Discussion 4B: Change of Basis, Inductors

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OH/HW Party: Tuesday 4-6pm
Worksheet: https://eecs16b.org/discussion/dis04B.pdf
Notes: https://tinyurl.com/justin16bnotes

Today:

1. Quick recap - diagonalizing a matrix system of differential equations
2. Mini lecture: change of basis, eigenbasis + diagonalization
3. Problems 1, 2 (lecture style)
4. Small break
5. Problems 3, 4

Quick Recap

Original Problem

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$$

Change of Coordinates Problem

$$\frac{d}{dt}\vec{\tilde{x}}(t) = V^{-1}AV\vec{\tilde{x}}(t) = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \vec{\tilde{x}}(t)$$

$$\vec{\tilde{x}}(0) = V^{-1}\vec{x}(0)$$

Change of Coordinates Solution

$$\tilde{x}_1(t) = \tilde{x}_1(0)e^{\lambda_1 t}$$

$$\vdots$$

$$\tilde{x}_n(t) = \tilde{x}_n(0)e^{\lambda_n t}$$

Original Solution

$$\vec{x}(t) = V\vec{\tilde{x}}(t)$$

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$$

$$\frac{d}{dt}\vec{x}(t) = V\Lambda V^{-1}\vec{x}(t)$$

Plug in diagonalization of A

$$V^{-1}\frac{d}{dt}\vec{x}(t) = \Lambda(V^{-1}\vec{x}(t))$$

Left-multiply both sides with V^{-1}

$$\frac{d}{dt}(V^{-1}\vec{x}(t)) = \Lambda(V^{-1}\vec{x}(t))$$

Derivative is a linear operator

$$\frac{d}{dt}\vec{\tilde{x}}(t) = \Lambda\vec{\tilde{x}}(t) = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \vec{\tilde{x}}(t)$$

New variable representing coordinates in the eigenbasis

$$\vec{\tilde{x}} = V^{-1}\vec{x}$$

Question:

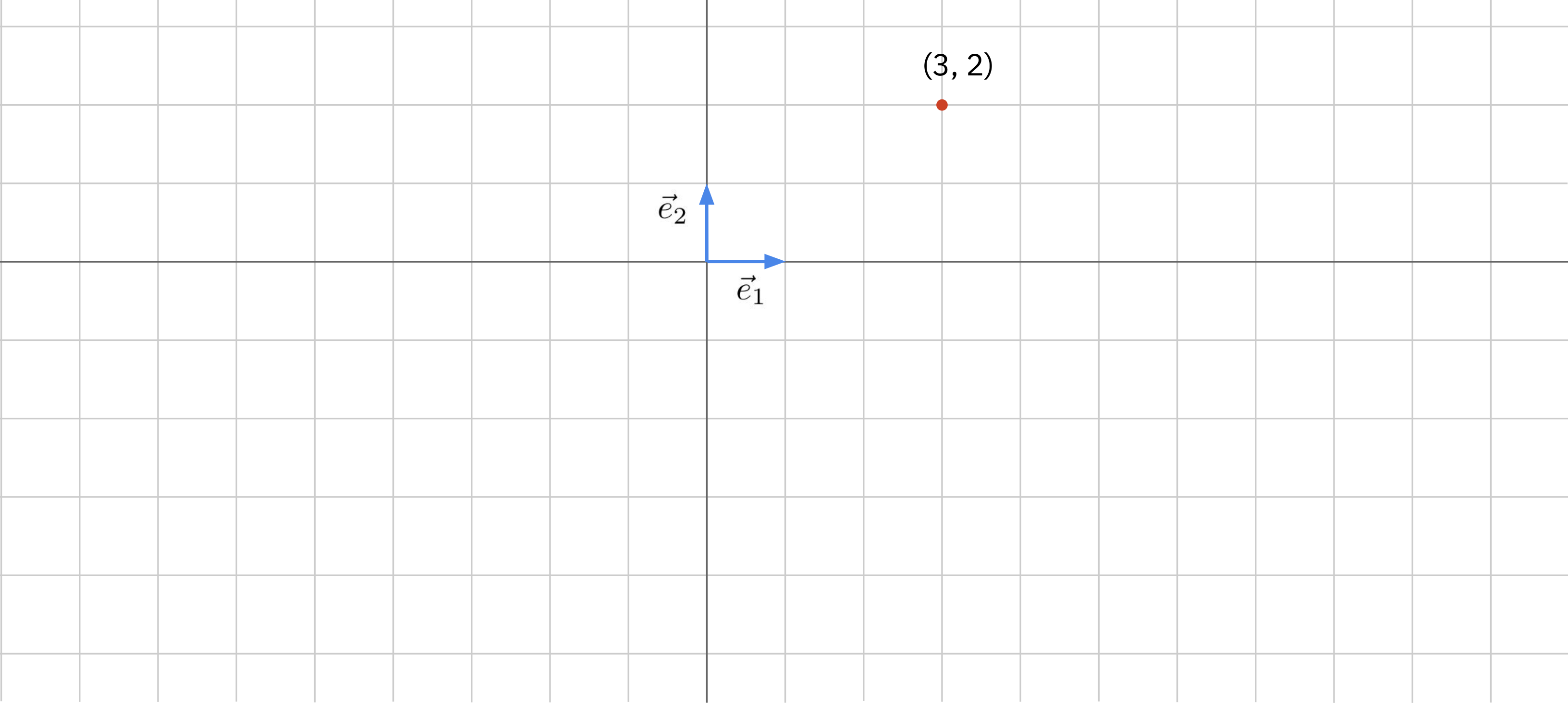
$A = V \Lambda V^{-1}$

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b} \longrightarrow \frac{d}{dt}\vec{\tilde{x}}(t) = \Lambda\vec{\tilde{x}}(t) + \vec{\tilde{b}}$$

$\vec{\tilde{b}} = V^{-1}\vec{b}$

$\frac{d}{dt}\tilde{x}_1 = \lambda_1 \tilde{x}_1 + \tilde{b}_1$

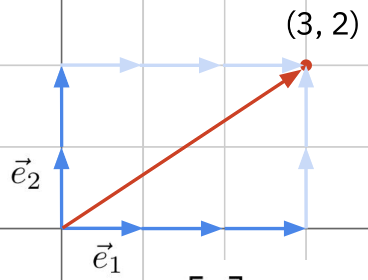
Change of Basis



Change of Basis

$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

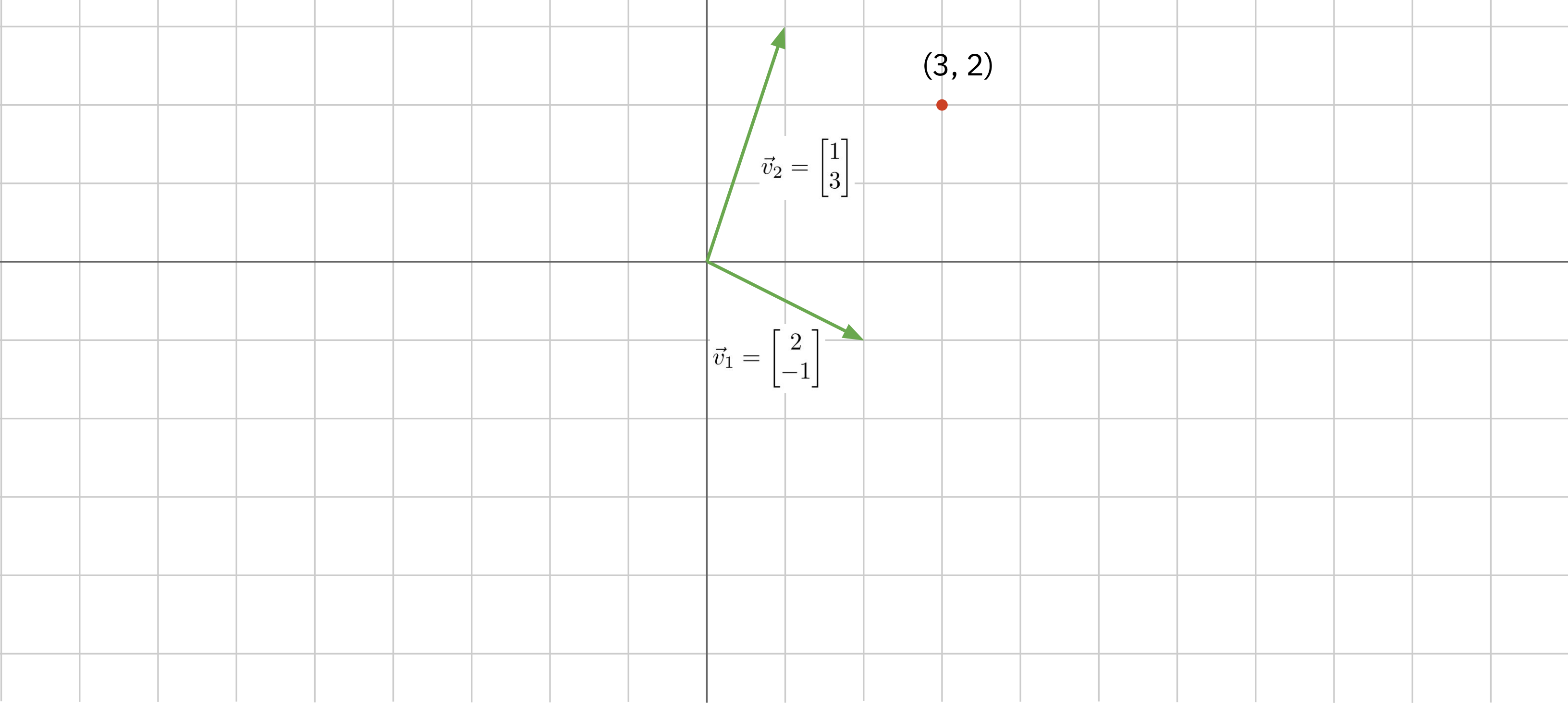
$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



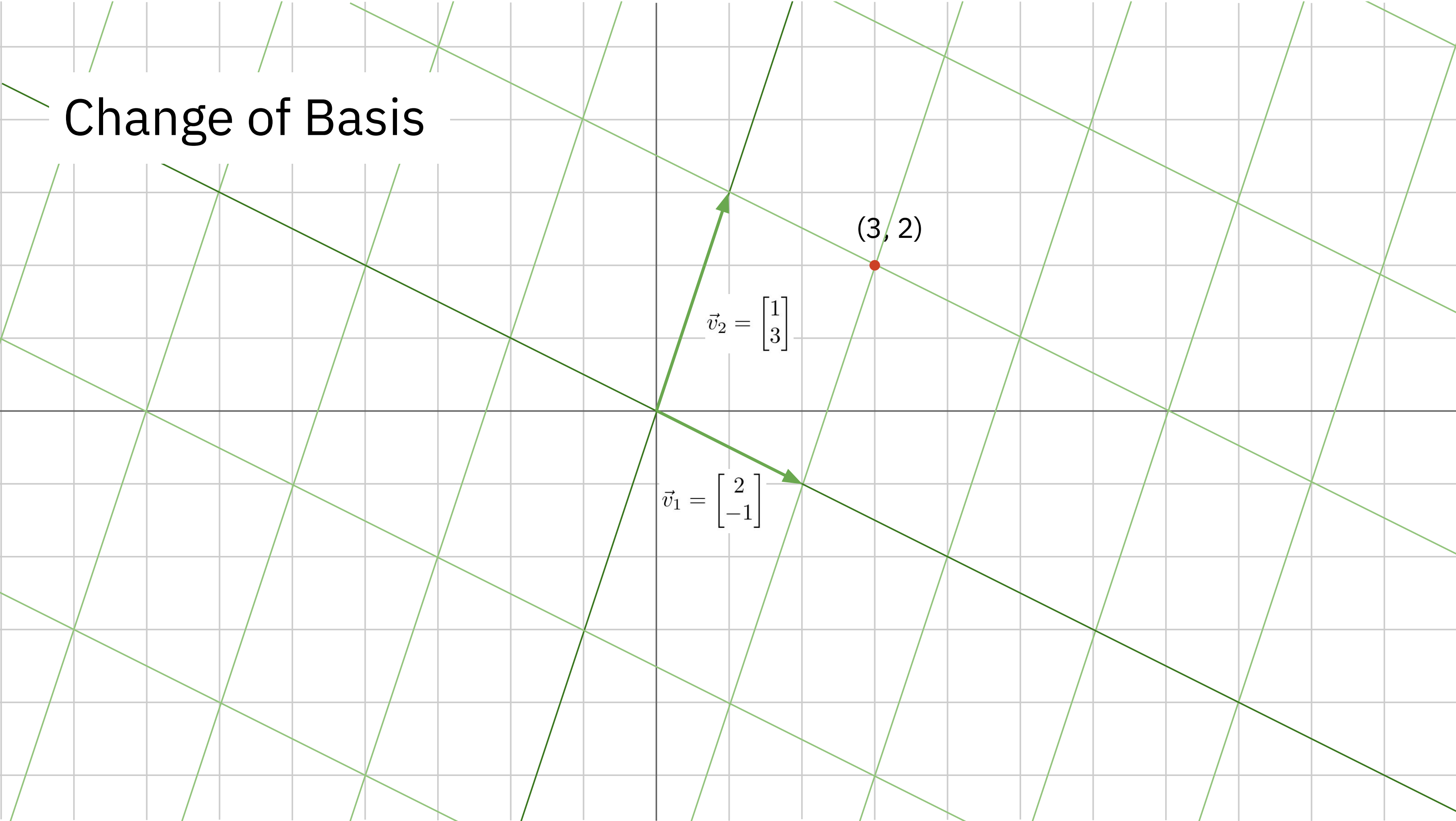
$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3\vec{e}_1 + 2\vec{e}_2$$

Coordinates in standard basis

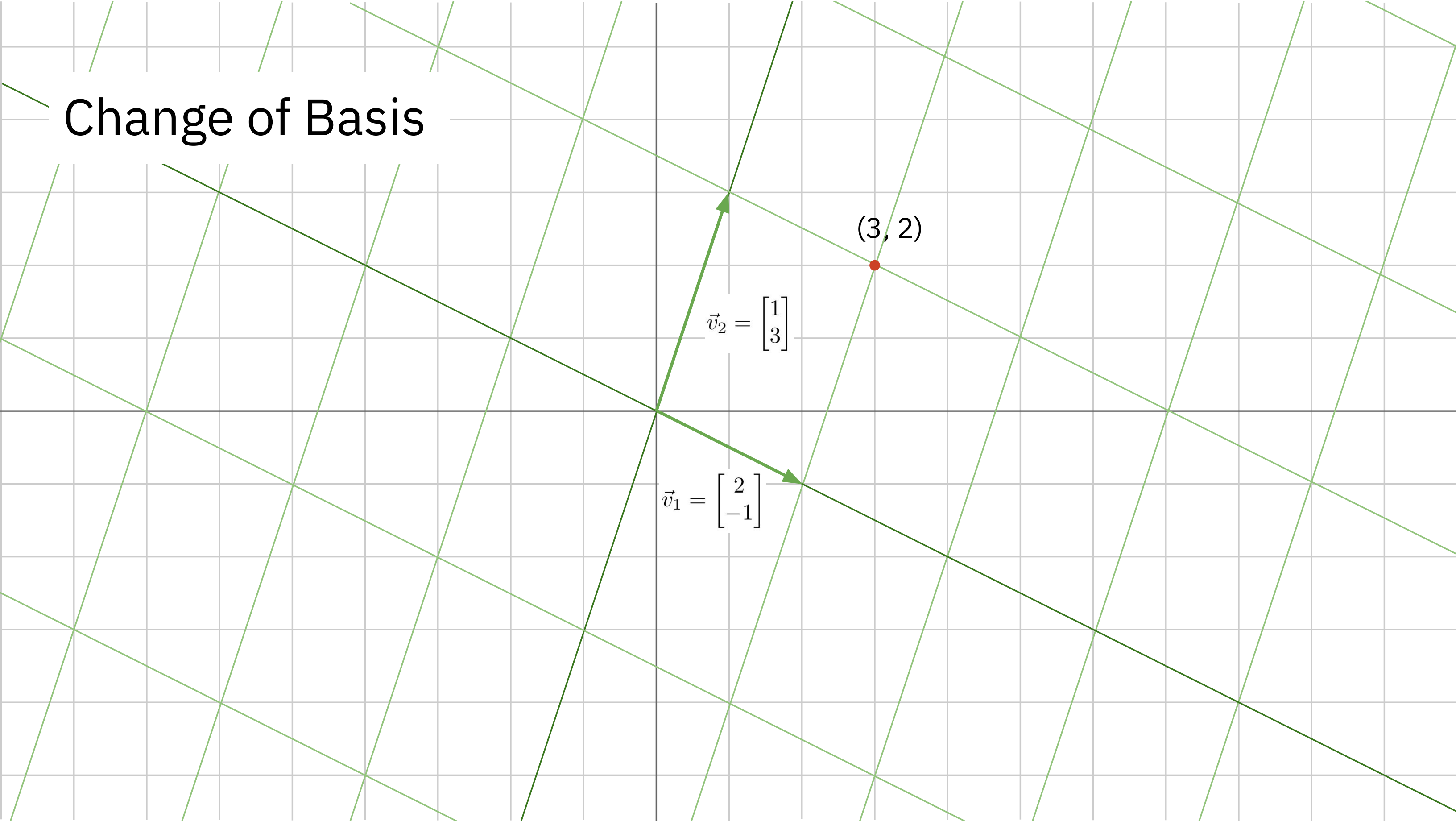
Change of Basis



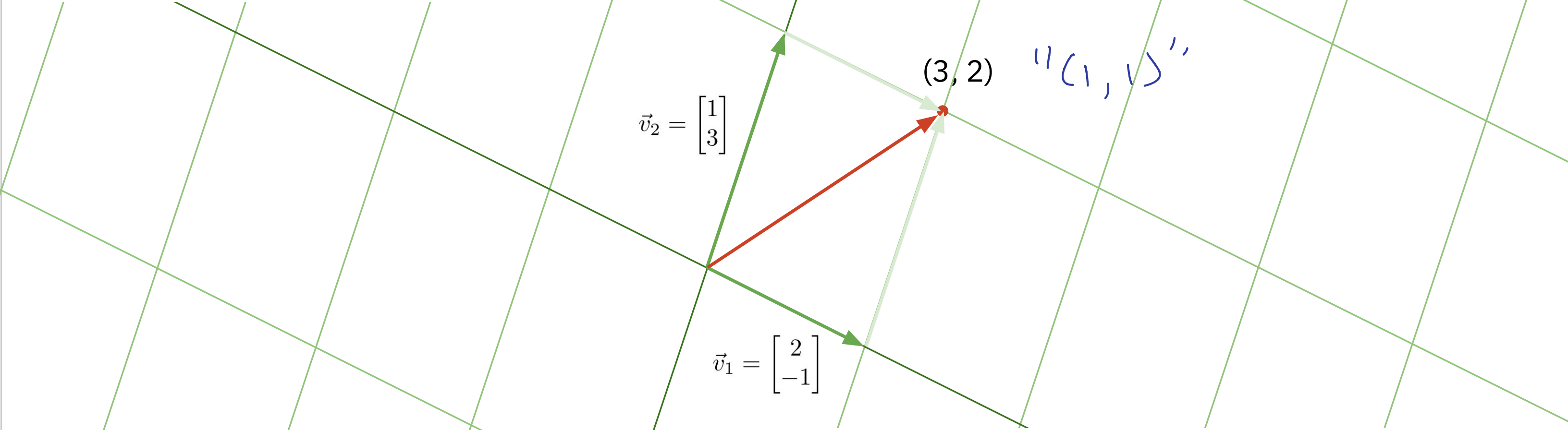
Change of Basis



Change of Basis



Change of Basis



$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = 1\vec{v}_1 + 1\vec{v}_2 = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Coordinates in new basis

Change of Basis: Standard to V-basis

$$\underset{std.}{\vec{x}} = a_v\vec{v}_1 + b_v\vec{v}_2 = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \begin{bmatrix} a_v \\ b_v \end{bmatrix} = \mathbf{V}\vec{x}_v$$

$$\begin{tikzcd} \vec{x}_{std} \arrow[r, shift left, "V^{-1}"] & \vec{x}_v \arrow[l, shift left, "V"] \end{tikzcd}$$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = a\vec{v}_1 + b\vec{v}_2 = \underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{x}_v}$$

$$\vec{x}_v = V^{-1}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\boxed{\begin{aligned} \vec{x}_{std} &= V\vec{x}_v \\ \vec{x}_v &= V^{-1}\vec{x}_{std} \end{aligned}}$$

Change of Basis: V-basis to U-basis

$$\vec{x} = \begin{bmatrix} | & | \\ \vec{e}_1 & \vec{e}_2 \\ | & | \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix} \begin{bmatrix} a_u \\ b_u \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \begin{bmatrix} a_v \\ b_v \end{bmatrix}$$

$$\vec{x} = I\vec{x} = \mathbf{V}\vec{x}_v = \mathbf{U}\vec{x}_u$$

(a) *Transformation From Standard Basis To Another Basis in* $\mathbb{R}^3$

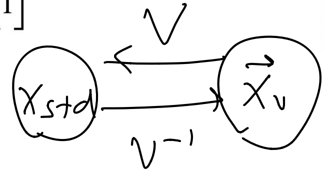
Calculate the coordinate transformation between the following bases:

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \overset{V^{-1}}{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e. find a matrix $\mathbf{T}$, such that $\vec{x}_v = \mathbf{T}\vec{x}_u$ where $\vec{x}_u$ contains the coordinates of a vector in a basis of the columns of $\mathbf{U}$ and $\vec{x}_v$ is the coordinates of the same vector in the basis of the columns of $\mathbf{V}$.

Let $\vec{x}_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and compute $\vec{x}_v$. Repeat this for $\vec{x}_u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Now let $\vec{x}_u = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. What is $\vec{x}_v$?

$$U\vec{x}_u = V\vec{x}_v$$



$$\vec{x}_v = \underbrace{V^{-1}}_{T}\vec{x}_u$$

$$\vec{x}_v = V^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \overset{V^{-1}}{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

(b) *Transformation Between Two Bases in* $\mathbb{R}^3$

Calculate the coordinate transformation between the following bases:

$$\mathbf{V} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \mathbf{W} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \end{bmatrix}}^{W^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e. find a matrix $\mathbf{T}$, such that $\vec{x}_w = \mathbf{T}\vec{x}_v$. Let $\vec{x}_v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and compute $\vec{x}_w$. Repeat this for $\vec{x}_v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Now let $\vec{x}_v = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. What is $\vec{x}_w$?

$$V\vec{x}_v = W\vec{x}_w$$

$$\vec{x}_w = W^{-1}\underbrace{V\vec{x}_v}_{\text{std. basis} \leftarrow V \text{ basis}}.$$

std. basis ← V basis

W-basis. ← std.

2. Diagonalization

(a) Consider a matrix $\mathbf{A}$, a matrix $\mathbf{V}$ whose columns are the eigenvectors of $\mathbf{A}$, and a diagonal matrix $\mathbf{\Lambda}$ with the eigenvalues of $\mathbf{A}$ on the diagonal (in the same order as the eigenvectors (or columns) of $\mathbf{V}$). From these definitions, show that

$$\mathbf{AV} = \mathbf{V\Lambda}$$

$A \in \mathbb{R}^{n \times n}$

n eigenvectors

$\vec{v}_1, \ldots, \vec{v}_n$

$\downarrow \qquad \downarrow$

$\lambda_1, \ldots, \lambda_n$

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$V = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}.$$

Eigenvector Definition

$$A\vec{v} = \lambda\vec{v}$$

$$A\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ A\vec{v}_1 & \cdots & A\vec{v}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} \lambda_1\vec{v}_1 & \cdots & \lambda_n\vec{v}_n \end{bmatrix} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

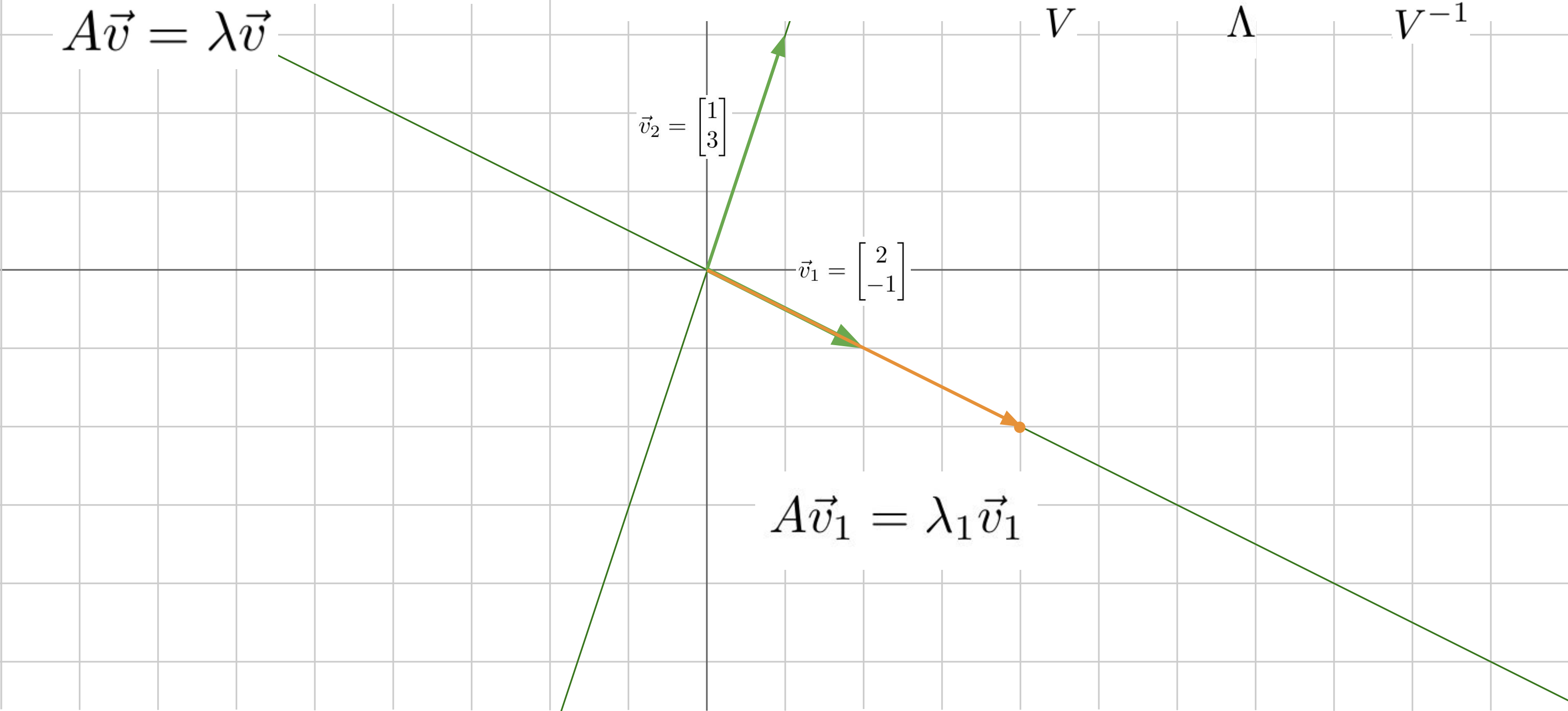
$$A V V^{-1} = V \Lambda V^{-1}$$

$$A = V \Lambda V^{-1}$$

Eigenvectors

$$A\vec{v} = \lambda\vec{v}$$

$$A = \begin{bmatrix} \frac{11}{7} & -\frac{6}{7} \\ -\frac{9}{7} & -\frac{4}{7} \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}^{-1}}_{V^{-1}}$$

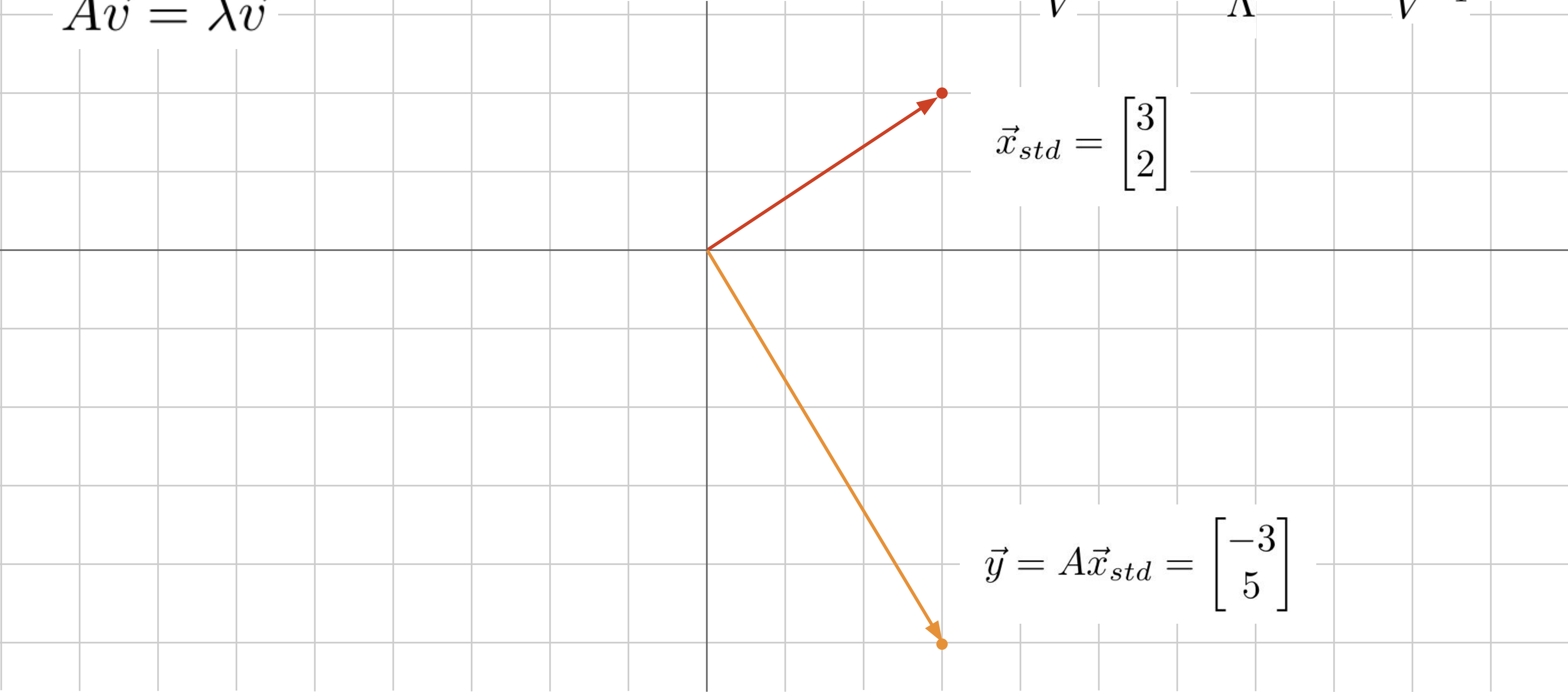


$$A\vec{v}_1 = \lambda_1\vec{v}_1$$

Eigenvectors

$$A\vec{v} = \lambda\vec{v}$$

$$A = \begin{bmatrix} \frac{11}{7} & -\frac{6}{7} \\ -\frac{9}{7} & -\frac{4}{7} \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}^{-1}}_{V^{-1}}$$



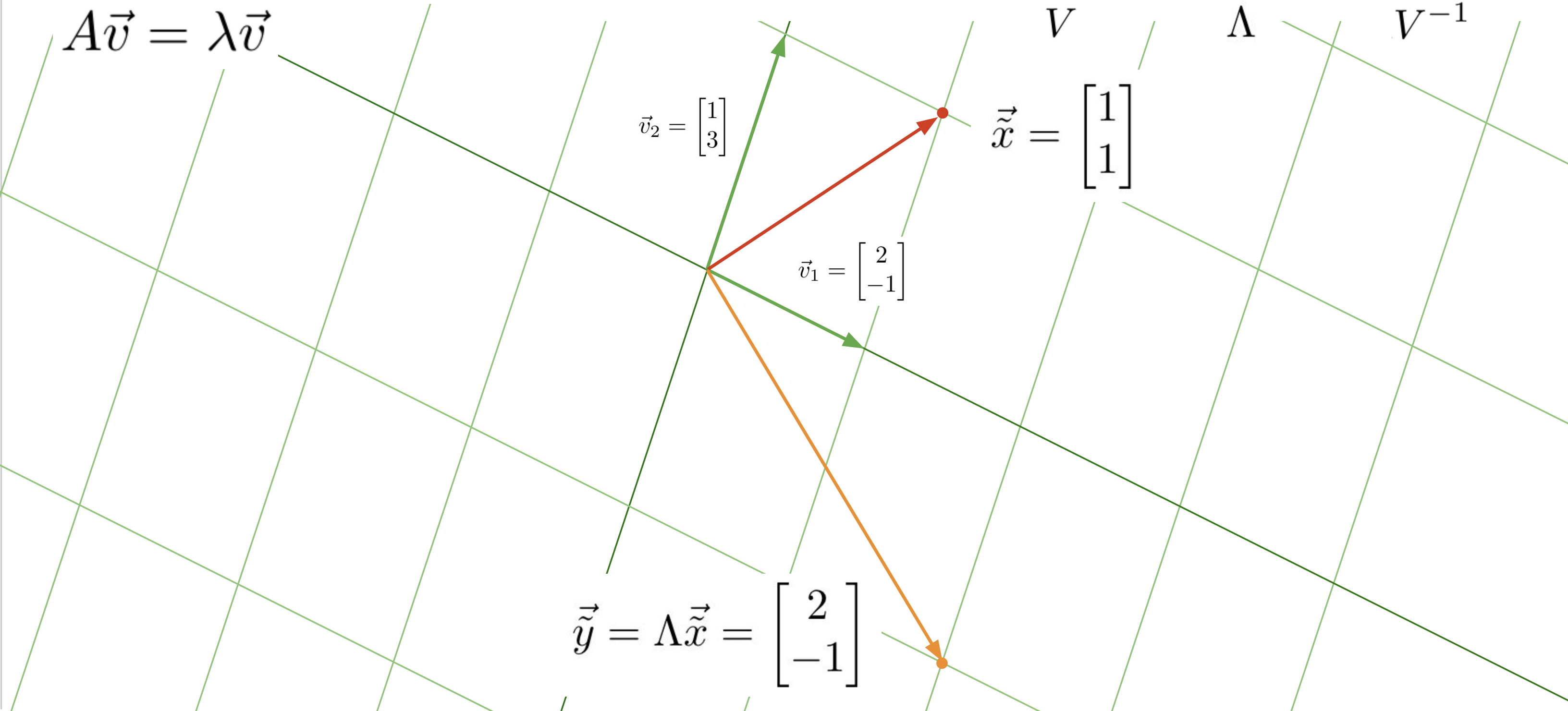
$$\vec{x}_{std} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\vec{y} = A\vec{x}_{std} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

Eigenvectors

$$A\vec{v} = \lambda\vec{v}$$

$$A = \begin{bmatrix} \frac{11}{7} & -\frac{6}{7} \\ -\frac{9}{7} & -\frac{4}{7} \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}^{-1}}_{V^{-1}}$$

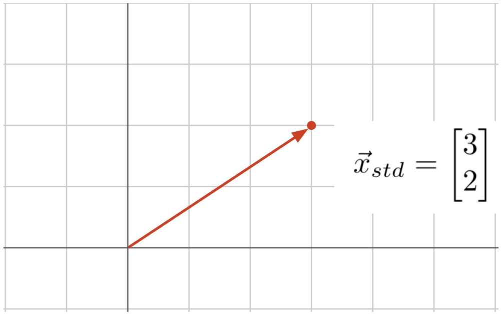


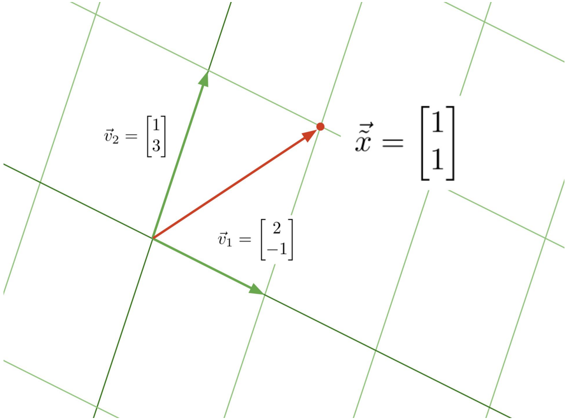
Diagonalization Viewed as Change of Basis

$$A = V\Lambda V^{-1}$$

$$A\vec{x}_{std} = V\Lambda \underline{V^{-1}\vec{x}_{std}}$$

Change of basis from standard to eigenbasis



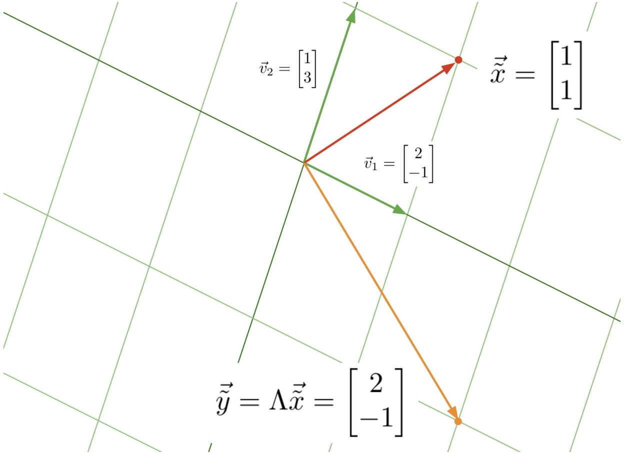


Diagonalization Viewed as Change of Basis

$$A = V \Lambda V^{-1}$$

$$\begin{aligned} A\vec{x}_{std} &= V \Lambda V^{-1} \vec{x}_{std} \\ &= V \underline{\Lambda \vec{\tilde{x}}} \end{aligned}$$

Diagonal linear transformation (each component scaled independently)



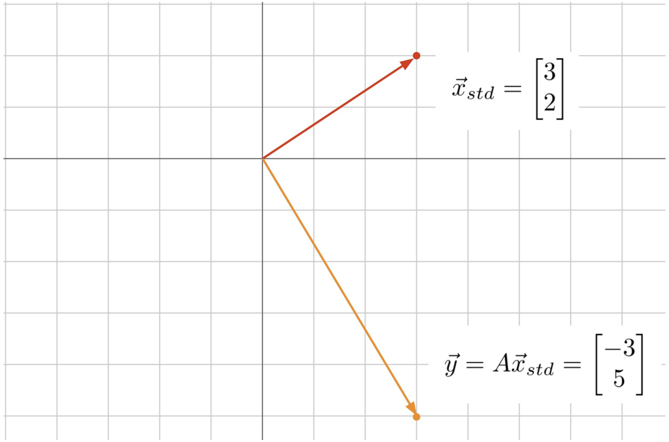
Diagonalization Viewed as Change of Basis

$$A = V\Lambda V^{-1}$$

$$\begin{aligned} A\vec{x}_{std} &= V\Lambda V^{-1}\vec{x}_{std} \\ &= V\Lambda\tilde{\vec{x}} \\ &= \underline{V\tilde{\vec{y}}} \quad = \vec{y} \end{aligned}$$

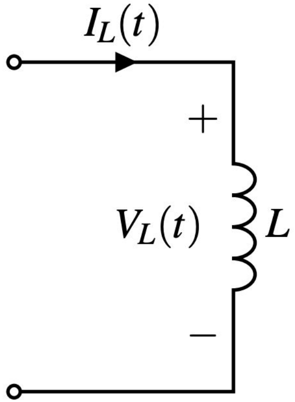
Change of basis from eigenbasis back to standard

$$\begin{tikzcd}[row sep=huge, column sep=huge]
\vec{x} \arrow[r, "\boxed{A}"] \arrow[d, shift right, "\boxed{V^{-1}}"'] & \vec{y} \arrow[d, shift left, "\boxed{V^{-1}}"] \\
\tilde{\vec{x}} \arrow[r, "\boxed{\Lambda}"'] \arrow[u, shift right, "\boxed{V}"'] & \tilde{\vec{y}} \arrow[u, shift left, "\boxed{V}"]
\end{tikzcd}$$

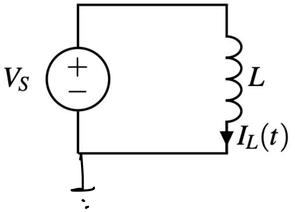


Inductor
Joins the Battle!

$$V_L(t) = L\frac{dI_L(t)}{dt}$$



When first studying capacitors, we analyzed a circuit where a current source was directly attached to a capacitor. In Figure 1, we form the equivalent "fundamental" circuit for an inductor:



$$L\frac{d}{dt}i_L(t) = V_L(t).$$

→ solving for $i_L(t)$.

Figure 1: Inductor in series with a voltage source.

(a) What is the current through an inductor as a function of time? If the inductance is $L = 3\text{H}$, what is the current at $t = 6\text{s}$? Assume that the voltage source turns from 0V to 5V at time $t = 0\text{s}$, and there's no current flowing in the circuit before the voltage source turns on.

$$i_L(0) = 0$$

$$\frac{di}{dt} = \frac{V_s}{L}$$

$$i_L(t) = \frac{V_s}{L}t + A$$

$$i_L(0) = \frac{V_s}{L}(0) + A$$

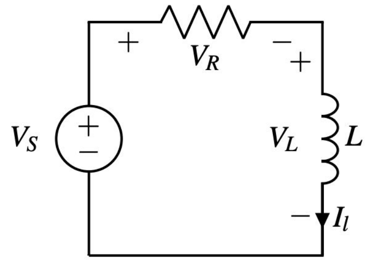
$$A = i_L(0) = 0.$$

$$i_L(t) = \frac{V_s}{L}t$$

$$i_L(6) = \frac{5}{3}(6) = 10A$$

Solve for the current $I_L(t)$ in the circuit over time, in terms of R, L, V_S, t.

$\underline{i_L(0) = 0}$



$\underline{\text{KVL}}$: $V_s - V_R - V_L = 0$

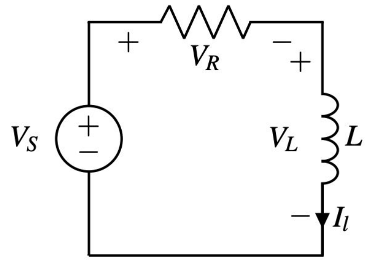
$$V_s - i_L R - L\frac{di_L}{dt} = 0.$$

$$\frac{di_L}{dt} = -\frac{R}{L} i_L + \frac{V_s}{L}.$$

$$i_L = \tilde{i} + \frac{V_s}{R}. \qquad \frac{d}{dt}\tilde{i} = -\frac{R}{L}\tilde{i} + \cancel{\frac{V_s}{L} - \frac{V_s}{L}}$$

$$\tilde{i} = \tilde{i}(0)e^{-\frac{R}{L}t} \qquad \tilde{i}(0) = \underbrace{i_L(0)}_{0} - \frac{V_s}{R} = -\frac{V_s}{R}$$

Solve for the current $I_L(t)$ in the circuit over time, in terms of R, L, V_S, t.



$$i_L = \tilde{i} + \frac{V_s}{R}$$

$$= -\frac{V_s}{R} e^{-\frac{R}{L}t} + \frac{V_s}{R}$$

$$= \frac{V_s}{R}\left(1 - e^{-\frac{R}{L}t}\right).$$

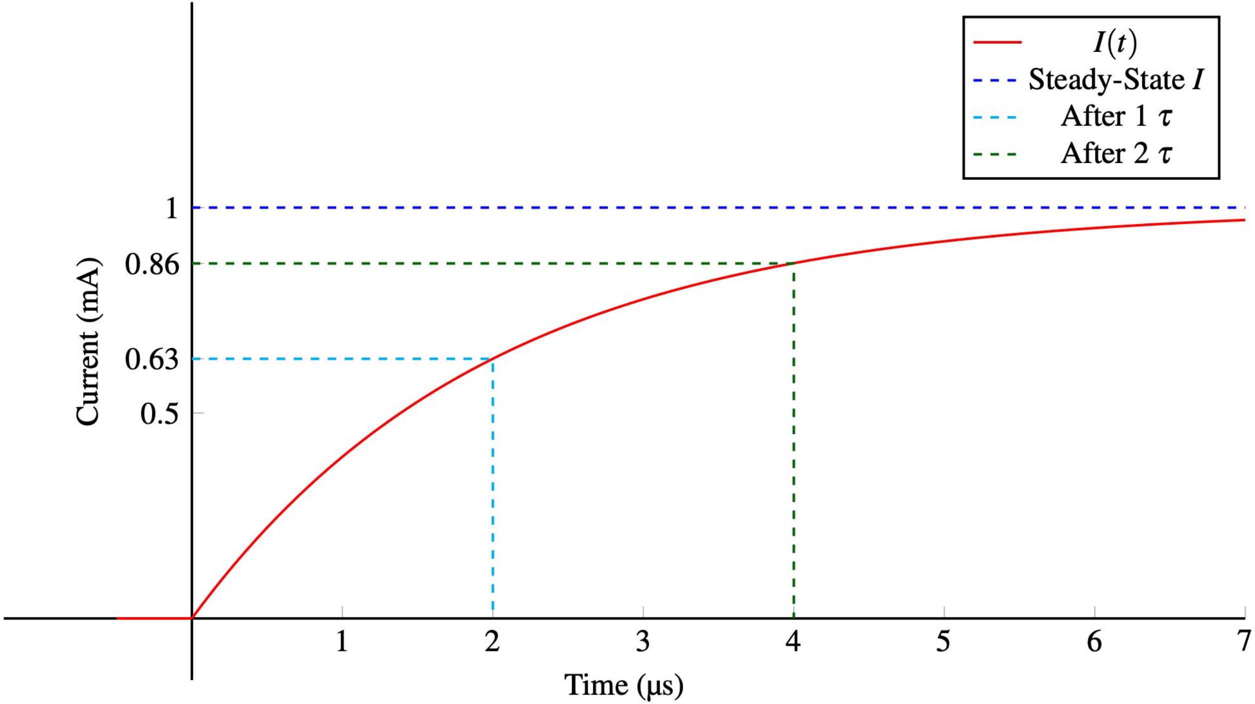


Figure 3: Transient Current in an RL circuit (with initial current $I(0) = 0$A.)

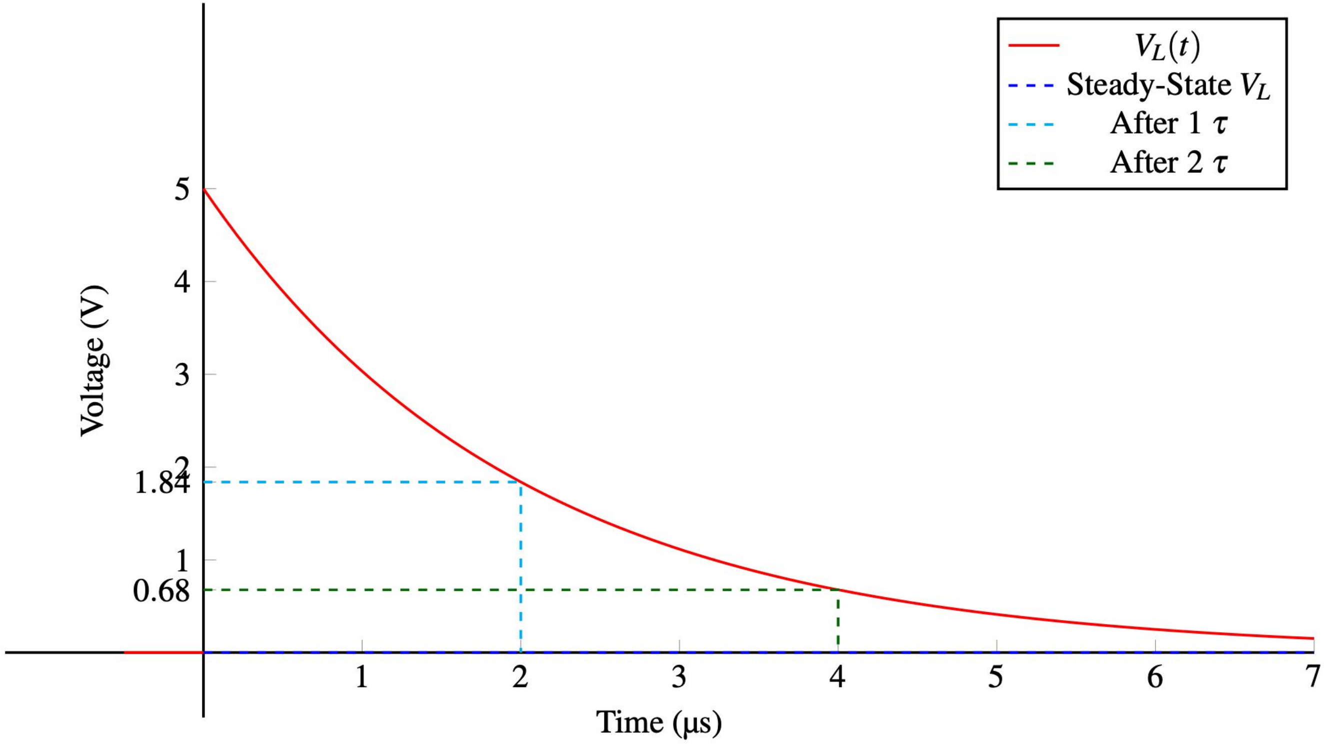


Figure 4: Transient Voltage across the inductor in an RL circuit (with initial current $I(0) = 0A$.)

4. Fibonacci Sequence

(a) The Fibonacci sequence is built as follows: the n-th number (F_n) is sum of the previous two numbers in the sequence. That is:

$$F_n = F_{n-1} + F_{n-2}$$

If the sequence is initialized with $F_1 = 0$ and $F_2 = 1$, then the first 11 numbers in the Fibonacci sequence are:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

$$\begin{bmatrix} F_3 \\ F_2 \end{bmatrix} = A \begin{bmatrix} F_2 \\ F_1 \end{bmatrix}$$

We can express this computation as a matrix multiplication:

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

What is $\mathbf{A}$?

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix}$$

$$\begin{bmatrix} F_4 \\ F_3 \end{bmatrix} = A \begin{bmatrix} F_3 \\ F_2 \end{bmatrix}$$

$$= A\left(A \begin{bmatrix} F_2 \\ F_1 \end{bmatrix}\right)$$

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = A^{n-2} \begin{bmatrix} F_2 \\ F_1 \end{bmatrix}.$$

$$A = V \Lambda V^{-1}$$

$$A^{n-2} = \underbrace{(V\Lambda \overbrace{V^{-1})(V}^{I}\Lambda \overbrace{V^{-1})\ldots(}^{I}\overbrace{(V}^{I}\Lambda V^{-1})}_{n-2 \text{ times}}$$

$$= V \underbrace{\Lambda \Lambda \Lambda \ldots \Lambda}_{n-2} V^{-1} \qquad \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}$$

$$= V \Lambda^{n-2} V^{-1}.$$

(b) Find the eigenvalues and corresponding eigenvectors of **A**.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Solve for eigenvalues:

$$\det(A - \lambda I) = 0$$

$$(1-\lambda)(-\lambda) - 1 = 0$$

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda_\pm = \frac{1 \pm \sqrt{(-1)^2 - 4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

Solve for eigenvectors: nullspace of $A - \lambda_1 I$

$$\left[\begin{array}{c:c} A - \lambda_1 I & \begin{matrix} 0 \\ 0 \end{matrix} \end{array}\right] = \left[\begin{array}{cc:c} 1 - \frac{1+\sqrt{5}}{2} & 1 & 0 \\ 1 & -\frac{1+\sqrt{5}}{2} & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc:c} \frac{1-\sqrt{5}}{2} & 1 & 0 \\ 1 & -\frac{1+\sqrt{5}}{2} & 0 \end{array}\right]$$

$$-\frac{(1+\sqrt{5})}{2} \cdot \frac{(1-\sqrt{5})}{2} = \left[\begin{array}{cc:c} \frac{1-\sqrt{5}}{2} & 1 & 0 \\ \frac{1-\sqrt{5}}{2} & 1 & 0 \end{array}\right]$$

$$= -\frac{(1-5)}{4} = 1$$

$$= \left[\begin{array}{ccc} \frac{1-\sqrt{5}}{2} & 1 & 0 \end{array}\right.$$

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

(c) Diagonalize $\mathbf{A}$ (that is, in the expression $\mathbf{A} = \mathbf{V\Lambda V}^{-1}$, solve for each component matrix.)

$$A = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}^{-1}$$

x_2 free: $x_2 = 1$

$$\frac{1-\sqrt{5}}{2} x_1 + 1 \underbrace{x_2}_{1} = 0$$

$$\frac{1-\sqrt{5}}{-12} x_1 + 1 = 0$$

$$x_1 = -\frac{2}{1-\sqrt{5}} \cdot \frac{1+\sqrt{5}}{1+\sqrt{5}}$$

$$= \frac{-2(1+\sqrt{5})}{1-5}$$

$$= \frac{1+\sqrt{5}}{2}$$

$$\vec{v}_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

(d) Use the diagonalized result to show that we can arrive at an analytical result for any F_n:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}$$