EECS 16B Spring 2021 — Designing Information Devices and Systems II — Discussion Worksheet

Discussion 1B

1 KVL/KCL Review

Kirchhoff's Circuit Laws are two important laws used for analyzing circuits. Kirchhoff's Current Law (KCL) says that the sum of all currents entering a node must equal 0. For example, in Figure 1, the sum of all currents entering node 1 is $I_1 - I_2 - I_3 = 0$. Assuming that I_1 and I_3 are known, we can easily obtain a solvable equation for V_x by applying Ohm's law: $I_1 - \frac{V_x}{R_1} - I_3 = 0$.

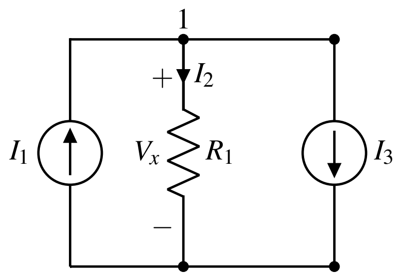


Figure 1: KCL Circuit

Kirchhoff's Voltage Law (KVL) states that the sum of all voltages in a circuit loop must equal 0. To apply KVL to the circuit shown in Figure 2, we can add up voltages in the loop in the counterclockwise direction, which yields $-V_1 + V_x + V_y = 0$. Using the relationships $V_x = i \cdot R_1$ and $i = I_1$, we can solve for all unknowns in this circuit. You can use these two laws to solve any circuit that is planar and linear.

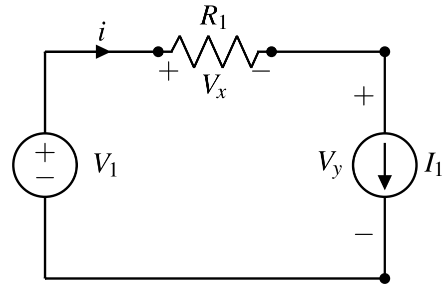


Figure 2: KVL Circuit

If you would like to review these concepts more in-depth, you can check out the EECS16A Fall 2020 course notes.

2 Op-amp Review

Figure 3 shows the equivalent model of an op-amp. It is important to note that this is the general model of an op-amp, so our op-amp golden rules cannot be applied to this unless certain conditions are met.

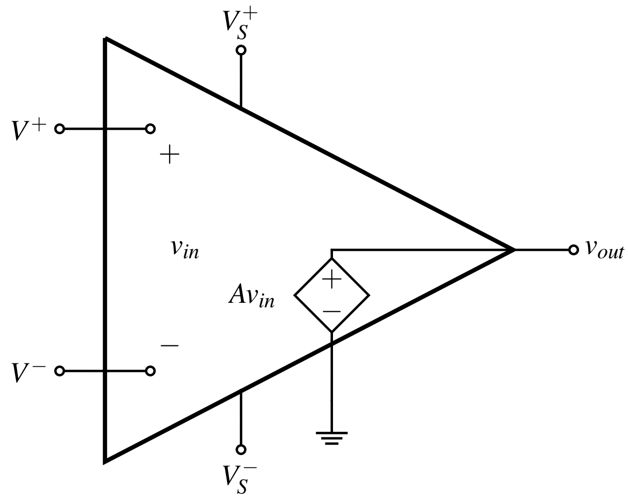


Figure 3: General Op-Amp Model

Conditions Required for the Golden Rules:

(a) $R_{in} \to \infty$

(b) $R_{out} \to 0$

(c) $A \to \infty$

(d) The op-amp must be operated in negative feedback.

When conditions (a)-(c) are met, the op-amp is considered ideal. Figure 4 shows an ideal op-amp in negative feedback, which can be analyzed using the Golden Rules.

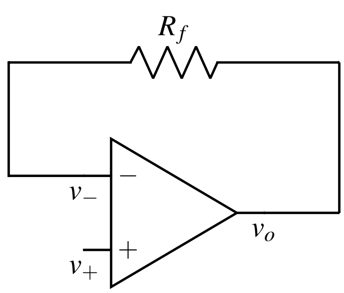


Figure 4: Ideal Op-Amp in Negative Feedback

Golden Rules of ideal op-amps in negative feedback:

(a) No current can flow into the input terminals ($I_- = 0$ and $I_+ = 0$). (This property follows from condition (a) and does not require negative feedback.)

(b) The $(+)$ and $(-)$ terminals are at the same voltage ($V_+ = V_-$).

If you would like to review these concepts more in-depth, you can check out op-amp introduction and op-amp negative feedback from the EECS16A course notes.

1. KVL/KCL Review

Use Kirchhoff's Laws on the circuit below to find V_x in terms of V_{in}, R_1, R_2, R_3.

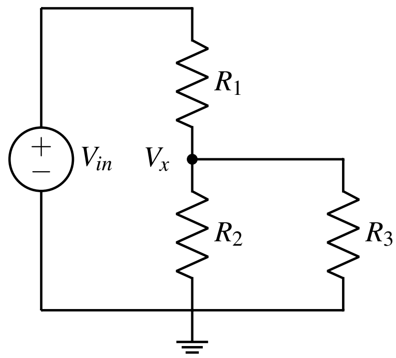


Figure 5: Example Circuit

(a) What is V_x?

Answer:

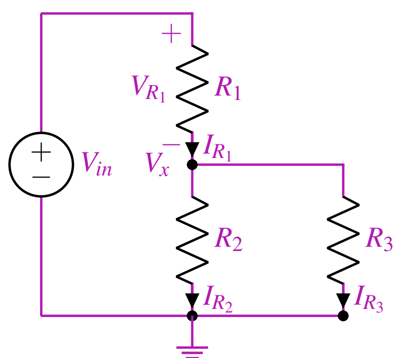


Figure 6

Applying KCL to the node at V_x, we get

$$\frac{V_{in} - V_x}{R_1} - \frac{V_x - 0}{R_2} - \frac{V_x - 0}{R_3} = 0$$

Solving this equation for V_x yields

$$V_x = V_{in} \frac{R_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

(b) As $R_3 \to \infty$, what is V_x? What is the name we used for this type of circuit?

Answer:

As $R_3 \to \infty$, the R_1R_2 term on the denominator will become insignificant, simplifying our expression.

$$\begin{aligned}\lim_{R_3\to\infty} V_x &= \lim_{R_3\to\infty} V \frac{R_2R_3}{R_1R_2+R_1R_3+R_2R_3} \\ &= V_{in}\frac{R_2R_3}{R_1R_3+R_2R_3} \\ &= V_{in}\frac{(R_2)R_3}{(R_1+R_2)R_3} \\ &= V_{in}\frac{R_2}{R_1+R_2}\end{aligned}$$

When $R_3 \to \infty$, it effectively becomes an open wire, which makes the rest of the circuit a voltage divider, or resistive divider.

2. Op-Amp Summer

Consider the following circuit (assume the op-amp is ideal):

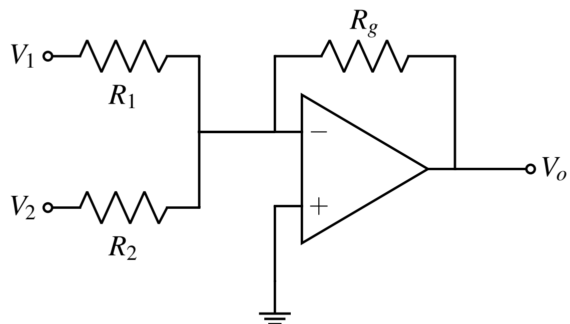


Figure 7: Op-amp Summer

What is the output V_o in terms of V_1 and V_2? You may assume that R_1, R_2, and R_g are known.

Answer:

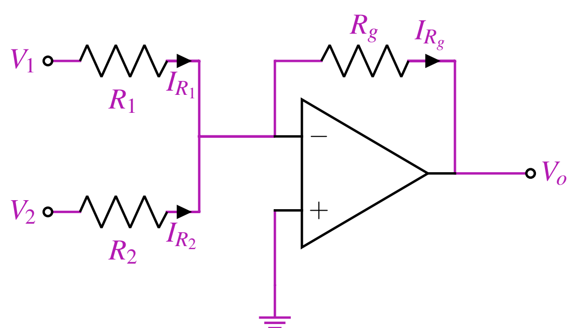


Figure 8

Let I_- be the current flowing into the (-) terminal of the op-amp

$$I_{R_g} + I_- = \frac{V_1}{R_1} + \frac{V_2}{R_2} = I_{R_g}$$

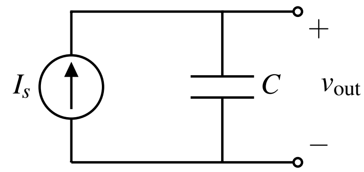
$$V_o = -R_g I_{R_g}$$

$$V_o = -\left(\frac{R_g}{R_1} \cdot V_1 + \frac{R_g}{R_2} \cdot V_2\right)$$

3. Current Sources And Capacitors (The following problem has been adapted from EECS16A Fall 20 Disc 9A.)

Recall charge has units of Coulombs (C), and capacitance is measured in Farads (F) $= \frac{\text{Coulomb}}{\text{Volt}}$.
It may also help to note metric prefix examples: $3\,\mu\text{F} = 3 \times 10^{-6}\text{F}$.

Given the circuit below, find an expression for $v_{\text{out}}(t)$ in terms of I_s, C, V_0, and t, where V_0 is the initial voltage across the capacitor at $t = 0$.



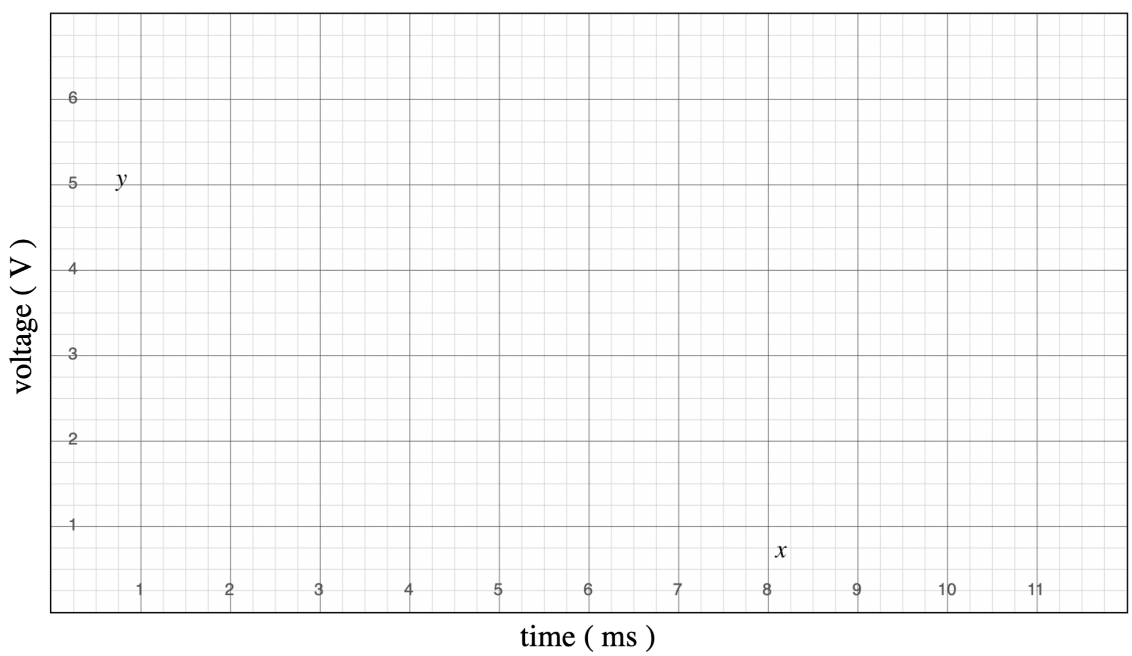
Then plot the function $v_{\text{out}}(t)$ over time on the graph below for the following conditions detailed below. Use the values $I_s = 1\text{mA}$ and $C = 2\,\mu\text{F}$.

(a) Capacitor is initially uncharged $V_0 = 0$ at $t = 0$.

(b) Capacitor has been charged with $V_0 = +1.5\text{V}$ at $t = 0$.

(c) **Practice:** Swap this capacitor for one with half the capacitance $C = 1\,\mu\text{F}$, which is initially uncharged $V_0 = 0$ at $t = 0$.

HINT: Recall the calculus identity $\int_a^b f'(x)dx = f(b) - f(a)$, where $f'(x) = \frac{df}{dx}$.



Answer:
The key here is to exploit the capacitor equation by taking its time-derivative

$$Q = C\,V_{out} \quad \longrightarrow \quad \frac{dQ}{dt} \equiv I_s = C\frac{dV_{out}}{dt}\ .$$

From here we can rearrange and show that
$$\frac{dV_{out}}{dt} = \frac{I_s}{C}$$
.
Thus the voltage has a constant slope!
Our solution is
$$V_{out}(t) = V_0 + \left(\frac{I_s}{C}\right) \cdot t$$

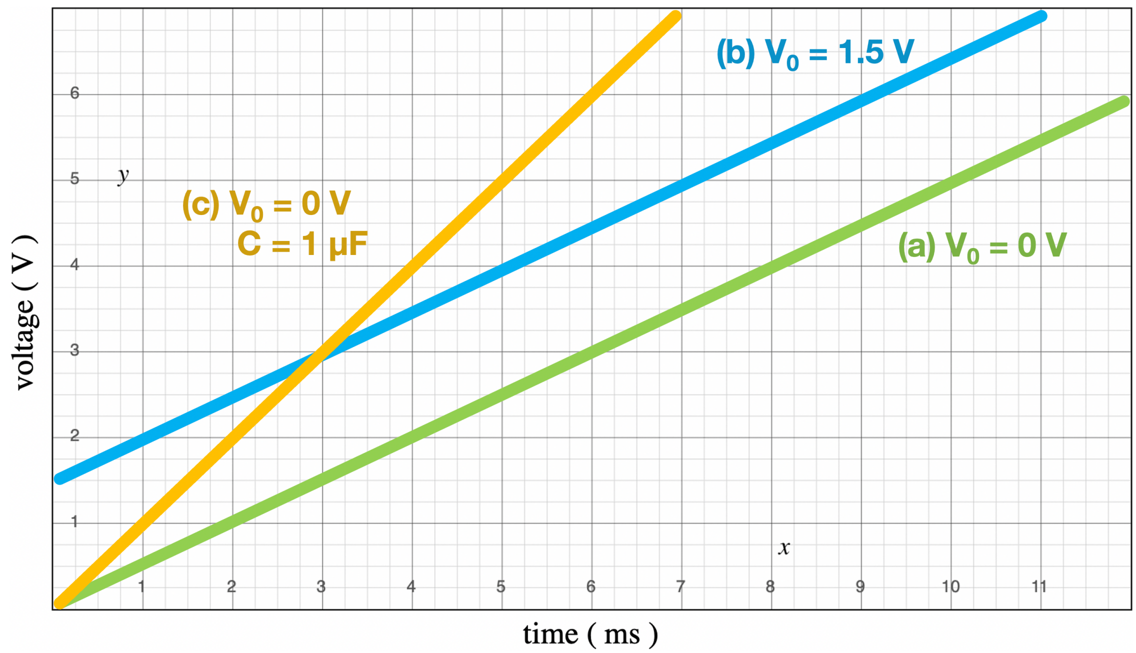
To be more mathematically formal, we are solving a differential equation that happens to return a linear function for $v_{out}(t)$:
$$\frac{dV_{out}}{dt} = \frac{I_s}{C} \quad \longrightarrow \quad \int_0^t \frac{dV_{out}}{dt} dt \equiv V_{out}(t) - V_{out}(0) \quad = \quad \int_0^t \frac{I_s}{C} dt \equiv \frac{I_s}{C} \int_0^t 1 dt \equiv \frac{I_s}{C} t$$

Thus we arrive at the same statement as seen earlier $V_{out}(t) = V_{out}(0) + \left(\frac{I_s}{C}\right) t$.

From this stage we can compute the slope of $V_{out}(t)$ for parts (a) and (b) along with the slope for (c), which should be twice as large.
$$\frac{I_s}{C} = \frac{1\text{mA}}{2\mu\text{F}} = \frac{1000\frac{\mu\text{C}}{\text{s}}}{2\frac{\mu\text{C}}{\text{V}}} = 500\frac{\text{V}}{\text{s}} = \left(\frac{1}{2}\right)\frac{\text{V}}{\text{ms}}$$

For part (c):
$$\frac{I_s}{C} = \frac{1\text{mA}}{1\mu\text{F}} = \frac{1000\frac{\mu\text{C}}{\text{s}}}{1\frac{\mu\text{C}}{\text{V}}} = 1000\frac{\text{V}}{\text{s}} = 1\frac{\text{V}}{\text{ms}}$$

When plotting, make sure to recall (a) and (c) start at the origin, while (b) has initially charged plates by $V_0 = 1.5V$. Results are shown below



4. Linear Algebra Review

For the following matrices, find the following properties:

i. What is the column space of the matrix?
ii. What is the null space of the matrix?
iii. What are the eigenvalues and corresponding eigenspaces for the matrix?

(a) $\begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$

Answer:

i. $\mathbb{R}^2$
ii. 0
iii. $\lambda_1 = 2$, $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = 3$, $v_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$

Answer:

i. $\text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$
ii. $\text{span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$
iii. $\lambda_1 = -3$, $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda_2 = 0$, $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$

For this matrix you are told that the eigenvalues are: 2, 1, and 0.

Answer:

i. $\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$
ii. $\text{span}\left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right)$

iii. $\lambda_1 = 2,\ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_2 = 1,\ v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_3 = 0,\ v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$