Discussion 3B: Solving Systems of Differential Equations

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OH/HW Party: Tuesday 4-6pm
Worksheet: https://eecs16b.org/discussion/dis03B.pdf
Notes: https://tinyurl.com/justin16bnotes

Today:

1. Quick recap + motivating example for systems of diff. eqs.
2. Linear algebra review (2x2 matrix inverses, diagonal matrices)
3. Go through the problem (lecture style)

Quick Recap

Guess and verify +
uniqueness

Homogeneous
Differential Equation

$$\boxed{\frac{d}{dt}x(t) = \lambda x(t)} \longrightarrow \boxed{x(t) = x(0)e^{\lambda t}}$$

Change of variables to recover a
homogeneous diff eq

Nonhomogeneous
Differential Equation
with Constant Input

$$\boxed{\frac{d}{dt}x(t) = \lambda x(t) + \alpha} \longrightarrow \boxed{x(t) = x(0)e^{\lambda t} + \frac{\alpha}{\lambda}(e^{\lambda t} - 1)}$$

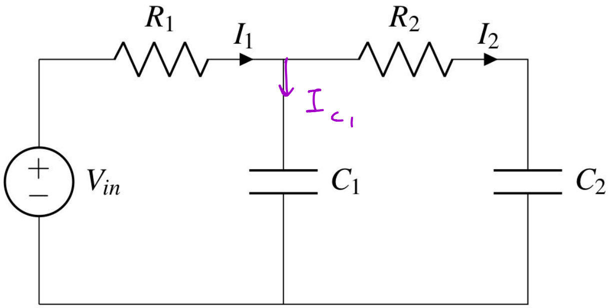
Discretize, solve a bunch of constant
input diff eqs & take the limit as $\Delta \to 0$

Nonhomogeneous
Differential Equation
with Nonconstant Input

$$\boxed{\frac{d}{dt}x(t) = \lambda x(t) + u(t)} \longrightarrow \boxed{x(t) = x(0)e^{\lambda t} + \int_0^t u(\tau)e^{\lambda(t-\tau)}\,d\tau}$$

Systems of Differential Equations

Motivating Example



KCL: $i_1 = i_{c_1} + i_2$

$i_2 = i_{c_2}$

$$\frac{V_{in} - V_{c_1}}{R_1} = C_1 \frac{dV_{c_1}(t)}{dt} + \frac{V_{c_1} - V_{c_2}}{R_2}$$

$$\frac{V_{c_1} - V_{c_2}}{R_2} = C_2 \frac{dV_{c_2}(t)}{dt}$$

Quick Recap

First Order Differential Equation with a Constant Input

Original Problem

$$\boxed{\frac{d}{dt}x(t) = \lambda x(t) + \alpha}$$

Change of Variables Problem

$$\frac{d}{dt}x(t) = \lambda x(t) + \alpha$$

$$\frac{d}{dt}(z(t) - \frac{\alpha}{\lambda}) = \lambda(z(t) - \frac{\alpha}{\lambda}) + \alpha$$

$$\frac{d}{dt}z(t) = \lambda z(t)$$

Change of Variables Solution

$$z(t) = z(0)e^{\lambda t}$$

Original Solution

$$\boxed{x(t) = x(0)e^{\lambda t} + \frac{\alpha}{\lambda}(e^{\lambda t} - 1)}$$

Linear Algebra Review

Inverse

$$A^{-1}A = AA^{-1} = I$$

$A \in \mathbb{R}^{n \times n}$ $\quad I = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{bmatrix}$ $\quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\boxed{\text{2x2 Matrix Inverse} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

Diagonal Matrix

$$D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$$

1. Changing Coordinates and Systems of Differential Equations

Suppose we have the pair of differential equations (valid for $t \geq 0$)

$$\frac{d}{dt}x_1(t) = -9x_1(t)$$

$$\frac{d}{dt}x_2(t) = -2x_2(t)$$

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

with initial conditions $x_1(0) = -1$ and $x_2(0) = 3$.

(a) Solve for $x_1(t)$ and $x_2(t)$ for $t \geq 0$.

$$x_1(t) = x_1(0)e^{-9t} = -e^{-9t}$$

$$x_2(t) = x_2(0)e^{-2t} = 3e^{-2t}$$

Suppose we are actually interested in a different set of variables with the following differential equations:

$$\frac{d}{dt}z_1(t) = -5z_1(t) + 2z_2(t)$$
$$\frac{d}{dt}z_2(t) = 6z_1(t) - 6z_2(t).$$

(b) Write out the above system of differential equations in matrix form. Assuming that the initial state $z(\vec{0}) = \begin{bmatrix} 7 & 7 \end{bmatrix}^T$, can we solve this system directly? NO.

$$\vec{z}(0) = \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$\underbrace{\frac{d}{dt}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}}_{\vec{z}(t)} = \underbrace{\begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}}_{\vec{z}(t).}$$

(c) Consider that in our frustration with the previous system of differential equations, we start hearing voices. These voices whisper to us that that we should try the following change of variables: y_1, y_2

$$z_1(t) = -y_1(t) + 2y_2(t)$$
$$z_2(t) = 2y_1(t) + 3y_2(t).$$

Write out this transformation in matrix form ($\vec{z} = V\vec{y}$).

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

$$V^{-1}\vec{z} = V^{-1}V\vec{y}.$$

$$V^{-1}\vec{z} = I\vec{y} = \vec{y}.$$

$$\begin{tikzcd} \vec{y} \arrow[r, bend right, "V"'] & \vec{z} \arrow[l, bend right, "V^{-1}"'] \end{tikzcd}$$

(d) How do the initial conditions for $z_i(t)$ translate into the initial conditions for $y_i(t)$?

1. By direct substitution:

$$\vec{y}(0) = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}$$

$$7 = z_1(0) = -y_1(0) + 2y_2(0) \qquad R_1$$

$$7 = z_2(0) = 2y_1(0) + 3y_2(0) \qquad R_2$$

$$R_2 + 2R_1$$

$$21 = 7y_2(0) \qquad \boxed{y_2(0) = 3}$$

$$-y_1(0) + 2(3) = 7$$

$$\boxed{y_1(0) = -1}$$

Initial Conditions

$$\boxed{\vec{z}(0) = \begin{bmatrix} 7 & 7 \end{bmatrix}^T}$$

Change of Variables

$$z_1(t) = -y_1(t) + 2y_2(t)$$
$$z_2(t) = 2y_1(t) + 3y_2(t).$$

(d) How do the initial conditions for $z_i(t)$ translate into the initial conditions for $y_i(t)$?

2. Using matrices and vectors:

$$\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix}$$

$$V^{-1} = \frac{1}{-3-4} \begin{bmatrix} 3 & -2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix}$$

2x2 Matrix Inverse

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$V\vec{y}(0) = \vec{z}(0)$$

$$V\vec{y}(0) = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

Initial Conditions

$$\boxed{z(\vec{0}) = \begin{bmatrix} 7 & 7 \end{bmatrix}^T}$$

$$V^{-1}V\vec{y}(0) = V^{-1}\begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$\vec{y}(0) = V^{-1}\begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Change of Variables

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}}^{V} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

(e) Rewrite the differential equations in terms of $y_i(t)$. Can we solve this system of differential equations?

1. By direct substitution:

$$\frac{d}{dt} y_1(t) = \dots$$

$$\frac{d}{dt} y_2(t) = \dots \quad \vec{z} = V\vec{y}$$

$$\underbrace{\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}}_{\vec{y}} = \underbrace{\begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix}}_{V^{-1}} \underbrace{\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}}_{\vec{z}}$$

$$\frac{d}{dt}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} -\frac{3}{7}z_1(t) + \frac{2}{7}z_2(t) \\ \frac{2}{7}z_1(t) + \frac{1}{7}z_2(t) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{7}\left(\frac{d}{dt}z_1\right) + \frac{2}{7}\left(\frac{d}{dt}z_2\right) \\ \frac{2}{7}\left(\frac{d}{dt}z_1\right) + \frac{1}{7}\left(\frac{d}{dt}z_2\right) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{7}(-5z_1 + 2z_2) + \frac{2}{7}(6z_1 - 6z_2) \\ \frac{2}{7}(-5z_1 + 2z_2) + \frac{1}{7}(6z_1 - 6z_2) \end{bmatrix}$$

Differential Equations

$$\frac{d}{dt}z_1(t) = -5z_1(t) + 2z_2(t)$$

$$\frac{d}{dt}z_2(t) = 6z_1(t) - 6z_2(t).$$

Change of Variables

$$z_1(t) = -y_1(t) + 2y_2(t)$$

$$z_2(t) = 2y_1(t) + 3y_2(t).$$

$$= \begin{bmatrix} \frac{27}{7} z_1 + \frac{-18}{7} z_2 \\ -\frac{4}{7} z_1 + \frac{-2}{7} z_2 \end{bmatrix}$$

$$= \begin{bmatrix} -9(y_1(t)) \\ -2(y_2(t)). \end{bmatrix}$$

$$\frac{d}{dt} y_1(t) = -9 y_1(t)$$

$$\frac{d}{dt} y_2(t) = -2 y_2(t)$$

$$y_1(t) = y_1(0) e^{-9t} = -e^{-9t}$$

$$y_2(t) = y_2(0) e^{-2t} = 3e^{-2t}$$

(e) Rewrite the differential equations in terms of $y_i(t)$. Can we solve this system of differential equations?

2. Using matrices and vectors:

$$\frac{d}{dt}\vec{z} = A\vec{z} \qquad \vec{z} = V\vec{y}$$

$$\frac{d}{dt}(V\vec{y}) = A(V\vec{y}). \qquad \frac{d}{dt}\vec{y} = ?$$

$$\cancel{V^{-1}}\left(\frac{d}{dt}\cancel{V}\,\vec{y}\right) = V^{-1}AV\vec{y}$$

$$\frac{d}{dt}\vec{y} = V^{-1}AV\vec{y}.$$

$$\frac{d}{dt}\vec{y} = \Lambda\vec{y}. = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix}\vec{y}.$$

Differential Equations

$$\frac{d}{dt}\left(\begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix}\right) = \overbrace{\begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}}^{A} \cdot \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix}$$

Change of Variables

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

(f) What are the solutions for $z_i(t)$?

1. Solve this with direct substitution:

$$y_1(t) = y_1(0)e^{-9t} = -e^{-9t}$$
$$y_2(t) = y_2(0)e^{-2t} = 3e^{-2t}$$

$$z_1(t) = -(-e^{-9t}) + 2(3e^{-2t}) = e^{-9t} + 6e^{-2t}$$
$$z_2(t) = 2(-e^{-9t}) + 3(3e^{-2t}) = -2e^{-9t} + 9e^{-2t}.$$

Change of Variables

$$z_1(t) = -y_1(t) + 2y_2(t)$$
$$z_2(t) = 2y_1(t) + 3y_2(t).$$

(f) What are the solutions for $z_i(t)$?

2. Using matrices and vectors:

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

$$\underline{\vec{z}} = V\underline{\vec{y}}$$

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix}$$

Change of Variables

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

$$\vec{x} = \alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n .$$

$$= \underbrace{\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}}_{\vec{\alpha}}$$

$$\vec{\alpha}' = \Lambda \vec{\alpha}$$

$$\vec{\alpha}' = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \vec{\alpha}$$

$$\begin{tikzcd} \vec{x} \arrow[r, "A"] \arrow[d, "V^{-1}"'] & A\vec{x} \\ \tilde{\vec{x}} \arrow[r, "\Lambda"'] & \Lambda \tilde{\vec{x}} \arrow[u, "V"'] \end{tikzcd}$$

$$A\vec{x} = A(\alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n)$$

$$= \alpha_1 \lambda_1 \vec{v}_1 + \dots + \alpha_n \lambda_n \vec{v}_n .$$

$$= \begin{bmatrix} & V & \end{bmatrix} \underbrace{\begin{bmatrix} \alpha_1 \lambda_1 \\ \vdots \\ \alpha_n \lambda_n \end{bmatrix}}_{\vec{\alpha}'}$$

Original Problem

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$$

$$\frac{d}{dt}\left(V\vec{\tilde{x}}(t)\right) = A\left(V\vec{\tilde{x}}(t)\right)$$

$$\frac{d}{dt}\vec{\tilde{x}}(t) = V^{-1}AV\vec{\tilde{x}}(t).$$

$$\vec{x}(t) = V\vec{\tilde{x}}(t).$$

Change of Coordinates Problem

$$\frac{d}{dt}\vec{\tilde{x}}(t) = V^{-1}AV\vec{\tilde{x}}(t) = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \vec{\tilde{x}}(t)$$

$$\Lambda$$

$$\vec{\tilde{x}}(0) = V^{-1}\vec{x}(0)$$

$$A = V\Lambda V^{-1}$$

$$V^{-1}A = \Lambda V^{-1}$$

Eigenbasis coordinates

$$V^{-1}AV = \Lambda$$

Original Solution

$$\vec{x}(t) = V\vec{\tilde{x}}(t)$$

$$A\vec{v} = \lambda\vec{v}$$

Change of Coordinates Solution

$$\tilde{x}_1(t) = \tilde{x}_1(0)e^{\lambda_1 t}$$

$$\vdots$$

$$\tilde{x}_n(t) = \tilde{x}_n(0)e^{\lambda_n t}$$

How to solve for the change of basis V?

To be continued...