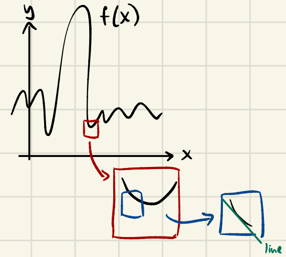
[Note 15]

Linearization: approximate nonlinear function around a point

→ key idea: functions, no matter how curved/nonlinear, look linear for a small region.

relative term

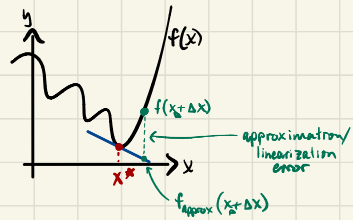
→ nonlinear modeling: very difficult (has applications in controls) [Prof. Murat Arcak's Research]



Key Results

1) Scalar 1D ($f(x): \mathbb{R} \to \mathbb{R}$)

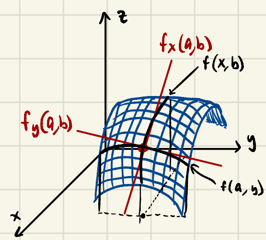
$$f(x) \approx f(x_*) + f'(x^*)(x - x^*)$$



2) Scalar, n-D ($f(x_1 \cdots x_n): \mathbb{R}^n \to \mathbb{R}$)

$$f(x,y) \approx f(x_*, y_*) + f_x(x_*, y_*)(x - x_*) + f_y(x_*, y_*)(y - y_*)$$

$\vdots$



lines $f_x(a,b)$ and $f_y(a,b)$ form a plane of approximation. (2 lines define a plane)

No thanks :)

3) Vector → Scalar ($f(\vec{x}, \vec{y}): \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$)

$$f(\vec{x}, \vec{y}) \approx f(\vec{x}_*, \vec{y}_*) + \overbrace{D_{\vec{x}} f}^{\left[\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n}\right]}\Big|_{(\vec{x}_*, \vec{y}_*)} \cdot (\vec{x} - \vec{x}_*) + \underbrace{D_{\vec{y}} f}_{\left[\frac{\partial f}{\partial y_1} \cdots \frac{\partial f}{\partial y_k}\right]}\Big|_{(x_*, y_*)} (\vec{y} - \vec{y}_*)$$

4) Vector → Vector ($\vec{f}(\vec{x}, \vec{y}): \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m$)

$$\vec{f}(\vec{x}, \vec{y}) = \vec{f}(\vec{x}_*, \vec{y}_*) + D_{\vec{x}} \vec{f}\Big|_{(\vec{x}_*, \vec{y}_*)} (\vec{x} - \vec{x}_*) + D_{\vec{y}} \vec{f}\Big|_{(\vec{x}_*, \vec{y}_*)} (\vec{y} - \vec{y}_*)$$

$$D_{\vec{x}} \vec{f} \to \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

EECS 16B Spring 2021 — Designing Information Devices and Systems II Discussion Worksheet — Discussion 13A

This discussion will recap a lot of the key concepts covered in lecture last week.

1. Linear Approximation

A common way to approximate a nonlinear function is to perform linearization near a point. In the case of a one-dimensional function $f(x)$, the linear approximation of $f(x)$ at a point $x_\star$ is given by

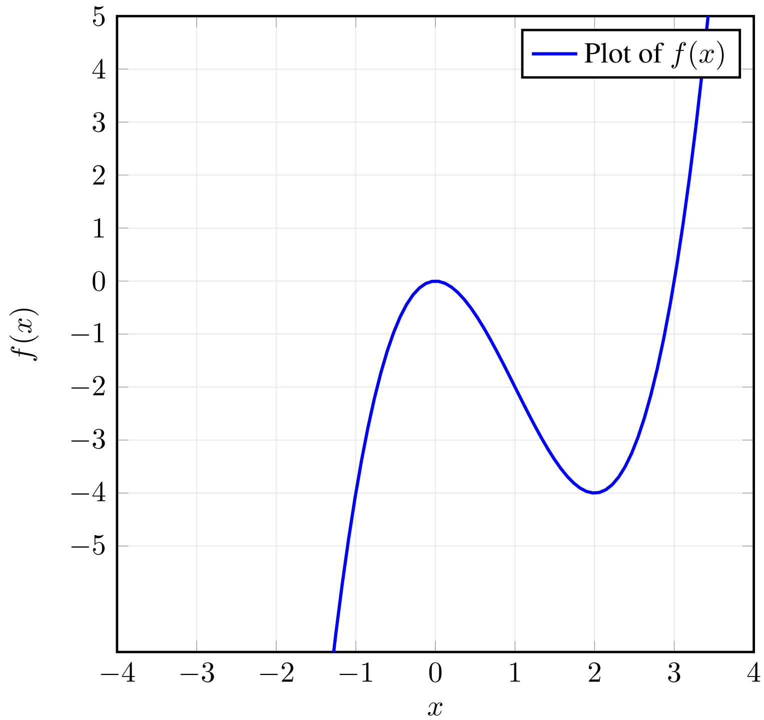
$$f(x) \approx f(x_\star) + f'(x_\star) \cdot (x - x_\star), \tag{1}$$

where $f'(x_\star) := \left.\frac{df(x)}{dx}\right|_{x=x_\star}$ is the derivative of $f(x)$ at $x = x_\star$.

Keep in mind that wherever we see $x_\star$, this denotes a *constant value* or operating point.

(a) Suppose we have the single-variable function $f(x) = x^3 - 3x^2$. We can plot the function $f(x)$ as follows:

$f'(x_\star) = 3x_\star^2 - 6x_\star$



i. Write the linear approximation of the function around an arbitrary point $x_\star$.

$f(x) \approx f(x_\star) + (3x_\star^2 - 6x_\star)(x - x_\star)$

ii. Use the expression above to linearize the function around the point $x = 1.5$. Draw the linearization into the plot of part i).

$f(x) \approx f(1.5) + (3 \cdot 1.5^2 - 6 \cdot 1.5)(x - 1.5)$

$f(x) \approx -3.375 - 2.25\,(x - 1.5)$

Now that we have this specific point's linearization, we understand how the function behaves around the point. Let's use this linearization to evaluate the function's approximation at $x = 1.7$ (based on our approximation at $x = 1.5$, we want to see how a $\delta = +0.2$ shift in the x value changes the corresponding $f(x)$ value). How does this approximation compare to the exact value of the function at $x = 1.7$?

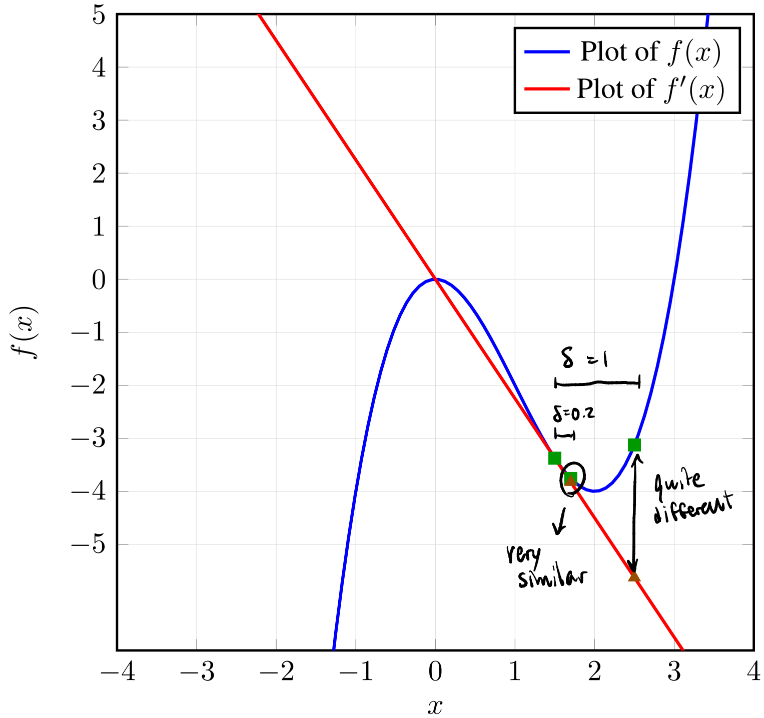
$$f(1.7) \approx -3.375 + (-2.25) \cdot (1.7 - 1.5) \tag{2}$$
$$\approx -3.375 - 0.45 \tag{3}$$
$$\approx -3.825 \tag{4}$$

Comparing to the exact value $f(1.7) = 1.7^3 - 3 \cdot 1.7^2 = -3.757$, we find that the difference is 0.068. Not too bad! What if we repeat with $\delta = 1$? To do so, we must use the approximation around $x = 1.5$ to compute $x = 2.5$, and compare to the exact value $f(2.5)$. How does our new approximation compare to the exact result?

$$f(2.5) \approx -3.375 + (-2.25) \cdot (2.5 - 1.5) \tag{5}$$
$$\approx -3.375 - 2.25 \tag{6}$$
$$\approx -5.625 \tag{7}$$

Comparing to the exact value $f(2.5) = 2.5^3 - 3 \cdot 2.5^2 = -3.125$, we find that the difference is much larger; the error jumped to 2.5! This is an error multiplication of $\frac{2.5}{0.068} \approx 37$, even though our δ only multiplied by 5. What happened?

Looking at the actual function, we see that the function has a significant curvature between our "anchor point" of $x_\star = 1.5$ and $x = 2.5$. Our linear model is unable to capture this curvature, and so we estimated $f(2.5)$ as if the function kept decreasing, as it did around $x = 1.5$ (where the slope was -2.25).



Now, we can extend this to higher dimensional functions. In the case of a two-dimensional function $f(x, y)$, the linear approximation of $f(x, y)$ at a point $(x_\star, y_\star)$ is given by

$$f(x,y) \approx f(x_\star, y_\star) + f_x(x_\star, y_\star) \cdot (x - x_\star) + f_y(x_\star, y_\star) \cdot (y - y_\star). \tag{8}$$

where $f_x(x_\star, y_\star)$ is the partial derivative of $f(x, y)$ with respect to x at the point $(x_\star, y_\star)$:

$$f_x(x_\star, y_\star) = \left.\frac{\partial f(x,y)}{\partial x}\right|_{(x_\star, y_\star)} \tag{9}$$

and $f_y(x_\star, y_\star)$ is the partial derivative of $f(x, y)$ with respect to y at the point $(x_\star, y_\star)$.

(b) Now, let's see how we can derive partial derivatives. When we are given a function $f(x, y)$, we calculate the partial derivative of f with respect to x by fixing y and taking the derivative with respect to x. Given the function $f(x, y) = x^2 y$, find the partial derivatives $f_y(x, y)$ and $f_x(x, y)$.

$f_x(x,y) = 2xy$
$f_y(x,y) = x^2$

(c) Write out the linear approximation of f near $(x_\star, y_\star)$.

$f(x,y) \simeq f(x_\star, y_\star) + 2x_\star y_\star (x - x_\star)$
$+ x_\star^2 (y - y_\star)$

Optional $\boxed{\text{Extra}}$ Practice for Partial Derivatives:

1) $f(x,y,z) = x^2 y + y^2 z + 3xyz$
$f_x = 2xy + 3yz$
$f_y = x^2 + 2yz + 3xz$
$f_z = y^2 + 3xy$

2) $f(x_1 \cdots x_n) = x_1 + x_1 x_2 + \cdots \prod_{i=1}^{n} x_i$
$f_{x_n} = x_1 x_2 \cdots x_{n-1} = \prod_{i=1}^{n-1} x_i$
$f_{x_{n-1}} = \underbrace{x_1 x_2 \cdots x_{n-2} x_n}_{\text{last term}} + \underbrace{x_1 x_2 \cdots x_{n-2}}_{\text{2nd}\to\text{last term}}$
$f_{x_2} = x_1 + x_1 x_3 + x_1 x_3 x_4 \cdots$
$f_{x_1} = 1 + x_2 + x_2 x_3 + \cdots$

(d) We want to see if the approximation arising from linearization of this function is reasonable for a point close to our point of evaluation. First, approximate $f(x, y)$ at the point $(2.01, 3.01)$ using $(x_\star, y_\star) = (2, 3)$, and compare the result to $f(2.01, 3.01)$.

$f(x,y) = x^2 y$

$f(2.01, 3.01)$

$2.01 = 2 + \delta_x$, $\delta_x = 0.01$

$3.01 = 3 + \delta_y$, $\delta_y = 0.01$

$\delta = \delta_x + \delta_y$

$(2+\delta_x)^2(3+\delta_y)$

exact value: $= 12 + 16\delta + 7\delta^2 + \delta^3 \Longrightarrow \boxed{12.160701}$

$\delta \quad \delta^2 \quad \delta^3$

approx value: $f(2.01, 3.01) = f(2,3) + 2 \cdot x_\star y_\star \overbrace{(x - x_\star)}^{\delta} + x_\star^2 \overbrace{(y - y_\star)}^{\delta}$

$= f(2,3) + 2 \cdot 2 \cdot 3 \cdot \delta + 2^2 \cdot \delta$

$= 12 + 12\delta + 4\delta$

$= 12 + 16\delta = \boxed{12.16}$

δ^2, δ^3, etc not captured by our linear model!

(e) Suppose we have now a vector-valued function $f(\vec{x},\vec{y})$, which takes in vectors $\vec{x} \in \mathbb{R}^n$, $\vec{y} \in \mathbb{R}^k$ and outputs a scalar $\in \mathbb{R}$. That is, $f(\vec{x},\vec{y})$ is $\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$. With this new model, how can we adapt our previous linearization method?

One way to linearize the function f is to do it for every single element in $\vec{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^\top$ and $\vec{y} = \begin{bmatrix} y_1 & y_2 & \dots & y_k \end{bmatrix}^\top$. Then, when we are looking at x_i or y_j, we fix everything else as constant. This would give us the linear approximation

$$f(\vec{x},\vec{y}) \approx f(\vec{x}_\star,\vec{y}_\star) + \sum_{i=1}^{n} \frac{\partial f(\vec{x},\vec{y})}{\partial x_i}(x_i - x_{\star,i}) + \sum_{j=1}^{k} \frac{\partial f(\vec{x},\vec{y})}{\partial y_j}(y_j - y_{\star,j}). \tag{10}$$

In order to simplify this equation, we can define the rows $D_{\vec{x}}$ and $D_{\vec{y}}$ as

$$D_{\vec{x}} f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \dots & \frac{\partial f}{\partial x_n} \end{bmatrix}, \quad \text{derivative /row} \tag{11}$$

$$D_{\vec{y}} f = \begin{bmatrix} \frac{\partial f}{\partial y_1} & \dots & \frac{\partial f}{\partial y_k} \end{bmatrix}. \tag{12}$$

Then, Equation (10) can be rewritten as

$$f(\vec{x},\vec{y}) \approx f(\vec{x}_\star,\vec{y}_\star) + \overbrace{\underbrace{(D_{\vec{x}} f)\Big|_{(\vec{x}_\star,\vec{y}_\star)}}_{1\times n} \cdot \underbrace{(\vec{x} - \vec{x}_\star)}_{n\times 1}}^{1\times 1} + \underbrace{(D_{\vec{y}} f)\Big|_{(\vec{x}_\star,\vec{y}_\star)}}_{1\times k} \cdot \underbrace{(\vec{y} - \vec{y}_\star)}_{k\times 1}. \tag{13}$$

Assume that $\boxed{n = k}$ and we define the function $f(\vec{x},\vec{y}) = \vec{x}^\top \vec{y} = \sum_{i=1}^{k} x_i y_i$. Find $D_{\vec{x}} f$ and $D_{\vec{y}} f$.

$$f(\vec{x},\vec{y}) = \vec{x}^T \vec{y}$$

$$= x_1 y_1 + x_2 y_2 + \cdots x_n y_n$$

$$D_{\vec{x}} f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}$$

$$= \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} = \vec{y}^T$$

$$D_{\vec{y}} f = \begin{bmatrix} \frac{\partial f}{\partial y_1} & \cdots & \frac{\partial f}{\partial y_n} \end{bmatrix} = \vec{x}^T$$

$$\boxed{\begin{aligned} f(\vec{x},\vec{y}) = f(\vec{x}_\star,\vec{y}_\star) &+ \vec{y}_\star^T (\vec{x} - \vec{x}_\star) \\ &+ \vec{x}_\star^T (\vec{y} - \vec{y}_\star) \end{aligned}}$$

(f) Following the above part, find the linear approximation of $f(\vec{x}, \vec{y})$ near $\vec{x}_\star = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{y}_\star = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Recall that $f(\vec{x}, \vec{y}) = \vec{x}^\top \vec{y} = \sum_{i=1}^{k} x_i y_i$.

$$\vec{x}_\star = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{y}_\star = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

approx:
$$f\left(\begin{bmatrix} 1+\delta_1 \\ 2+\delta_2 \end{bmatrix}, \begin{bmatrix} -1+\delta_3 \\ 2+\delta_4 \end{bmatrix}\right) \approx \vec{x}_\star^\top \vec{y}_\star + \vec{y}_\star^\top (\vec{x} - \vec{x}_\star) + \vec{x}_\star^\top (\vec{y} - \vec{y}_\star)$$

$$(\vec{x} - \vec{x}_\star) = \begin{bmatrix} 1+\delta_1 \\ 2+\delta_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

$$= 3 + \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} \delta_3 \\ \delta_4 \end{bmatrix}$$

$$= \boxed{3 - \delta_1 + 2\delta_2 + \delta_3 + 2\delta_4}$$

exact:
$$f\left(\begin{bmatrix} 1+\delta_1 \\ 2+\delta_2 \end{bmatrix}, \begin{bmatrix} -1+\delta_3 \\ 2+\delta_4 \end{bmatrix}\right) = (1+\delta_1)(-1+\delta_3) + (2+\delta_2)(2+\delta_4)$$

$$= \boxed{3 - \delta_1 + 2\delta_2 + \delta_3 + 2\delta_4 + \underline{\delta_1\delta_3 + \delta_2\delta_4}}$$

$\delta_1\delta_3$, $\delta_2\delta_4$ etc not captured by our linear model!

(g) When the function $\vec{f}(\vec{x},\vec{y}) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m$ takes in vectors and outputs a vector (rather than a scalar), we can view each dimension in $\vec{f}$ independently as a separate function f_i, and linearize each of them:

$$\vec{f}(\vec{x},\vec{y}) = \begin{bmatrix} f_1(\vec{x},\vec{y}) \\ f_2(\vec{x},\vec{y}) \\ \vdots \\ f_m(\vec{x},\vec{y}) \end{bmatrix} \approx \begin{bmatrix} f_1(\vec{x}_\star,\vec{y}_\star) + D_{\vec{x}} f_1 \cdot (\vec{x}-\vec{x}_\star) + D_{\vec{y}} f_1 \cdot (\vec{y}-\vec{y}_\star) \\ f_2(\vec{x}_\star,\vec{y}_\star) + D_{\vec{x}} f_2 \cdot (\vec{x}-\vec{x}_\star) + D_{\vec{y}} f_2 \cdot (\vec{y}-\vec{y}_\star) \\ \vdots \\ f_m(\vec{x}_\star,\vec{y}_\star) + D_{\vec{x}} f_m \cdot (\vec{x}-\vec{x}_\star) + D_{\vec{y}} f_m \cdot (\vec{y}-\vec{y}_\star) \end{bmatrix} \tag{14}$$

We can rewrite this in a clean way with the *Jacobian*:

$$D_{\vec{x}}\vec{f} = \begin{bmatrix} D_{\vec{x}} f_1 \\ D_{\vec{x}} f_2 \\ \vdots \\ D_{\vec{x}} f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}, \tag{15}$$

and similarly

stack of scalar-output cases

$$D_{\vec{y}}\vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_k} \end{bmatrix}. \tag{16}$$

$\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$

$\to \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m = \left.\begin{bmatrix} \mathbb{R} \\ \mathbb{R} \\ \vdots \\ \mathbb{R} \end{bmatrix}\right\} m$

Then, the linearization becomes

$$\vec{f}(\vec{x},\vec{y}) \approx \vec{f}(\vec{x}_\star,\vec{y}_\star) + (D_{\vec{x}}\vec{f})\Big|_{(\vec{x}_\star,\vec{y}_\star)} \cdot (\vec{x}-\vec{x}_\star) + \underbrace{(D_{\vec{y}}\vec{f})\Big|_{(\vec{x}_\star,\vec{y}_\star)} \cdot (\vec{y}-\vec{y}_\star)}. \tag{17}$$

not present in example question.

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{f}(\vec{x}) = \begin{bmatrix} x_1^2 x_2 \\ x_1 x_2^2 \end{bmatrix}$. Find $D_{\vec{x}}\vec{f}$, applying the definition above.

f_1

f_2

$$D_{\vec{x}} f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 x_2 & x_1^2 \\ x_2^2 & 2x_1 x_2 \end{bmatrix}$$

(h) Compare the approximation of $\vec{f}$ at the point $\begin{bmatrix}2.01\\3.01\end{bmatrix}$ using $\vec{x}_\star = \begin{bmatrix}2\\3\end{bmatrix}$ versus $\vec{f}\left(\begin{bmatrix}2.01\\3.01\end{bmatrix}\right)$. Recall the definition that $\vec{f}(\vec{x}) = \begin{bmatrix}x_1^2x_2\\x_1x_2^2\end{bmatrix}$.

$\delta_1 = \delta_2 = 0.01$

exact:

$$\vec{f}\left(\begin{bmatrix}2.01\\3.01\end{bmatrix}\right) = \begin{bmatrix}(2+\delta)^2(3+\delta)\\(2+\delta)(3+\delta)^2\end{bmatrix} = \begin{bmatrix}12 + 16\delta + 7\delta^2 + \delta^3\\18 + 21\delta + \underbrace{8\delta^2 + \delta^3}_{\times}\end{bmatrix} = \boxed{\begin{matrix}12.160701\\18.210801\end{matrix}}$$

close!

approx:

$$\vec{f}\left(\begin{bmatrix}2.01\\3.01\end{bmatrix}\right) \approx \vec{f}\begin{bmatrix}2\\3\end{bmatrix} + \begin{bmatrix}12 & 4\\9 & 12\end{bmatrix}\begin{bmatrix}\delta\\\delta\end{bmatrix} = \begin{bmatrix}12+16\delta\\18+21\delta\end{bmatrix} = \boxed{\begin{matrix}12.16\\18.21\end{matrix}}$$

$$D_{\vec{x}}\vec{f}\Big|_{(2,3)} = \begin{bmatrix}2x_1x_2 & x_1^2\\x_2^2 & 2x_1x_2\end{bmatrix} = \begin{bmatrix}12 & 4\\9 & 12\end{bmatrix}$$

(i) **Practice:** Let $\vec{x}$ and $\vec{y}$ be vectors with 2 rows, and let $\vec{w}$ be another vector with 2 rows. Let $\vec{f}(\vec{x},\vec{y}) = \vec{x}\vec{y}^\top\vec{w}$. Find $D_{\vec{x}}\vec{f}$ and $D_{\vec{y}}\vec{f}$.

(j) **Practice:** Continuing the above part, find the linear approximation of $\vec{f}$ near $\vec{x} = \vec{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and with $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

These linearizations are important for us because we can do many easy computations using linear functions.

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