EECS 16B Designing Information Devices and Systems II
Spring 2021 Discussion Worksheet Discussion 4A

The relevant notes for this discussion is Note 3A.

1. Changing Coordinates and Systems of Differential Equations, II

In the previous discussion we analyzed and solved a pair of differential equations where the variables of interest were coupled.

$$\frac{d}{dt}z_1(t) = -5z_1(t) + 2z_2(t)$$
$$\frac{d}{dt}z_2(t) = 6z_1(t) - 6z_2(t).$$

We solved this system by using a coordinate transformation that gave us a decoupled system of equations. In the last discussion we were simply handed this transformation, but in this discussion we will construct the transformation for ourselves.

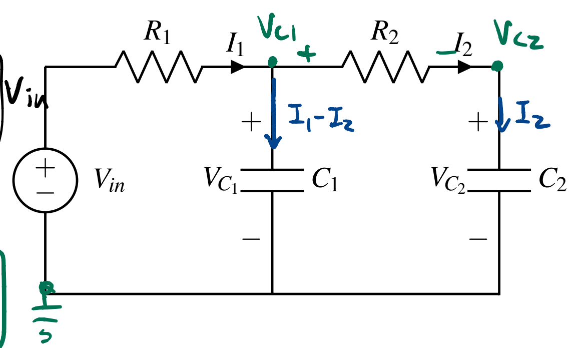
We will focus our explorations on the voltages across the capacitors in the following circuit.

a)* from (1) and (2)

$$\begin{bmatrix} \frac{d}{dt}V_{C_1}(t) \\ \frac{d}{dt}V_{C_2}(t) \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_1}\right) & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix}\begin{bmatrix} V_{C_1}(t) \\ V_{C_2}(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1C_1} \\ 0 \end{bmatrix} V_{in}$$

$$= \begin{bmatrix} -(3+2) & 2 \\ 6 & -6 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \frac{d}{dt}V_{C_1}(t) \\ \frac{d}{dt}V_{C_2}(t) \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} V_{C_1}(t) \\ V_{C_2}(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} V_{in}}$$



Part a): keep in mind we want equations of the form:

$$\frac{d}{dt}V_{C_1}(t) = \alpha_1 V_{C_1}(t) + \beta_1 V_{C_2}(t) + \gamma_1 V_{in}(t)$$
$$\frac{d}{dt}V_{C_2}(t) = \alpha_2 V_{C_2}(t) + \beta_2 V_{C_2}(t) + \gamma_2 V_{in}(t)$$

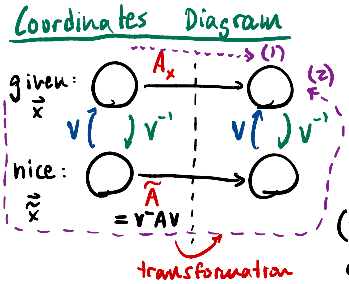
$V_{in} - V_{C_1} = I_1 R_1$

Figure 1: Two dimensional system: a circuit with two capacitors, like the one in lecture.

(a) Write the system of differential equations governing the voltages across the capacitors V_{C_1}, V_{C_2}. Use the following values: $C_1 = 1\mu F, C_2 = \frac{1}{3}\mu F, R_1 = \frac{1}{3}M\Omega, R_2 = \frac{1}{2}M\Omega$.

Motivation and Key Concepts

Coordinates Diagram



Paths
(1) hard !!
(2) easy :)

hard = coupled
easy = decoupled

(see 6/A notes for deriving form of $\tilde{A}$)

→ want to derive V (previously given)

→ similar to lec3B.

Ohm's Law + KCL + $I_C = C\frac{dV_C}{dt}$

$$\begin{cases} V_{C_2} = V_{C_1} - I_2 R_2 \\ I_2 = C_2 \frac{d}{dt}V_{C_2} \\ I_1 = I_2 + C_1\frac{d}{dt}V_{C_1} \\ V_{in} - I_1 R_1 = V_{C_1} \end{cases}$$

$$\begin{cases} I_1 = \frac{V_{in}}{R_1} - \frac{V_{C_1}}{R_1} \\ I_2 = \frac{V_{C_1}}{R_2} - \frac{V_{C_2}}{R_2} \end{cases}$$

$$\frac{d}{dt}V_{C_1} = \frac{I_1 - I_2}{C_1}$$

$$= \frac{1}{C_1}\left(\frac{V_{in}}{R_1} - \frac{V_{C_1}}{R_1} - \frac{V_{C_1}}{R_2} + \frac{V_{C_2}}{R_2}\right)$$

(1) $\frac{d}{dt}V_{C_1} = -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right)V_{C_1} + \left(\frac{1}{R_2C_1}\right)V_{C_2} + \frac{1}{R_1C_1}V_{in}$

(2) $\frac{d}{dt}V_{C_2} = \frac{1}{C_2}(I_2)$

$= \frac{1}{R_2C_2}V_{C_1} - \frac{1}{R_2C_2}V_{C_2} + 0V_{in}$

continued at a)*

Process: given circuit, form: $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + Bu(t)$

if A isn't nice (= decoupled):

→ change coords to basis where A is nice (by construction)

then solve in nice basis, and transform back.

(b) Suppose also that V_{in} was at 7 Volts for a long time, and then transitioned to be 0 Volts at time $t = 0$. This "new" system of differential equations (valid for $t \geq 0$)

no input term $V_{in}(t)$ for $t>0$! $[V_{in}(t \geq 0) = 0]$

$$\begin{cases} \frac{d}{dt}y_1(t) = -5y_1(t) + 2y_2(t) & (1) \\ \frac{d}{dt}y_2(t) = 6y_1(t) - 6y_2(t) & (2) \end{cases}$$

with initial conditions $y_1(0) = 7$ and $y_2(0) = 7$.

Write out the differential equations and initial conditions in matrix/vector form.

$$\begin{bmatrix} \frac{d}{dt}y_1(t) \\ \frac{d}{dt}y_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}}_{A_y} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

(c) Find the eigenvalues λ_1, λ_2 and eigenspaces for the matrix corresponding to the differential equation matrix above.

$A_y\vec{v} = \lambda\vec{v}$

$A_y\vec{v} - \lambda\vec{v} = 0$

$(A_y - \lambda I)\vec{v} = 0$

$\det(A - \lambda I) = 0$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow ad-bc = \det$ of 2×2

$\det\left(\begin{bmatrix} -5-\lambda & 2 \\ 6 & -6-\lambda \end{bmatrix}\right) = 0$

$(-5-\lambda)(-6-\lambda) - 2\cdot 6 = 0$

$30 + 11\lambda + \lambda^2 - 12 = 0$

$\lambda^2 + 11\lambda + 18 = 0$

$(\lambda+9)(\lambda+2) = 0$

$\Rightarrow \boxed{\lambda_1 = -9,\ \lambda_2 = -2}$

Null$(A+9I)$?

$\begin{bmatrix} -5+9 & 2 \\ 6 & -6+9 \end{bmatrix} \vec{v}_{\lambda_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\downarrow$

$4x+2y$

$4(-1)+2(2)=0$ ✓

$\rightarrow \begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix} \vec{v}_{\lambda_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\boxed{\vec{v}_{\lambda_1} = \alpha \begin{bmatrix} -1 \\ 2 \end{bmatrix}}$

Null$(A+2I)$?

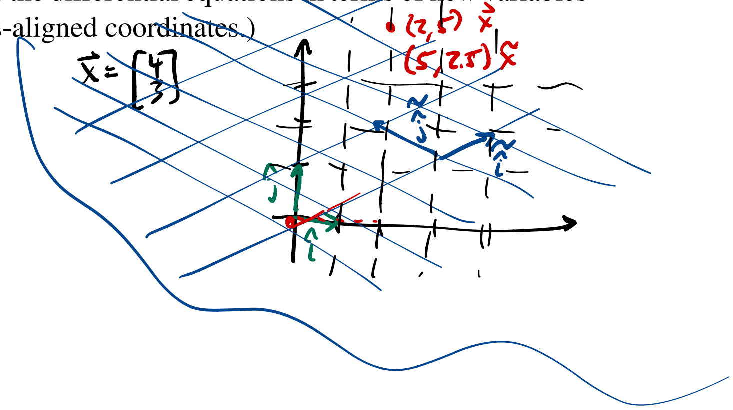
$\begin{bmatrix} -5+2 & 2 \\ 6 & -6+2 \end{bmatrix} \vec{v}_{\lambda_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -3 & 2 \\ 6 & -4 \end{bmatrix} \vec{v}_{\lambda_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \boxed{\vec{v}_{\lambda_2} = \beta \begin{bmatrix} 2 \\ 3 \end{bmatrix}}$

If the diagram is confusing, please don't worry (it was purely for some visual intuition). See the following for more:
→ 3Blue1Brown Lin. Alg. Matrix Vid. → Chapter 4B of the EECS16A Compendium

(d) Change coordinates into the eigenbasis to re-express the differential equations in terms of new variables $z_{\lambda_1}(t)$, $z_{\lambda_2}(t)$. (These variables represent eigenbasis-aligned coordinates.)



$$\underline{\vec{y} = \vec{v}_{\lambda_1} z_{\lambda_1} + \vec{v}_{\lambda_2} z_{\lambda_2}} \to \vec{y} = V\vec{z}$$

$$\vec{y} = \begin{bmatrix}-1 & 2\\ 2 & 3\end{bmatrix}\begin{bmatrix}z_{\lambda_1}\\ z_{\lambda_2}\end{bmatrix} \qquad \Rightarrow \vec{z} = V^{-1}\vec{y}$$

$$V = \begin{bmatrix}-1 & 2\\ 2 & 3\end{bmatrix} \Rightarrow V^{-1} = \begin{bmatrix}-3/7 & 2/7\\ 2/7 & 1/7\end{bmatrix}$$

$$(\tilde{A}) = A_{z_y} = V^{-1} A_y V \qquad A_y = \begin{bmatrix}-5 & 2\\ 6 & -6\end{bmatrix}$$

$$\begin{bmatrix}-9 & 0\\ 0 & -2\end{bmatrix}$$

$$\frac{d}{dt} z_{\lambda_1}(t) = -9 z_{\lambda_1}(t)$$
$$\frac{d}{dt} z_{\lambda_2}(t) = -2 z_{\lambda_2}(t)$$

$$\boxed{\begin{bmatrix}\frac{d}{dt} z_{\lambda_1}(t)\\ \frac{d}{dt} z_{\lambda_2}(t)\end{bmatrix} = \begin{bmatrix}-9 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}z_{\lambda_1}(t)\\ z_{\lambda_2}(t)\end{bmatrix}}$$

uncoupled!

(e) Solve the differential equation for $z_{\lambda_i}(t)$ in the eigenbasis.

$$z_{\lambda_1}(t) = c_1 e^{-9t}$$
$$z_{\lambda_2}(t) = c_2 e^{-2t}$$

$$\boxed{\vec{z}_\lambda(t) = \begin{bmatrix}-1 e^{-9t}\\ 3 e^{-2t}\end{bmatrix}}$$

$$\vec{z}_\lambda(0) = \begin{bmatrix}-3/7 & 2/7\\ 2/7 & 1/7\end{bmatrix}\begin{bmatrix}7\\ 7\end{bmatrix} = \begin{bmatrix}-1\\ 3\end{bmatrix}$$

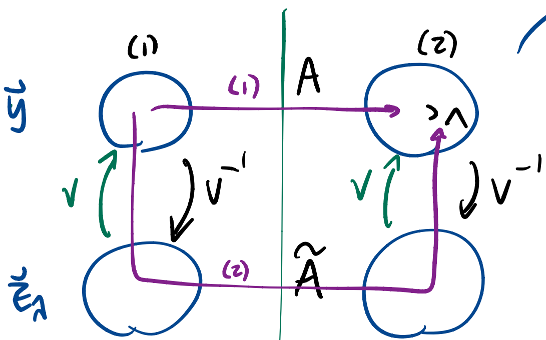
$$\vec{z}_\lambda(t) = V^{-1}\vec{y}(t) \to \vec{z}_\lambda(0) = V^{-1}\vec{y}(0)$$
$$\vec{y} = V\vec{z}_\lambda(t)$$

(f) Convert your solution back into the original coordinates to find $y_i(t)$.

$$\vec{y}(t) = V\vec{z}_\lambda(t) = \begin{bmatrix}-1 & 2\\ 2 & 3\end{bmatrix}\begin{bmatrix}-1e^{-9t}\\ 3e^{-2t}\end{bmatrix} = \boxed{\begin{bmatrix} e^{-9t}+6e^{-2t}\\ -2e^{-9t}+9e^{-2t}\end{bmatrix} = \vec{y}(t)}$$

Post-Disc Q/A:

remember: we apply matrix multiplications on the left.



$\tilde{A}$ is the transformation "equivalent" for the eigenbasis-aligned coords.

$$A = V\tilde{A}V^{-1}$$
$$V^{-1}A = \tilde{A}V^{-1}$$
$$V^{-1}AV = \tilde{A}$$

$$\tilde{A} = V^{-1}AV$$

why? $\rightarrow$

$$V\tilde{A}V^{-1}$$
$$= V(V^{-1}AV)V^{-1}$$
$$= \cancel{VV^{-1}}A\cancel{VV^{-1}}$$
$$= A$$

$$\boxed{\frac{d}{dt}\vec{y}(t) = A\,\vec{y}(t)}$$

$$\boxed{\frac{d}{dt}\vec{z}_\lambda(t) = \tilde{A}\,z_\lambda(t)}$$

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