EECS 16B Spring 2021 — Designing Information Devices and Systems II — Discussion Worksheet

Discussion 3A

For this discussion, Note 2 is helpful.

1. Differential equations with piecewise constant inputs

Working through this question will help you understand better differential equations with inputs. Along the way, we will also touch a bit on going from continuous-time into a discrete-time view. This problem also provides a vehicle to review relevant concepts from calculus.

(a) Consider the scalar system

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t). \tag{1}$$

Our goal is to solve this system (find an appropriate function $x(t)$) for general inputs $u(t)$. To do this, we will start with a piecewise constant $u(t)$; we already have the tools to solve this system, which we will do in the first few parts. Later in the worksheet, we will extend this to general $u(t)$.

Suppose that $x(t)$ is continuous (in real systems, this is almost always true). Further suppose that our input $u(t)$ of interest is piecewise constant over durations of width Δ. In other words:

$$u(t) = u(i\Delta) = u[i] \text{ if } t \in [i\Delta, (i+1)\Delta). \tag{2}$$

In keeping with this notation, we will use the notation

$$x_d[i] = x(i\Delta).$$

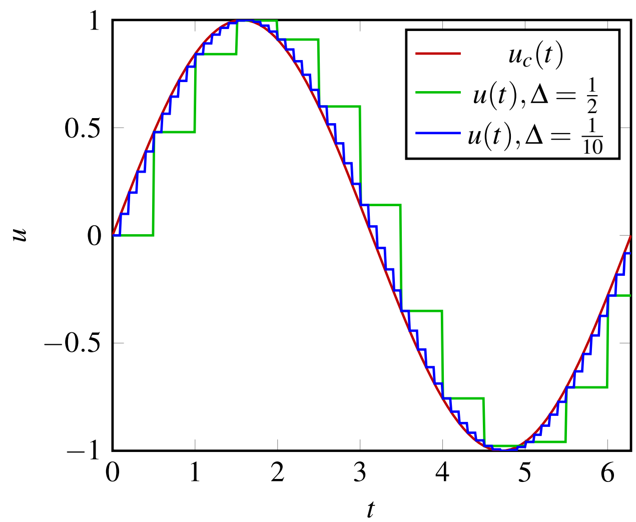

Figure 1: An example of a discrete input where the limit as the time-step Δ goes to 0 approaches a continuous function. The red line, the original signal $u_c(t) = \sin(t)$, is traced almost exactly by the blue line, which has a small time-step, and not nearly as well by the green line, which has a large time-step.

The first step to analyzing this system is to discover its behavior across a time-step with constant input, since we already know how to solve these kinds of systems.

Given that we know the value of $x(i\Delta) = x_d[i]$, compute $x_d[i+1] = x((i+1)\Delta)$.

Hint: For $t \in [i\Delta, (i+1)\Delta)$, the system is

$$\frac{d}{dt}x(t) = \lambda x(t) + u[i].$$

Also see Note 2.

Here is a solution to this system, which may help with visual intuition:

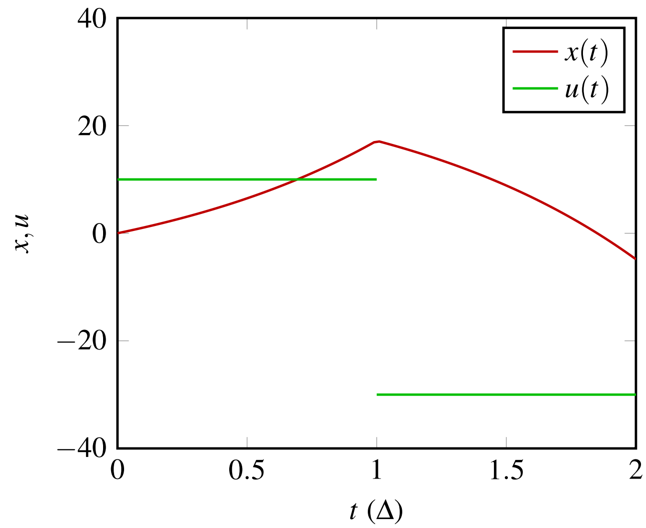

Figure 2: An example of a solution to this diff. eq. system. In this case $\lambda = 1, u[0] = 10, u[1] = -30$.

(b) Now that we've found a one-step recurrence for $x_d[i+1]$ in terms of $x_d[i]$, we want to get an expression for $x_d[i]$ in terms of the original value $x(0) = x_d[0]$, and all the inputs u. This is so that we can eventually convert this function for $x_d[i]$ into a function for $x(t)$.

Unroll the implicit recursion you derived in the previous part to write $x_d[i+1]$ as a sum that involves $x_d[0]$ and the $u[j]$ for $j = 0, 1, \ldots, i$.

For this part, feel free to just consider the discrete-time system in a simpler form

$$x_d[i+1] = ax_d[i] + bu[i] \tag{3}$$

and you don't need to worry about what a and b actually are in terms of λ and Δ.
(*Hint*: What is $x_d[1]$ in terms of $x_d[0]$? What is $x_d[2]$ in terms of (only) $x_d[0]$? What about $x_d[3]$? Can you find a pattern?)

(c) For a given time t in continuous real time, what is the discrete i interval that corresponds to it?
(*Hint*: $\lfloor x \rfloor$ is the largest integer smaller than x.)

(d) Here's the first payoff! Use the results of part (a) and (b) to give an approximate expression for $x(t)$ for any t, in terms of $x_d[0] = x(0)$ and the inputs $u[j]$. You can assume that Δ is small enough that $x(t)$ does not change too much (is approximately constant) over an interval of length Δ.

(*Hint*: The assumption we just made allows us to approximate $x(t) \approx x\left(\Delta\left\lfloor \frac{t}{\Delta} \right\rfloor\right) = x_d\left[\left\lfloor \frac{t}{\Delta} \right\rfloor\right]$.)

(e) Now, we are going to turn this around. Suppose that the $u[i]$ is actually a sample of a desired input $u_c(t)$ in continuous time. Namely, suppose that $u[i] = u_c(i\Delta)$.

To clarify, $u(t)$ is a piecewise constant function; $u[i]$ is the discrete input that constructs $u(t)$; and $u_c(t)$ is the underlying input $u[i]$ is sampled from.

The underlying goal is to find an expression for $x(t)$ in the limit $\Delta \to 0$, in terms of $u_c(t)$ and the initial condition $x(0)$. To this end, start by substituting an appropriate value of u_c for u in the result from part (d). (Note: don't take any limits in this problem; just do the substitution.)

(f) We want to take the limit $\Delta \to 0$ of our (discrete-time) expression and thus get a continuous-time function, but right now our discrete-time expression itself is pretty complicated. Let's simplify it by making some approximations which become exact in the limit.

Further approximate the previous expression by considering the following two estimates:

i. Let $n = \left\lfloor \frac{t}{\Delta} \right\rfloor \approx \frac{t}{\Delta}$ where needed and treat $\Delta \approx \frac{t}{n}$. This is a meaningful approximation when we think about n large enough.

ii. Treat $\frac{1-e^{-\lambda\Delta}}{\lambda} \approx \Delta$. This is a meaningful approximation when we think about Δ small enough. One can derive this estimate by using Taylor's theorem from calculus, but it's not required here.

(*Hint*: Use the first estimate to get rid of "floor" terms, then use both estimates to simplify further.)

(g) Here's our second payoff! We now obtain a continuous-time expression for $x(t)$, completing the transition into continuous-time. Take the limit of $x(t)$ as $\Delta \to 0$ or equivalently as $n \to \infty$. What is the expression you get for $x(t)$?

(*Hint*: Remember your definition of definite integrals as limits of Riemann sums in calculus.)

(h) Verify the analytic solution for $x(t)$ found in (g) for $u(t) = 0$ and for $u(t) = u_0$.

(i) Verify the analytic solution found in (g) by plugging it back into the differential equation.

(j) If input $u(t)$ is the linear sum of two other inputs $u(t) = c_1u_1(t) + c_2u_2(t)$, what does the solution look like? What does this mean?

This problem brings together many different concepts and uses a lot of notation. As such, it may be difficult to fully comprehend everything the first time. *It is perfectly fine* to go back and spend more time on the problem until you completely understand it. Being able to quickly analyze complex mathematical problems like this is part of the vaunted "mathematical maturity" that this class helps you foster. As the semester continues, you will find that these kinds of problems will seem progressively easier, both to understand quickly and to solve. But it won't happen without practice.

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