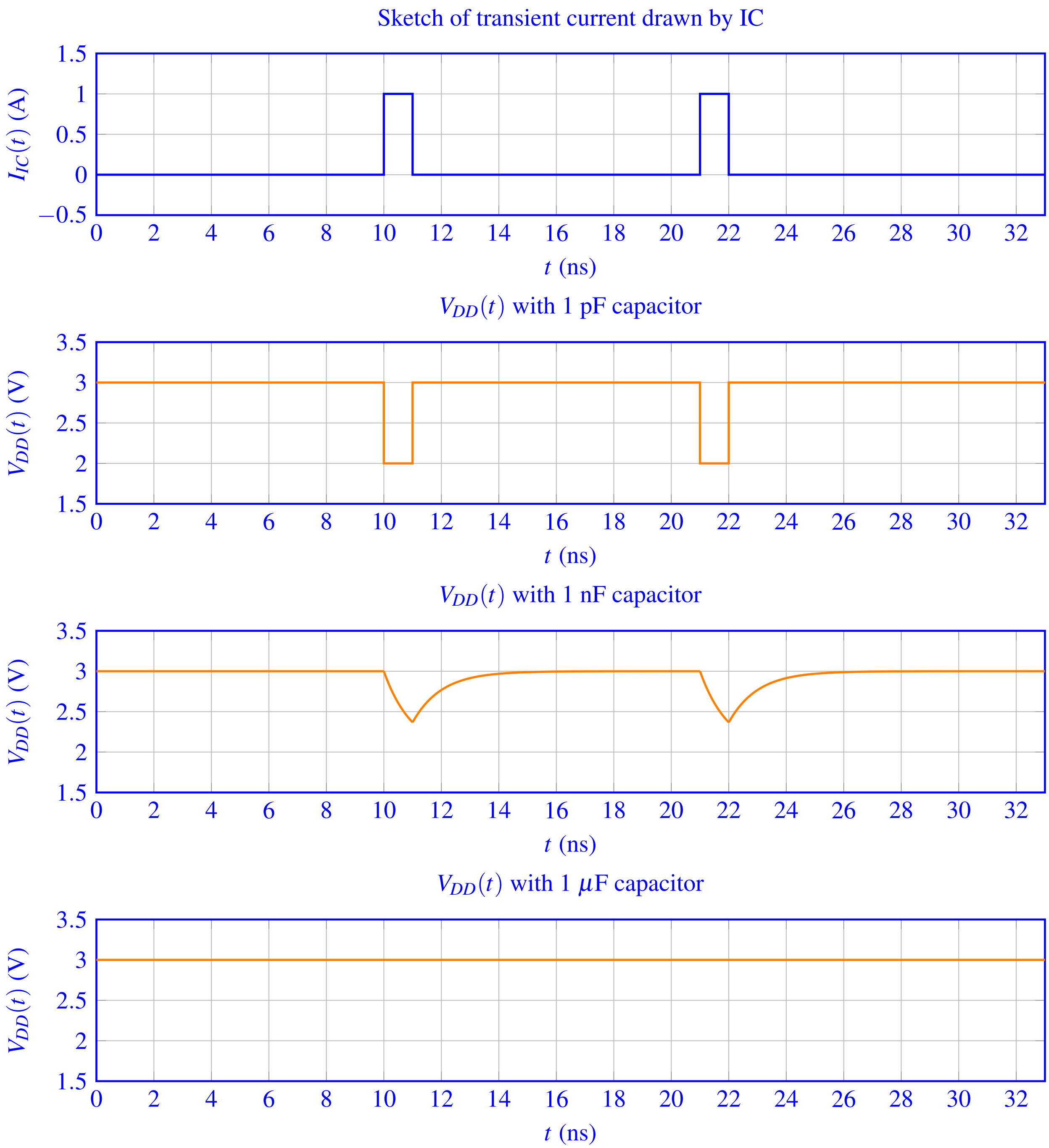
$$V_{DD}(t) = V_S - Ri_0(1 - e^{-\frac{t}{RC}}) \tag{65}$$

You can compute the voltage at the end of the pulse by plugging in $t = t_p$ and the R and C values for your scenario. This voltage will serve as the initial condition for the next piecewise constant section. The process of simplifying the general piecewise differential equation and solving for A can be performed repeatedly to determine the shape of the plot for further pulses.

In general: each of the three curves will tend towards a final value $V_{DD} = V_s$, growing exponentially slower towards this goal as time progresses. However, on each time interval t_p, the current source will start drawing charge from both V_s–whose current decreases as time proceeds–and C–whose charge, and therefore whose potential to contribute voltage, tends to increase with time. With each t_p, V_{DD} decreases nonlinearly, as there are both exponential and linear factors contributing to the rise and fall of the voltage.

With a lower capacitance, we will see the capacitor charge and discharge faster with time–this means that V_{DD} will fluctuate more/change more drastically on t_p; as you increase capacitance, this fluctuation is less evident, as C has more charge to pull from, and will thus be less affected by the change of charge incurred by the current source.

The final sketches should look something like this:



Note that if your solutions contain just the correct plots without much verbose, textual explanation of the plots, then you still deserve full credit.

The idea here is to see the effect that the capacitors have on $V_{DD}(t)$ when viewed at the time scale of the current spikes.

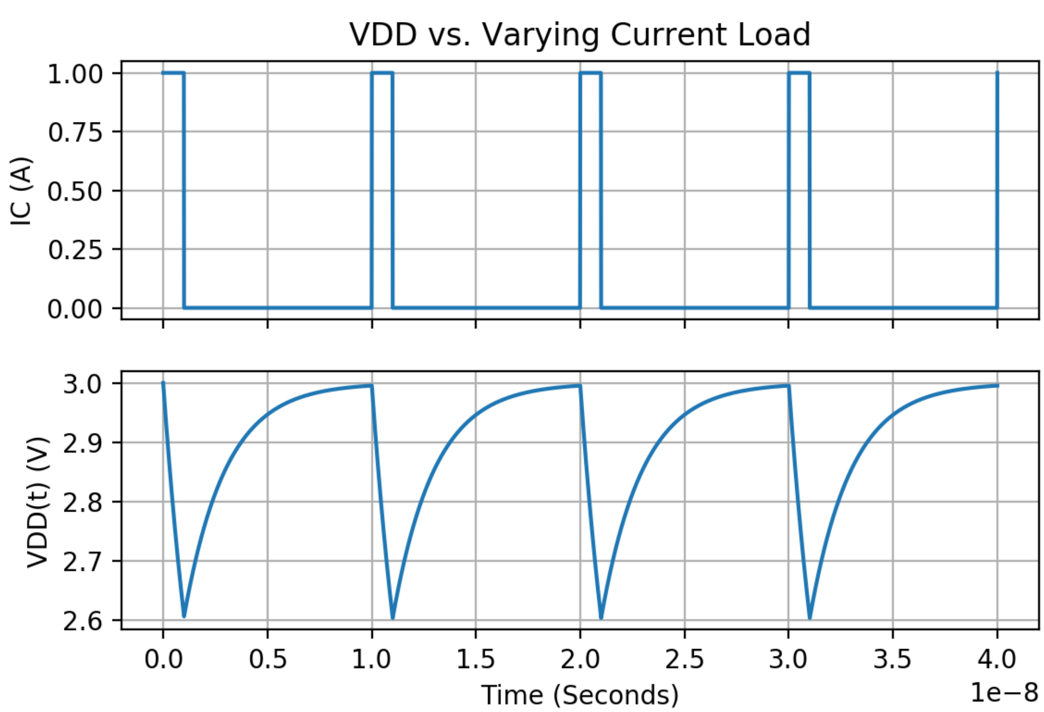
- the 1 pF capacitor causes the RC circuit to have a time constant of $\tau = RC = 1\text{ps}$, that is $1\text{picosecond} = 10^{-12}$ seconds, and the effect that this has on $V_{DD}(t)$ is invisible at the nanosecond time scale. For this reason, we can conclude that the 1 pF capacitor would not be adequate to mitigate the noise that the IC will put on the power supply.

- the 1 nF capacitor causes the RC circuit to have a time constant of $\tau = RC = 1$ns. This is a long enough time scale that the effect on $V_{DD}(t)$ will be visible. At the end of the 1 ns current spike, $V_{DD}(t)$ will have dropped from 3 V to $2 + \exp(-1) \approx 2.37$ V. This means that the 1 nF capacitor is actually reducing the power supply noise a little bit, but not much.
- the 1 μF capacitor causes the RC circuit to have a time constant of $\tau = RC = 1\mu$s. This time constant is 1000 times longer than the duration of the current spike. At the end of the current spike, $V_{DD}(t)$ will have dropped by only one millivolt, so at the scale at which these sketches are drawn, there is no visible change. The 1 μF capacitor has almost totally removed the power supply noise.

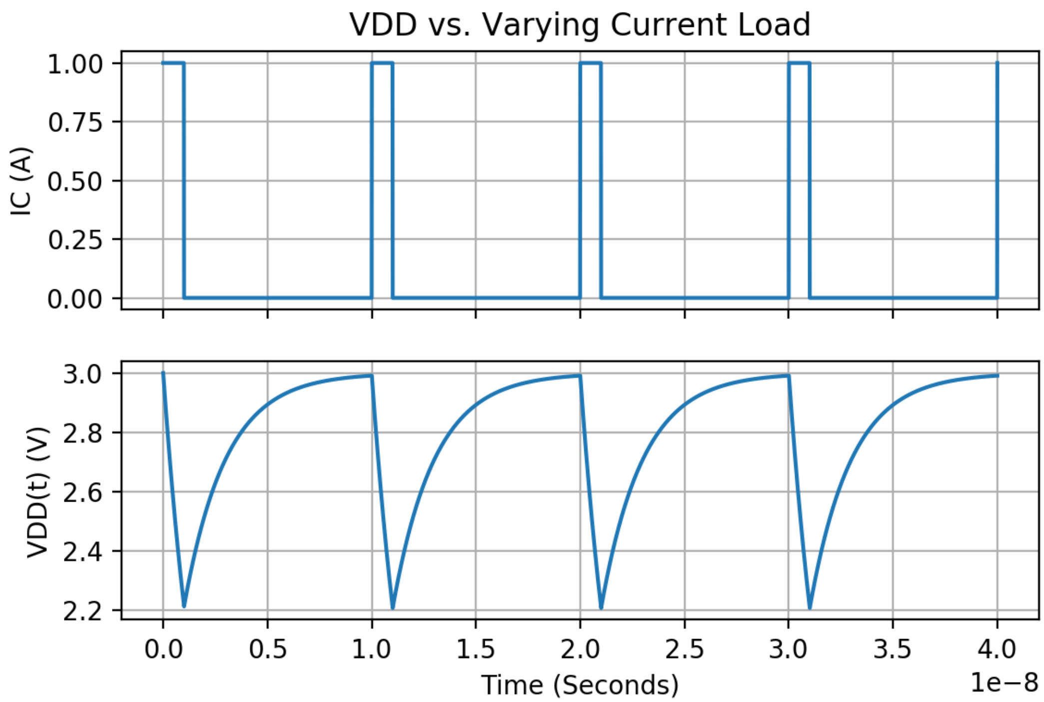
(c) Launch the attached Jupyter notebook to interact with a simulated version of this IC power supply. Try to simulate the scenarios outlined in the previous parts. For one of these scenarios, keep the RC time constant fixed, but vary the relative value of R vs. C (e.g. compare $R = 1, C = 2 \times 10^{-9}$ to the case where $R = 2, C = 1 \times 10^{-9}$. You may use scientific notations 2e-9 and 1e-9 in the iPython notebook). **Is it better to have a lower R or lower C value for a fixed RC time constant when attempting to minimize supply noise? Give an intuitive explanation for why this might be the case.**

Solution: A lower resistance and higher capacitance leads to smaller variation in the supply voltage with each current spike. One intuitive way to see this is to think about where the charge comes from whenenver the current source turns on. The charge comes from the capacitor and from the voltage source through the resistor. By $Q = CV$ for a constant amount of charge drawn, a larger capacitor results in lower voltage change. By $V = IR$ for a constant amount of current drawn through the resistor, a larger resistor leads to a larger voltage drop.

In the case where $R = 1$ and $C = 2e - 9$, we get the following plot:



In the case where $R = 2$ and $C = 1e-9$, we get the following plot:



Notice that the shape of the V_{DD} curves is the same because the RC constant is the same. However they drop to different voltages by the end of each pulse.

6. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) **What sources (if any) did you use as you worked through the homework?**

(b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

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