EECS 16B Spring 2021 — Designing Information Devices and Systems II — Discussion Worksheet

Discussion 12A

In this discussion, we practice computing the SVD for a "wide" matrix (more columns than rows) and for a "tall" matrix (more rows than columns). There is also an associated jupyter notebook on Datahub that will serve useful to confirm the numerical calculations (specifically for performing Gram-Schmidt).

Also, note that the techniques and insights communicated in this discussion are conveyed in Note 13, sec. 3.

1. Computing the SVD: A "Tall" Matrix Example

Define the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}.$$

(a) In this part, we will find the full SVD of A in steps.

Answer: In this subpart to calculate the full SVD, we will follow the algorithm of Note 13, sec. 3. We select Method 1 (computing using $A^\top A$) since A is "tall", and $A^\top A$ is smaller than $AA^\top$.

(i) Compute $A^\top A$ and find its eigenvalues.

Answer: First, we compute

$$A^\top A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \tag{1}$$

$$= \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}. \tag{2}$$

The eigenvalues of $A^\top A$ are the roots of $(\lambda - 9)^2 - 81 = 0$; $\lambda_1 = 18$, and $\lambda_2 = 0$.

(ii) Find orthonormal eigenvectors $\vec{v}_i$ (right singular vectors, columns of V).

Answer: We can find the corresponding (unit) eigenvectors for the above eigenvalues in the usual way, by computing null $\left(A^\top A - \lambda_1 I\right)$ and null $\left(A^\top A - \lambda_2 I\right)$. This yields that:

$$\vec{v}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}. \tag{3}$$

(iii) Find singular values, $\sigma_i = \sqrt{\lambda_i}$.

Answer: A has one nonzero singular value $\sqrt{18} = 3\sqrt{2}$, and the other singular value is zero.

(iv) Use $\vec{v}_i$ to find orthonormal $\vec{u}_i$ (for nonzero σ).

Answer: We obtain:

$$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix} \tag{4}$$

To complete a basis of $\mathbb{R}^3$ as required for the full SVD, we can do Gram-Schmidt using the Jupyter notebook to get

$$\vec{u}_2 = \begin{bmatrix} \frac{\sqrt{8}}{3} \\ \frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \end{bmatrix} \qquad \vec{u}_3 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \tag{5}$$

(v) Use the previous parts to write the full SVD of A.

Answer: Finally, we compose this information, and write that A can be decomposed as:

$$A = \underbrace{3\sqrt{2}\begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{\text{compact SVD}} = \underbrace{\begin{bmatrix} -\frac{1}{3} & \frac{\sqrt{8}}{3} & 0 \\ \frac{2}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{2}{3} & -\frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{\text{full SVD}}. \tag{6}$$

The full SVD representation of A is given below. Note that the full SVD and compact SVD represent the same matrix; the compact form merely omits the columns/rows of U or V which will hit the zero entries of Σ.

(vi) Use the Jupyter notebook to run the code cell that calls `numpy.linalg.svd` on A. What is the result? Does it match our result above?

Answer: The SVD that Jupyter notebook gives is different because of the non-uniqueness of Gram-Schmidt. We can extend a given set of vectors to an orthonormal basis in an infinite number of ways, so the SVD is not unique. Furthermore, it is important to note that the extended columns of U only ever multiply with the zero-entries of Σ. So, they cannot impact the final result of A. However, it is still critical that all the columns of U are in fact mutually orthogonal and normalized.

(b) Find the rank of A.

Answer: A has 1 nonzero singular value. So A has rank 1.

(c) Find a basis for the range (or column space) of A.

Answer: We know if $A = U\Sigma V^\top$ is an SVD, then the columns of U with nonzero corresponding singular values are a basis for the column space of A. Any columns corresponding to $\sigma = 0$ cannot add to the span. Therefore, matching terms with the SVD of A,

$$\text{range}(A) = \text{span}\left\{\begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}\right\}$$

(d) Find a basis for the null space of A.

Answer: We know if $A = U\Sigma V^\top$ is a compact SVD, then the columns of V with corresponding singular values equal to 0 are a basis for the null space of A. Thus

$$\text{null}(A) = \text{span}\left\{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\right\}$$

(e) We now want to create the SVD of $A^\top$. Rather than repeating all of the steps in the algorithm, feel free to use the jupyter notebook for this subpart (which defines a `numpy.linalg.svd` command). What are the relationships between the matrices composing A and the matrices composing $A^\top$?

Answer: We know that A has an SVD representation of $U\Sigma V^\top$ as we solved for above. One natural approach to solving for the SVD of $A^\top$ is to take the transpose of the SVD terms, and "reassign variables". That is, we can say that $A^\top$ has SVD $U_\star \Sigma_\star V_\star^\top$, and to find how these new $\star$ variables relate to the originals, we write:

$$A^\top = \left(U\Sigma V^\top\right)^\top = V\Sigma^\top U^\top$$

Now, pattern-matching, we can say that $U_\star = V$, $\Sigma_\star = \Sigma^\top$, $V_\star^\top = U^\top \implies V_\star = U$. Note how the roles have exchanged, and Σ is transposed.

We can write now write the full SVD of $A^\top$ (feel free to confirm that the multiplication yields the right result):

$$A^\top = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{\sqrt{8}}{3} & \frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

2. Computing the SVD: A "Wide" Matrix Example

Define the matrix

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}.$$

(a) In this part, we will find the full SVD of A in steps.

Answer: In this subpart to calculate the full SVD, we will follow the algorithm of Note 13, sec. 3. We select Method 2 (computing using $AA^\top$) since A is "wide", and $AA^\top$ is smaller than $A^\top A$.

(i) Compute $AA^\top$ and find its eigenvalues.

Answer:

$$AA^\top = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix} \tag{8}$$

Next, we find the eigenvalues of the above matrix.

$$\det(A - \lambda I) = \lambda^2 - 34\lambda + 225 = (\lambda - 25)(\lambda - 9) = 0$$

Hence, the eigenvalues are $\lambda_1 = 25$ and $\lambda_2 = 9$.

(ii) Find orthonormal eigenvectors $\vec{u}_i$ (left singular vectors, columns of U). Feel free to use the associated Jupyter notebook to perform Gram-Schmidt for this part, if needed.

Answer: Finding $\text{null}\left(AA^\top - \lambda_1 I\right)$ and $\text{null}\left(AA^\top - \lambda_2 I\right)$ will give us $\vec{u}_1$ and $\vec{u}_2$ respectively.

Hence, $\vec{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$.

(iii) Find the singular values, $\sigma_i = \sqrt{\lambda_i}$.

Answer: The singular values are $\sigma_1 = \sqrt{25} = 5$ and $\sigma_2 = \sqrt{9} = 3$.

(iv) Use $\vec{u}_i$ to find orthonormal $\vec{v}_i$ (for nonzero σ). Feel free to use the associated Jupyter notebook to perform Gram-Schmidt for this part, if needed.

Answer: These right singular vectors compose the columns of V:

$$\vec{v}_1 = \frac{1}{\sigma_1} A^\top u_1 \tag{9}$$

$$= \frac{1}{5} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \tag{10}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \tag{11}$$

Similarly, we get $\vec{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \end{bmatrix}$. Extending to the $\mathbb{R}^3$ basis using Gram-Schmidt (as required for the full SVD), we get $\vec{v}_3 = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$. We compose $V^\top$ using these vectors.

(v) Use the previous parts to write the full SVD of A.

Answer: So the full SVD representation of A is given below (along with the compact SVD for comparison). Note that the full SVD and compact SVD represent the same matrix; the compact form merely omits the columns/rows of U or V which will hit the zero entries of Σ.

$$A = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{18}} & \frac{4}{\sqrt{18}} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}}_{\text{full SVD}} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{18}} & \frac{4}{\sqrt{18}} \end{bmatrix}}_{\text{compact SVD}} \tag{12}$$

(b) Find the rank of A, using the computed full SVD.

Answer: Since there are 2 nonzero singular values, the rank of A is 2.

We could also directly observe A, and note that we have 3 columns, but the first 2 are linearly independent, and they therefore span $\mathbb{R}^2$. The last column must lie in this span and is therefore linearly dependent with the first 2 columns.

(c) Find a basis for the range (or columnspace) of A.

Answer: The columns of U corresponding to nonzero singular values will form $\text{range}(A)$. Here, we pattern-match to the SVD we derived, and find that it's both column vectors, so we write that:

$$\text{range}(A) = \text{span}\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$$

(d) Find a basis for the nullspace of A.

Answer: The nullspace can be found using the rows of $V^\top$ (columns of V) which correspond to zero singular values. Here, this is the 3rd row of $V^\top$. So we write that:

$$\text{null}(A) = \text{span}\left\{ \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \right\}$$

Note: The nullspace of A is given by the orthogonal complement of space spanned by the columns of the V matrix. Hence, we can find the nullspace by finding the eigenvector(s) corresponding to the zero eigenvalue of the $A^\top A$ matrix as well.

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