Disc 3A — "Slow Paced"

Announcements

- 16A Review Session — 5-6:30PM today
- Self-grades and resubmissions due tomorrow!
- HW 3 released, due this Friday
- Lite Lab 1 due today
- Hit me up anytime at eecs16b-sp21@berkeley.edu

1. Differential equations with piecewise constant inputs

Working through this question will help you understand better differential equations with inputs. Along the way, we will also touch a bit on going from continuous-time into a discrete-time view. This problem also provides a vehicle to review relevant concepts from calculus.

(a) Consider the scalar system

$$\overbrace{\frac{d}{dt}x(t)}^{\text{Rate of change}} = \overbrace{\lambda x(t)}^{\text{Scaled signal}} + \underbrace{u(t)}_{\text{Input}}. \tag{1}$$

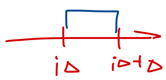
Goal: model the change in signal $x(t)$ as affected by input $u(t)$

Our goal is to solve this system (find an appropriate function $x(t)$) for general inputs $u(t)$. To do this, we will start with a piecewise constant $u(t)$; we already have the tools to solve this system, which we will do in the first few parts. Later in the worksheet, we will extend this to general $u(t)$.

Suppose that $x(t)$ is continuous (in real systems, this is almost always true). Further suppose that our input $u(t)$ of interest is piecewise constant over durations of width Δ. In other words:

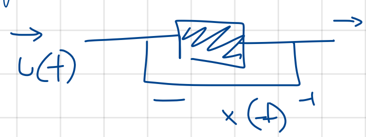
$$u(t) = u(i\Delta) = u[i] \text{ if } t \in [i\Delta, (i+1)\Delta). \tag{2}$$

In keeping with this notation, we will use the notation



$$x_d[i] = x(i\Delta).$$

Example for intuition: think of measuring the voltage $x(t)$ across some circuit element at time t, while also administering some input voltage $u(t)$ at time t.



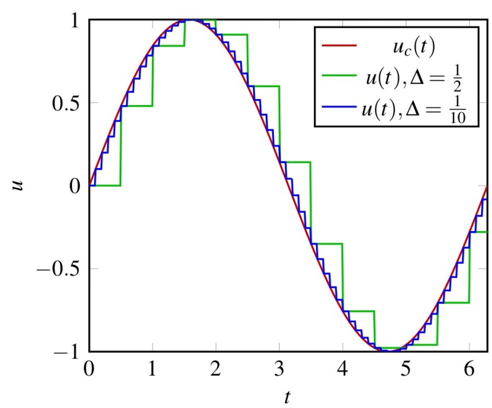


Figure 1: An example of a discrete input where the limit as the time-step Δ goes to 0 approaches a continuous function. The red line, the original signal $u_c(t) = \sin(t)$, is traced almost exactly by the blue line, which has a small time-step, and not nearly as well by the green line, which has a large time-step.

Here is a solution to this system, which may help with visual intuition:

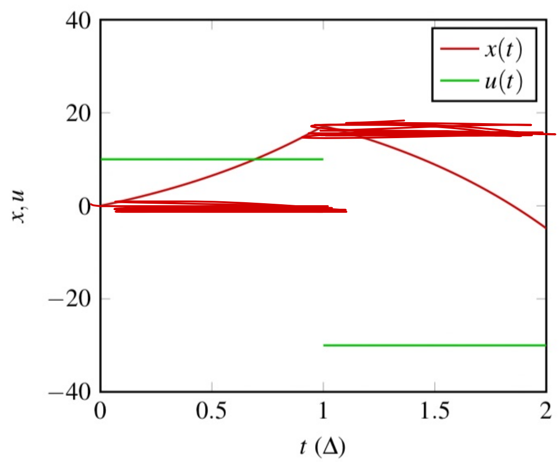


Figure 2: An example of a solution to this diff. eq. system. In this case $\lambda = 1, u[0] = 10, u[1] = -30$.

(a)

The first step to analyzing this system is to discover its behavior across a time-step with constant input, since we already know how to solve these kinds of systems.

Given that we know the value of $x(i\Delta) = x_d[i]$, compute $x_d[i+1] = x((i+1)\Delta)$.

Hint: For $t \in [i\Delta, (i+1)\Delta)$, the system is

$$\frac{d}{dt}x(t) = \lambda x(t) + u[i].$$

Notation

$x(t = i\cdot\Delta) = x_d[i]$

$x_d[i+1]$

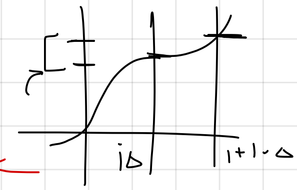
$u(i\Delta) = u[i] = u_d[i]$

$\frac{d}{dt}x(t) = \lambda x(t) + u[i]$

↳ Given we know $x(i\Delta)$,

$$x((i+1)\Delta) = \underline{x(i\Delta)} + \int_{t=i\Delta}^{t=(i+1)\Delta} \boxed{\frac{d}{dt}x(t)}\, dt$$

← overall change



From Note 2, pg 1 eq 1

Dif eq of form	$\frac{d}{dx}x(t) = \lambda x(t) + u(t)$
has soln	$x(t) = \alpha\cdot e^{\lambda(t-i\Delta)} + \beta$

α and β are fns of $x(t), u(t), \lambda$

Differential equation for time $t \in [i\Delta, (i+1)\Delta]$:

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t) = \lambda x(t) + u[i]$$

⇒ Want to fit to solution $x(t) = \alpha\cdot e^{\lambda(t-i\Delta)} + \beta$

$$\frac{d}{dt}x(t) = \lambda\cdot\alpha\cdot e^{\lambda(t-i\Delta)} = \lambda x(t) + u[i]$$

$$\Rightarrow \cancel{\lambda\cdot\alpha\, e^{\lambda(t-i\Delta)}} = \lambda\cdot(\cancel{\alpha\, e^{\lambda(t-i\Delta)}} + \beta) + u[i]$$

$$0 = \lambda\beta + u[i] \Rightarrow \boxed{\beta = -\frac{u[i]}{\lambda}}$$

Now, use initial condition:

$$\underset{\text{given}}{x(i\Delta)} = \alpha\, e^{\lambda(\overset{0}{i\Delta - i\Delta})} + \beta$$

$\alpha\cdot 1$

$$x(i\Delta) = \alpha + \beta \Rightarrow \alpha = x(i\Delta) - \beta$$

$$\boxed{\alpha = x(i\Delta) + \frac{u[i]}{\lambda}}$$

$$x(t) = \alpha\cdot e^{\lambda(t-i\Delta)} + \beta$$

$$x(t) = \left(x(i\Delta) + \frac{u[i]}{\lambda}\right)\cdot e^{\lambda(t-i\Delta)} - \frac{u[i]}{\lambda}$$

$$= x(i\Delta)\cdot e^{\lambda(t-i\Delta)} + \frac{u[i]}{\lambda}\cdot e^{\lambda(t-i\Delta)} - \frac{u[i]}{\lambda}$$

$(i+1)\Delta - i\Delta = i\Delta + \Delta - i\Delta = \Delta$

$x(t = (i+1)\cdot\Delta)$

$$= x(i\Delta)\cdot e^{\lambda(t-i\Delta)} + u[i]\cdot\frac{1}{\lambda}\left(e^{\lambda(t-i\Delta)} - 1\right)$$

$$x_d[i+1] = x_d[i]\cdot e^{\lambda(\cancel{(i+1)\Delta}^{\Delta} - i\Delta)} + u[i]\cdot\frac{1}{\lambda}\left(e^{\lambda\cdot\Delta} - 1\right)$$

$$\boxed{\Rightarrow x_d[i+1] = x_d[i]\cdot e^{\lambda\Delta} + \frac{u[i]}{\lambda}\cdot\left(e^{\lambda\Delta} - 1\right)}$$

(b) Now that we've found a one-step recurrence for $x_d[i+1]$ in terms of $x_d[i]$, we want to get an expression for $x_d[i]$ in terms of the original value $x(0) = x_d[0]$, and all the inputs u. This is so that we can eventually convert this function for $x_d[i]$ into a function for $x(t)$.

Unroll the implicit recursion you derived in the previous part to write $x_d[i+1]$ as a sum that involves $x_d[0]$ and the $u[j]$ for $j = 0, 1, \ldots, i$.

For this part, feel free to just consider the discrete-time system in a simpler form

$$x_d[i+1] = ax_d[i] + bu[i] \tag{4}$$

and you don't need to worry about what a and b actually are in terms of λ and Δ.
(*Hint*: What is $x_d[1]$ in terms of $x_d[0]$? What is $x_d[2]$ in terms of (only) $x_d[0]$? What about $x_d[3]$? Can you find a pattern?)

From previous part: $x_d[i+1] = x_d[i] \cdot \underbrace{e^{\lambda\Delta}}_{a} + u[i] \cdot \underbrace{\frac{(e^{\lambda\Delta} - 1)}{\lambda}}_{b}$

$x_d[1] = a \cdot x_d[0] + b \cdot u[0]$

$x_d[2] = a \cdot x_d[1] + b \cdot u[1] = a \cdot (a \cdot x_d[0] + b \cdot u[0]) + b \cdot u[1]$

$= a^2 \cdot x_d[0] + u[0] \cdot ab + u[1] \cdot b$

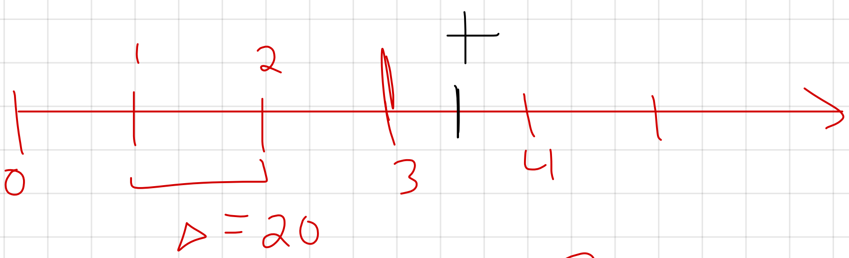
$x_d[3] = a \cdot x_d[2] + b \cdot u[2]$

$= a \cdot (a^2 \cdot x_d[0] + u[0] \cdot ab + u[1] \cdot b) + b \cdot u[2]$

$= a^3 \cdot x_d[0] + u[0] \cdot a^2 b + u[1] \cdot ab + u[2] \cdot b$

Using a summation: $\boxed{x_d[i] = a^i \cdot x_d[0] + \sum_{j=0}^{i-1} u[j] \cdot a^{i-1-j} \cdot b}$

(c) For a given time t in continuous real time, what is the discrete i interval that corresponds to it?
(*Hint*: $\lfloor x \rfloor$ is the largest integer smaller than x.)



$t = \underbrace{3.5}\cdot\underbrace{20}$

$= 70$

$i = ? = \text{floor}\left(\frac{t}{\Delta}\right) = \left\lfloor \frac{t}{\Delta} \right\rfloor$

$\boxed{i = \left\lfloor \frac{t}{\Delta} \right\rfloor}$

(d) Here's the first payoff! Use the results of part (a) and (b) to give an approximate expression for $x(t)$ for any t, in terms of $x_d[0] = x(0)$ and the inputs $u[j]$. You can assume that Δ is small enough that $x(t)$ does not change too much (is approximately constant) over an interval of length Δ.

(*Hint*: The assumption we just made allows us to approximate $x(t) \approx x\left(\Delta\left\lfloor\frac{t}{\Delta}\right\rfloor\right) = x_d\left[\left\lfloor\frac{t}{\Delta}\right\rfloor\right]$.)

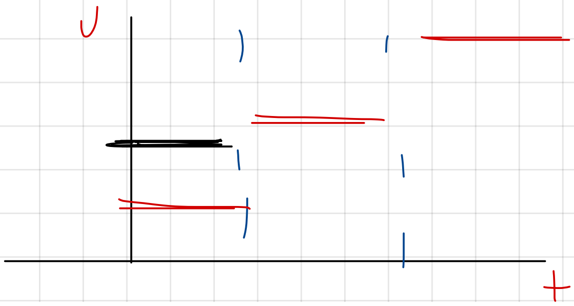
↳ from $i = \lfloor \frac{t}{\Delta} \rfloor$ ⓒ

↳ from part ⓑ: $x_d[i] = a^i x_d[0] + b\left(\sum_{j=0}^{i-1} a^{i-j-1} \cdot u[j]\right)$, $i = \lfloor \frac{t}{\Delta} \rfloor$

ⓑ $a = e^{\lambda\Delta}$, $b = (e^{\lambda\Delta} - 1)/\lambda$

$$x(t) \approx x\left[\Delta \cdot \left\lfloor \tfrac{t}{\Delta} \right\rfloor\right] = x_d\left[\left\lfloor \tfrac{t}{\Delta} \right\rfloor\right] = a^{\lfloor \frac{t}{\Delta} \rfloor} \cdot x_d[0] + b \cdot \left(\sum_{j=0}^{\lfloor \frac{t}{\Delta} \rfloor - 1} a^{\lfloor \frac{t}{\Delta} \rfloor - j - 1} \cdot u[j]\right)$$

$$\boxed{\Rightarrow x(t) \approx x_d\left[\left\lfloor \tfrac{t}{\Delta} \right\rfloor\right] = \left(e^{\lambda\Delta}\right)^{\lfloor \frac{t}{\Delta} \rfloor} \cdot x_d[0] + \frac{(e^{\lambda\Delta} - 1)}{\lambda} \cdot \left(\sum_{j=0}^{\lfloor \frac{t}{\Delta} \rfloor - 1} \left(e^{\lambda\Delta}\right)^{\lfloor t/\Delta \rfloor - j - 1} \cdot u[j]\right)}$$



λ

(e) Now, we are going to turn this around. Suppose that the $u[i]$ is actually a sample of a desired input $u_c(t)$ in continuous time. Namely, suppose that $u[i] = u_c(i\Delta)$.

To clarify, $u(t)$ is a piecewise constant function; $u[i]$ is the discrete input that constructs $u(t)$; and $u_c(t)$ is the underlying input $u[i]$ is sampled from. **Solution:** In the terms of the earlier analogy, suppose $u_c(t)$ is a voltage signal (i.e. from another circuit) being read by the computer. At time $t = i\Delta$ the computer reads the input $u_c(i\Delta) = u[i]$ and provides the piecewise constant input $u(t)$ to the original circuit like before.

Notation

$U_c(t)$: underlying signal

$u(t)$: sampled

$t \in [i\Delta, (i+1)\Delta)$

$u[i]$

The underlying goal is to find an expression for $x(t)$ in the limit $\Delta \to 0$, in terms of $u_c(t)$ and the initial condition $x(0)$. To this end, start by substituting an appropriate value of u_c for u in the result from part (d). (Note: don't take any limits in this problem; just do the substitution.)

$$\Rightarrow x(t) \approx x_d\left[\left\lfloor \tfrac{t}{\Delta} \right\rfloor\right] = \left(e^{\lambda\Delta}\right)^{\lfloor \frac{t}{\Delta} \rfloor} \cdot x_d[0] + \frac{(e^{\lambda\Delta} - 1)}{\lambda} \cdot \left(\sum_{j=0}^{\lfloor \frac{t}{\Delta} \rfloor - 1} \left(e^{\lambda\Delta}\right)^{\lfloor t/\Delta \rfloor - j - 1} \cdot u[j]\right)$$

Substitute $u[j] \longrightarrow U_c(t = j \cdot \Delta)$

$$x(t) \approx x_d\left[\left\lfloor \tfrac{t}{\Delta} \right\rfloor\right] = \left(e^{\lambda\Delta}\right)^{\lfloor \frac{t}{\Delta} \rfloor} \cdot x_d[0] + \frac{(e^{\lambda\Delta} - 1)}{\lambda} \cdot \left(\sum_{j=0}^{\lfloor \frac{t}{\Delta} \rfloor - 1} \left(e^{\lambda\Delta}\right)^{\lfloor t/\Delta \rfloor - j - 1} \cdot U_c(j\Delta)\right)$$

(f) We want to take the limit $\Delta \to 0$ of our (discrete-time) expression and thus get a continuous-time function, but right now our discrete-time expression itself is pretty complicated. Let's simplify it by making some approximations which become exact in the limit.

Further approximate the previous expression by considering the following two estimates:

i. Let $n = \lfloor \frac{t}{\Delta} \rfloor \approx \frac{t}{\Delta}$ where needed and treat $\Delta \approx \frac{t}{n}$. This is a meaningful approximation when we think about n large enough.

ii. Treat $\frac{1-e^{-\lambda\Delta}}{\lambda} \approx \Delta$. This is a meaningful approximation when we think about Δ small enough. One can derive this estimate by using Taylor's theorem from calculus, but it's not required here.

(*Hint*: Use the first estimate to get rid of "floor" terms, then use both estimates to simplify further.)

$n = \lfloor \frac{t}{\Delta} \rfloor \approx \frac{t}{\Delta}$

From part e:

$$x(t) \approx x_d\left[\lfloor \tfrac{t}{\Delta} \rfloor\right] = \left(e^{\lambda\Delta}\right)^{\lfloor \frac{t}{\Delta} \rfloor} \cdot x_d[0] + \frac{(e^{\lambda\Delta}-1)}{\lambda} \cdot \left(\sum_{j=0}^{\lfloor \frac{t}{\Delta} \rfloor - 1} \left(e^{\lambda\Delta}\right)^{\lfloor t/\Delta \rfloor - j - 1} \cdot u_c(j\Delta)\right)$$

$$\approx e^{(\lambda\Delta \cdot \frac{t}{\Delta})} \cdot x_d[0] + \frac{(e^{\lambda\Delta}-1)}{\lambda} \sum_{j=0}^{n-1} e^{\lambda\Delta \cdot (\frac{t}{\Delta} - j - 1)} \cdot u_c(j\Delta)$$

$$\approx e^{\lambda t} \cdot x_d[0] + \frac{(e^{\lambda\Delta}-1)}{\lambda} \sum_{j=0}^{n-1} e^{\lambda t - \lambda\Delta j - \lambda\Delta} \cdot u_c(j\Delta)$$

$$= e^{\lambda t} \cdot x_d[0] + \frac{(e^{\lambda\Delta}-1)}{\lambda} \cdot e^{\lambda t - \lambda\Delta} \cdot \sum e^{-\lambda\Delta j} u_c(j\Delta)$$

$$= e^{\lambda t} \cdot x_d[0] + \frac{1}{\lambda} \cdot e^{\lambda t} \cdot \left(e^{0} - e^{-\lambda\Delta}\right)$$

$$\hookrightarrow e^{\lambda t} \cdot \underbrace{\frac{1 - e^{-\lambda\Delta}}{\lambda}}_{\Delta}$$

$$\boxed{x(t) = e^{\lambda t} x_d[0] + e^{\lambda t} \Delta \cdot \sum e^{-\lambda\Delta j} u_c(j\Delta)}$$

in next few subproblems...

$$\boxed{x(t) = e^{\lambda t} x_d[0] + e^{\lambda t} \int_0^t e^{-\lambda t} u_c(\tau)\, d\tau}$$

(g) Here's our second payoff! We now obtain a continuous-time expression for $x(t)$, completing the transition into continuous-time. Take the limit of $x(t)$ as $\Delta \to 0$ or equivalently as $n \to \infty$. What is the expression you get for $x(t)$?

(*Hint*: Remember your definition of definite integrals as limits of Riemann sums in calculus.)