Disc 4A — Slow-Paced Notes

Announcements

- HW 3 self-grades + resubmissions due tomorrow
- OH are continuing, HW party on Tues/Thurs 4-6 PM
- Lab lite: expected to go to one OH to receive support! Lab Lite 2 due tonight!
- HW 4 released, due Friday 2/12
- Grading lite opt-in due Friday 2/12

Mini-Lecture: Vector Differential Equations (Note 3A for more info!)

Given a differential equations of form:

$$\underbrace{\begin{aligned} \frac{d}{dt}x_1(t) &= a_{11}x_1(t) + a_{12}x_2(t) + b_1 \\ \frac{d}{dt}x_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + b_2 \end{aligned}}$$

In general, manipulate circuit diff eqs to have this form.

Set up a linear algebraic expression:

$$\begin{bmatrix} \frac{d}{dt}x_1 \\ \frac{d}{dt}x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{\vec{b}}$$

$$\Rightarrow \frac{d\vec{x}(t)}{dt} = A\vec{x} + \vec{b}$$

Now that we have a linear expression, can manipulate with linear algebra!

In today's discussion: eigenbasis of a matrix A

↳ Eigenbasis: Set of vectors V, where each $v \in V$ satisfies $\underline{Av = \lambda v}$, for some $\lambda \in \mathbb{R}$

define V: $\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_k \\ | & | & & | \end{bmatrix}$, where k is the rank of the eigenbasis

↳ Small note — $\text{span}(V)$ is called the eigenspace

Knowing the definitive property of an eigenvector/eigenvalue pair lets us figure out how to find them:

$$Av = \lambda v$$
$$\Rightarrow Av - \lambda v = 0$$
$$\Rightarrow (A - \lambda I)v = 0 \quad \Rightarrow v \text{ is in the null space of } A - \lambda I!$$

$\Rightarrow$ To find vectors v in the eigenbasis, <u>solve for the null space of $A - \lambda I$</u>

Note on eigenspace — for any vector $v' \in \text{span}(V) \Rightarrow v'$ can be expressed as $v' = V\alpha$ or as $v' = V\Lambda\alpha'$

but $V\Lambda = AV$, since $Av = \lambda v \quad \forall v \in V$

$\Rightarrow$ any $v' \in \text{span}(V)$ can be expressed as $AV\alpha'$

$\Rightarrow v' \in \text{span}(V)$ is also in $\text{span}(AV)$

V is a change of basis for A

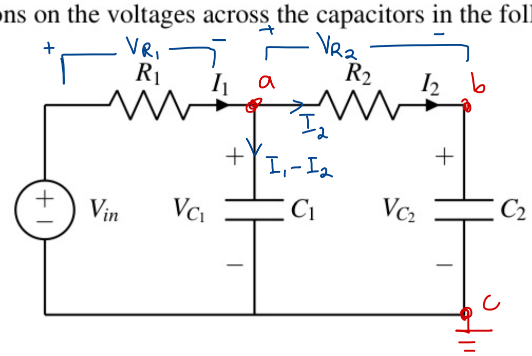
1. Changing Coordinates and Systems of Differential Equations, II

In the previous discussion we analyzed and solved a pair of differential equations where the variables of interest were coupled.

$$\frac{d}{dt}z_1(t) = -5z_1(t) + 2z_2(t)$$
$$\frac{d}{dt}z_2(t) = 6z_1(t) - 6z_2(t).$$

We solved this system by using a coordinate transformation that gave us a decoupled system of equations. In the last discussion we were simply handed this transformation, but in this discussion we will construct the transformation for ourselves.

We will focus our explorations on the voltages across the capacitors in the following circuit.



$I_2 = C_2 \cdot \frac{d}{dt} V_{c2}$

$I_1 = I_2 + C_1 \cdot \frac{d}{dt} V_{c_1}$

Figure 1: Two dimensional system: a circuit with two capacitors, like the one in lecture.

(a) Write the system of differential equations governing the voltages across the capacitors V_{C_1}, V_{C_2}. Use the following values: $C_1 = 1\mu\text{F}, C_2 = \frac{1}{3}\mu\text{F}, R_1 = \frac{1}{3}\text{M}\Omega, R_2 = \frac{1}{2}\text{M}\Omega$.

V_{c_1} difference in voltage btw points a and c

$\Rightarrow V_{c_1} = V_{in} - V_{R_1}$

$\Rightarrow V_{c_1} = V_{in} - I_1 \cdot R_1$

$\Rightarrow I_1 R_1 = V_i - V_{c_1} \quad \Rightarrow I_1 = \frac{V_{in}}{R_1} - \frac{V_{c_1}}{R_1}$ ①

V_{c_2}: difference in voltage btw points b and c

- $V_{c_2} = V_{c_1} - V_{R_2}$

$\Rightarrow V_{c_2} = V_{c_1} - I_2 \cdot R_2$

$\Rightarrow I_2 \cdot R_2 = V_{c_1} - V_{c_2} \quad \Rightarrow I_2 = \frac{V_{c_1}}{R_2} - \frac{V_{c_2}}{R_2}$ ②

- $I_2 = C_2 \cdot \frac{d}{dt} V_{c_2} \quad \Rightarrow C_2 \frac{d}{dt} V_{c_2} = \frac{V_{c_1}}{R_2} - \frac{V_{c_2}}{R_2}$

$\Rightarrow \frac{d}{dt} V_{c_2} = \frac{1}{R_2 C_2}(V_{c_1} - V_{c_2})$ ③

- $I_1 = I_2 + C_1 \cdot \frac{d}{dt} V_{c_1} \quad \Rightarrow \frac{d}{dt} V_{c_1} = \frac{1}{C_1}(I_1 - I_2) = \frac{1}{C_1}\left(\frac{V_{in}}{R_1} - \frac{V_{c_1}}{R_1} - \frac{V_{c_1}}{R_2} + \frac{V_{c_2}}{R_2}\right)$

$\Rightarrow \frac{d}{dt} V_{c_2} = -\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right) V_{c_1} + \frac{V_{in}}{R_1 C_1} + \frac{V_{c_2}}{R_2 C_1}$ ④

$$\begin{matrix} ③ \\ ④ \end{matrix}\begin{bmatrix} \frac{d}{dt} V_{C_1}(t) \\ \frac{dt}{dt} V_{C_2}(t) \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right) & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix} \begin{bmatrix} V_{C_1}(t) \\ V_{C_2}(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1C_1} \\ 0 \end{bmatrix} \cdot V_{in}(t)$$

Units note!

$1F = 1\frac{sec}{\Omega}$

(since $I = C\frac{dV}{dt}$

$\Rightarrow Amps = F \cdot \frac{Amp\,\Omega}{sec}$

$\Rightarrow F = \frac{sec}{\Omega}$

Now, plug in values! $C_1 = 1\mu F$, $C_2 = \frac{1}{3}\mu F$, $R_1 = \frac{1}{3}M\Omega$, $R_2 = \frac{1}{2}M\Omega$

$$\Rightarrow -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right) = -\left(\frac{1}{\frac{1}{3}\cdot 1} + \frac{1}{\frac{1}{2}\cdot 1}\right) = -(3+2) = -5$$

$$\frac{1}{R_2C_1} = 2 \qquad \frac{1}{R_2C_2} = \frac{1}{\frac{1}{2}\frac{1}{3}} = 6 \qquad -\frac{1}{R_2C_2} = -6 \qquad \frac{1}{R_1C_1} = 3$$

$$\Rightarrow \begin{bmatrix} \frac{d}{dt} V_{C_1}(t) \\ \frac{d}{dt} V_{C_2}(t) \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \cdot \begin{bmatrix} V_{C_1}(t) \\ V_{C_2}(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} \cdot V_{in}(t)$$

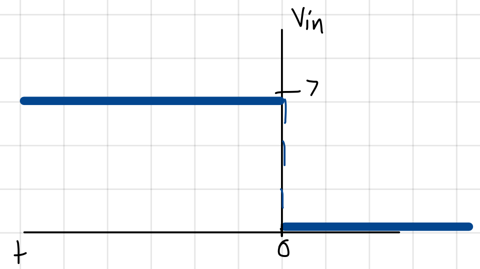
(b) Suppose also that V_{in} was at 7 Volts for a long time, and then transitioned to be 0 Volts at time $t = 0$. This "new" system of differential equations (valid for $t \geq 0$)

$$\frac{d}{dt}y_1(t) = -5y_1(t) + 2y_2(t) \tag{1}$$

$$\frac{d}{dt}y_2(t) = 6y_1(t) - 6y_2(t) \tag{2}$$

with initial conditions $y_1(0) = 7$ and $y_2(0) = 7$.

Write out the differential equations and initial conditions in matrix/vector form.



$$\begin{bmatrix} \frac{d}{dt} y_1(t) \\ \frac{d}{dt} y_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}}_{A_y} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \quad ; \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

← System of linear equations!

(c) Find the eigenvalues λ_1, λ_2 and eigenspaces for the matrix corresponding to the differential equation matrix above.

↳ How do we usually find eigenvalues for some matrix A?

Values λ that solve $\det(A - \lambda I) = 0$ (since $\vec{v}$ corresponding to λ has property $A\vec{v} = \lambda\vec{v}$) bc $\vec{v} \in \mathcal{N}(A - \lambda I)$

$$\Rightarrow \det\left(\begin{bmatrix} -5-\lambda & 2 \\ 6 & -6-\lambda \end{bmatrix}\right) = 0 \Rightarrow (5+\lambda)(6+\lambda) - 2\cdot 6 = 0$$

$$\Rightarrow 30 + 11\lambda + \lambda^2 - 12 = 0$$

$$\Rightarrow \lambda^2 + 11\lambda + 18 = 0$$

$$↳ (\lambda+9)(\lambda+2) = 0$$

$\Rightarrow \lambda_1 = -9$, $\lambda_2 = -2$ solve our eigenvalue equation!

For λ_1: $(A - \lambda_1 I)\vec{v}_1 = 0 \Rightarrow \begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}\vec{v}_1 = 0 \Rightarrow \boxed{\vec{v}_1 = \alpha\begin{bmatrix} -1 \\ 2 \end{bmatrix}}$

For λ_2: $(A - \lambda_2 I)\vec{v}_2 = 0 \Rightarrow \begin{bmatrix} -3 & 2 \\ 6 & -4 \end{bmatrix}\vec{v}_2 = 0 \Rightarrow \boxed{\vec{v}_2 = \alpha\begin{bmatrix} 2 \\ 3 \end{bmatrix}}$

(d) Change coordinates into the eigenbasis to re-express the differential equations in terms of new variables $z_{\lambda_1}(t)$, $z_{\lambda_2}(t)$. (These variables represent eigenbasis-aligned coordinates.)

eigenvectors:

$\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\vec{y}$ can be written in terms of the eigenvectors of A.

$$\vec{y} = z_1 \vec{v}_1 + z_2 \vec{v}_2$$

coordinates in eigenbasis

$$\text{or } \vec{y} = \underbrace{\begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}}_{\text{eigenvectors}} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\vec{y} = V\vec{z}$$

$$V^{-1}\vec{y} = V^{-1}V\vec{z} = \vec{z}$$

$$V = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$$

$$\boxed{\text{Equations for moving between basis:} \quad V^{-1}\vec{y} = \vec{z} \qquad \vec{y} = V\vec{z}}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$V^{-1} = \frac{1}{-7}\begin{bmatrix} 3 & -2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -3/7 & 2/7 \\ 2/7 & 1/7 \end{bmatrix}$$

```tikzcd
\vec{y} \arrow[r, "A"] & \frac{d\vec{y}}{dt} \arrow[d, "V^{-1}"] \\
\vec{z} \arrow[u, "V"] & \frac{d\vec{z}}{dt}
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we defined this as $V\vec{z}$

$$\frac{d\vec{y}}{dt} = A\vec{y} = AV\vec{z}$$

$$V^{-1}\frac{d\vec{y}}{dt} = V^{-1}AV\vec{z}$$

$$\frac{d\vec{z}}{dt} = V^{-1}AV\vec{z}$$

V^{-1} $\quad$ A $\quad$ V

$$\frac{d\vec{z}}{dt} = \begin{bmatrix} -3/7 & 2/7 \\ 2/7 & 1/7 \end{bmatrix}\begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}\vec{z}$$

$$\frac{d\vec{z}}{dt} = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix}\vec{z}$$

(e) Solve the differential equation for $z_{\lambda_i}(t)$ in the eigenbasis.

$V^{-1}\vec{y} = \vec{z}$

$\vec{y} = V\vec{z}$

Initial Conditions for $\vec{z}$

$$\vec{z}(0) = V^{-1}\vec{y}(0)$$

$$\vec{z}(0) = \begin{bmatrix} -3/7 & 2/7 \\ 2/7 & 1/7 \end{bmatrix} \begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\frac{d\vec{z}}{dt} = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix} \vec{z}$$

$$\vec{z}(t) = \begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix}$$

(f) Convert your solution back into the original coordinates to find $y_i(t)$.

$$\vec{y}(t) = V\vec{z}(t) = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix} = \begin{bmatrix} e^{-9t} + 6e^{-2t} \\ -2e^{-9t} + 9e^{-2t} \end{bmatrix}$$

original coordinates

eigenbasis coordinates