Key Concepts + Review of SVD Structure.

Full SVD of a general matrix $\underset{m\times n}{A}$:

$$\underset{[m\times n]}{A} = \underset{[m\times m]}{U}\,\underset{[m\times n]}{\Sigma}\,\underset{[n\times n]}{V^T}$$

rank r

$r \le \min(n,m)$

$\Sigma_{A,\text{wide}}$ $(m\le n)$: $\left[\begin{array}{ccc|c} \sigma_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_n & \end{array}\right]$

(σ_i could be zero)

$\Sigma_{A,\text{tall}}$ $(m\ge n)$: $\left[\begin{array}{ccc} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \\ \hline & 0 & \end{array}\right]$

$[\sigma_1 \ge \sigma_2 \cdots \sigma_r > 0]$

$r\times r$ square sublock of nonzero σ values.

$U_n \equiv U_{m-r}$

$$= m\underbrace{\begin{bmatrix} \underbrace{U_r}_{r} & \underbrace{U_n}_{m-r} \end{bmatrix}}_{m} \; m\underbrace{\left[\begin{array}{ccc|c} \sigma_1 & 0\;0 & & \\ 0 & \ddots & & 0 \\ 0 & & \sigma_r & \\ \hline & 0 & 0 & \end{array}\right]}_{\underbrace{\;r\;\;\; m-r\;}_{m}\;\; n-m} \; n\begin{bmatrix} V_r^T \\ V_n^T \end{bmatrix}\begin{array}{l} \}\,r \\ \}\,n-r \end{array}$$

assume $\sigma_i \neq 0$

(else, fit into 0 block)

$V_n^T \equiv V_{m-r}^T$

orthonormal $\rightarrow U$, Σ, V^T

$U_r \approx U_{\text{rank}}$ [non zero σ]

$U_n \approx U_{\text{null}}$ [zero σ]

most general structure is above.

Properties [Note 13, Section 5]

1) columnspace of U_r = col space of A | $\text{col}(U_R) = \text{col}(A)$ ☆
2) columnspace of U_n $\perp$ colspace of A | $\text{col}(U_n) \perp \text{col}(A)$

   (U_n: remaining $m-r$ columns)

3) columnspace of V_R = rowspace of V_R^T $\perp$ nullspace of A | $\text{col}(V_r) \perp \text{null}(A)$
4) columnspace of V_n = nullspace of A. | $\text{col}(V_n) = \text{null}(A)$ ☆

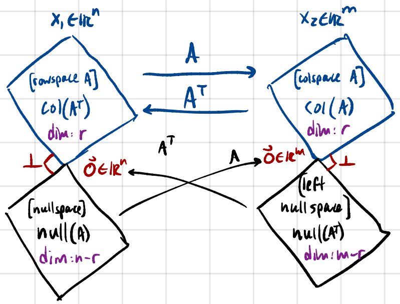
Geometric Interpretation (rotate, scale, rotate): $A\vec{x} = U\left(\Sigma\left(V^T\vec{x}\right)\right)$ [dis 12B]

EXTRA [not in scope]: Fundamental Theorem of Linear Algebra.

$A \in \mathbb{R}^{m\times n}$

rank$(A) = r$

$\mathbb{R}^n \longleftrightarrow \mathbb{R}^m$



EECS 16B Designing Information Devices and Systems II
Spring 2021 Discussion Worksheet **Discussion 12A**

In this discussion, we practice computing the SVD for a "wide" matrix (more columns than rows) and for a "tall" matrix (more rows than columns). There is also an associated jupyter notebook on Datahub that will serve useful to confirm the numerical calculations (specifically for performing Gram-Schmidt).

Also, note that the techniques and insights communicated in this discussion are conveyed in $\boxed{\text{Note 13, sec. 3}}$.

1. Computing the SVD: A "Tall" Matrix Example

Define the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}.$$

(a) In this part, we will find the full SVD of A in steps.

(i) Compute $A^\top A$ and find its eigenvalues.

$$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \boxed{\begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix} = A^TA}$$

$$\det\left(\begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix} - \lambda I\right) = 0$$

$$= \det\left(\begin{bmatrix} 9-\lambda & -9 \\ -9 & 9-\lambda \end{bmatrix}\right) = (9-\lambda)^2 - (-9\cdot -9)$$

$$\Rightarrow (\lambda - 9)^2 - 81 = 0$$

$$\lambda^2 - 18\lambda = 0$$

$$\Rightarrow \boxed{\begin{matrix}\lambda_1 = 18 \\ \lambda_2 = 0\end{matrix}}$$

(ii) Find orthonormal eigenvectors $\vec{v}_i$ (right singular vectors, columns of V).

$$A^TA - \lambda_1 I = \begin{bmatrix} -9 & -9 \\ -9 & -9 \end{bmatrix}$$

$$\Rightarrow \text{null}(A^TA - \lambda_1 I)$$

$$= \text{null}\left(\begin{bmatrix} -9 & -9 \\ -9 & -9 \end{bmatrix}\right)$$

$$\Rightarrow -9x_1 - 9x_2 = 0$$

$$\Rightarrow x_1 = -x_2$$

$$A^TA - \lambda_2 I = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

$$\Rightarrow \boxed{\vec{v}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}$$

(iii) Find singular values, $\sigma_i = \sqrt{\lambda_i}$.

$$\lambda_1 = 18 \quad \lambda_2 = 0 \Rightarrow \sigma_1 = \sqrt{\lambda_1} = \sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}, \quad \sigma_2 = \sqrt{0} = 0$$

$$\sigma_1 = 3\sqrt{2} \quad \sigma_2 = 0$$

(iv) Use $\vec{v}_i$ to find orthonormal $\vec{u}_i$ (for nonzero σ). [Derived in dis 11B, Q1d]

want to use $\vec{v}_i$ to find $\vec{u}_i$

$$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1$$

$$= \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix} = \vec{u}_1$$

$$\left[\vec{u}_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right] \rightarrow$$

Gram-Schmidt (first extension)

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix} \right) \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(-\frac{1}{3}\right) \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/9 \\ -2/9 \\ 2/9 \end{bmatrix}$$

$$= \begin{bmatrix} 8/9 \\ 2/9 \\ -2/9 \end{bmatrix} \xrightarrow{\text{normalize}} \begin{bmatrix} \sqrt{8}/3 \\ 2/(3\sqrt{8}) \\ -2/(3\sqrt{8}) \end{bmatrix}$$

$\hookrightarrow$ norm $= \frac{\sqrt{8}}{3}$

$$= \begin{bmatrix} \sqrt{8}/3 \\ 1/(3\sqrt{2}) \\ -1/(3\sqrt{2}) \end{bmatrix}$$

$$u_2 = \begin{bmatrix} \sqrt{8}/3 \\ 1/(3\sqrt{2}) \\ -1/(3\sqrt{2}) \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

(v) Use the previous parts to write the full SVD of A.

$$A = \underbrace{\begin{bmatrix} -1/3 & \sqrt{8}/3 & 0 \\ 2/3 & 1/(3\sqrt{2}) & 1/\sqrt{2} \\ -2/3 & -1/(3\sqrt{2}) & 1/\sqrt{2} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{V^T} = \text{full SVD}$$

(vi) Use the Jupyter notebook to run the code cell that calls `numpy.linalg.svd` on A. What is the result? Does it match our result above?

The result does not match; Gram-Schmidt is not unique.

(b) Find the rank of A.

\# nonzero σ_i $= 1$

$\boxed{\text{rank}(A) = 1}$

(c) Find a basis for the range (or column space) of A.

$\text{col}(A) = \text{col}(U_r)$

($\Leftarrow$ first column of U)

$\text{col}(A) = \text{span}(\vec{u}_1) = \boxed{\text{span}\left\{\begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}\right\}}$

(d) Find a basis for the null space of A.

$\text{col}(V_n) = \text{null}(A)$

(second row of V^T) $= \boxed{\text{span}\left\{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\right\}}$

(e) We now want to create the SVD of $A^\top$. Rather than repeating all of the steps in the algorithm, feel free to use the jupyter notebook for this subpart (which defines a `numpy.linalg.svd` command). What are the relationships between the matrices composing A and the matrices composing $A^\top$?

$A = U \Sigma V^T$

$A^T = (U \Sigma V^T)^T = V \Sigma^T U^T$

$A^T = U_* \Sigma_* V_*^T$

$U_* = V$

$\Sigma_* = \Sigma^T$

$V_*^T = U^T \Rightarrow V_* = U$

$$\boxed{A^T = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 & -2/3 \\ \sqrt{8}/3 & 1/3\sqrt{2} & 1/3\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}$$

2. Computing the SVD: A "Wide" Matrix Example

Define the matrix

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}.$$

(a) In this part, we will find the full SVD of A in steps.

(i) Compute $AA^\top$ and find its eigenvalues.

$$\boxed{AA^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}}$$

$$(17-\lambda)^2 - 64 = 0$$

$$\lambda^2 + 34\lambda - 64 + 17^2 = 0$$

$$\boxed{\begin{aligned} \lambda_1 &= 25 \\ \lambda_2 &= 9 \end{aligned}}$$

(ii) Find orthonormal eigenvectors $\vec{u}_i$ (left singular vectors, columns of U). Feel free to use the associated Jupyter notebook to perform Gram-Schmidt for this part, if needed.

$$\text{null}(AA^T - \lambda_1 I) = \text{null}\left(\begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix}\right) = \vec{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\text{null}(AA^T - \lambda_2 I) = \text{null}\left(\begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}\right) = \vec{u}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

(iii) Find the singular values, $\sigma_i = \sqrt{\lambda_i}$.

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{25} = 5$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{9} = 3$$

(iv) Use $\vec{u}_i$ to find orthonormal $\vec{v}_i$ (for nonzero σ). Feel free to use the associated Jupyter notebook to perform Gram-Schmidt for this part, if needed. [dB llB Gl c/d]

$$A = U \Sigma V^T$$

$$2\times3 = 2\times2 \quad 2\times3 \quad \underset{r \times c}{3\times3}$$

$$\vec{v}_i = \frac{1}{\sigma_i} A^T \vec{u}_i$$

$$\vec{v}_1 = \frac{1}{5}\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1/\sqrt{18} \\ -1/\sqrt{18} \\ 4/\sqrt{18} \end{bmatrix}$$

Gram-Schmidt

$$\vec{v}_3 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

(v) Use the previous parts to write the full SVD of A.

$$\boxed{A = \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & -1/\sqrt{18} & 4/\sqrt{18} \\ -2/3 & 2/3 & 1/3 \end{bmatrix}}_{V^T}}$$

(b) Find the rank of A, using the computed full SVD.

rank = # nonzero σ

$\boxed{\text{rank}(A) = 2}$

(c) Find a basis for the range (or columnspace) of A.

$\text{range}(A) = \text{span}\{\vec{u}_1, \vec{u}_2\}$

$$\boxed{\text{range}(A) = \text{span}\left\{\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\right\} = \mathbb{R}^2}$$

(d) Find a basis for the nullspace of A.

$\text{null}(A) = \text{span}\{\vec{v}_3\}$

$$\boxed{\text{null}(A) = \text{span}\left\{\begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}\right\}}$$

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