Homework 13

This homework is due on Friday, December 2, 2022 at 11:59PM. Self-grades and HW Resubmissions are due the following Friday, December 9, 2022 at 11:59PM.

1. **Linearizing for understanding amplification**

   Linearization isn't just something that is important for control, robotics, machine learning, and optimization — it is one of the standard tools used across different areas, including circuits.

   The circuit below is a voltage amplifier, where the element inside the box is a bipolar junction transistor (BJT). You do not need to know what a BJT is to do this question.

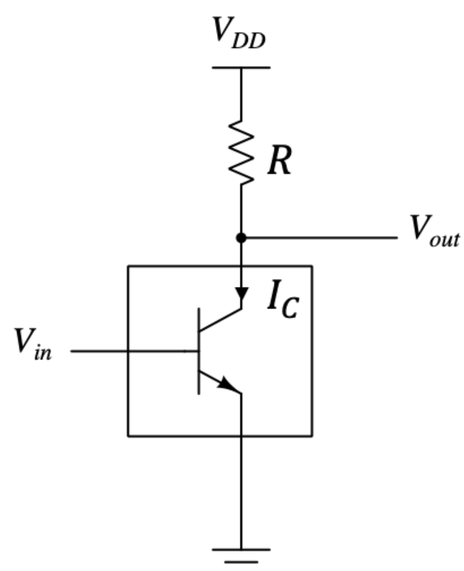

Figure 1: Voltage amplifier circuit using a BJT

   The BJT in the circuit can be modeled quite accurately as a nonlinear, voltage-controlled current source, where the collector current I_C is given by:

$$I_C(V_{\text{in}}) = I_S \cdot e^{\frac{V_{\text{in}}}{V_{\text{TH}}}}, \tag{1}$$

   where V_{TH} is the thermal voltage. We can assume $V_{\text{TH}} = 26\,\text{mV}$ at room temperature. I_S is a constant whose exact value we are not giving you because we want you to find ways of eliminating it in favor of other quantities whenever possible.

   The goal of this circuit is to pick a particular point $(V_{\text{in}}^\star, V_{\text{out}}^\star)$ so that any small variation δV_{in} in the input voltage V_{in} can be amplified to a relatively larger variation δV_{out} in the output voltage V_{out}. In other words, if $V_{\text{in}} = V_{\text{in}}^\star + \delta V_{\text{in}}$ and $V_{\text{out}} = V_{\text{out}}^\star + \delta V_{\text{out}}$, then we want the magnitude of the 'amplification gain' given by $\left|\frac{\delta V_{\text{out}}}{\delta V_{\text{in}}}\right|$ to be large. We're going to investigate this amplification using linearization.

(NOTE: in this problem, δV is single variable indicating a small variation in V, not $\delta \times V$.)

(a) **Write a symbolic expression for V_{out} as a function of I_C, V_{DD} and R in Fig 1.**

Solution:

$$V_{\text{out}} = V_{\text{DD}} - RI_C \tag{2}$$

since we have a voltage drop of $I_C R$ across the resistor and the top voltage is V_{DD}.

(b) Now let's linearize I_C in the neighborhood of an input voltage $V_{\text{in}}^\star$ and a specific $I_C^\star$. Assume that you have a found a particular pair of input voltage $V_{\text{in}}^\star$ and current $I_C^\star$ that satisfy the current equation (1).

We can look at nearby input voltages and see how much the current changes. We can write the linearized expression for the collector current around this point as:

$$I_C(V_{\text{in}}) = I_C(V_{\text{in}}^\star) + g_m(V_{\text{in}} - V_{\text{in}}^\star) = I_C^\star + g_m\,\delta V_{\text{in}} \tag{3}$$

where $\delta V_{\text{in}} = V_{\text{in}} - V_{\text{in}}^\star$ is the change in input voltage, and g_m is the slope of the local linearization around $(V_{\text{in}}^\star, I_C^\star)$. **What is g_m here as a function of $I_C^\star$ and V_{TH}?**

(HINT: Find g_m by taking the appropriate derivative around the operating point. You should recognize a part of your equation is equal to the current operating point $I_C^\star = I_C(V_{\text{in}}^\star)$, so your final form should not depend on I_S. Also, note that in circuits terminology, "operating point" is defined to be the point around which we linearize input-output relationship.)

Solution: We start out by writing out the linearization form that we are looking for:

$$I_C(V_{\text{in}}) = I_C^\star + g_m\,\delta V_{\text{in}} \tag{4}$$

Now, taking the first derivative of I_C around $V_{\text{in}}^\star$:

$$g_m = \frac{\mathrm{d}I_C(V_{\text{in}}^\star)}{\mathrm{d}V_{\text{in}}} \tag{5}$$

$$= \frac{1}{V_{\text{TH}}} I_S e^{\frac{V_{\text{in}}\star}{V_{\text{TH}}}} \tag{6}$$

$$= \frac{I_C^\star}{V_{\text{TH}}} \tag{7}$$

where in the last line, we recognize that $I_C^\star = I_S e^{\frac{V_{\text{in}}\star}{V_{\text{TH}}}}$, and therefore knowledge of I_S is not required to determine g_m if $I_C^\star$ and V_{TH} are known.

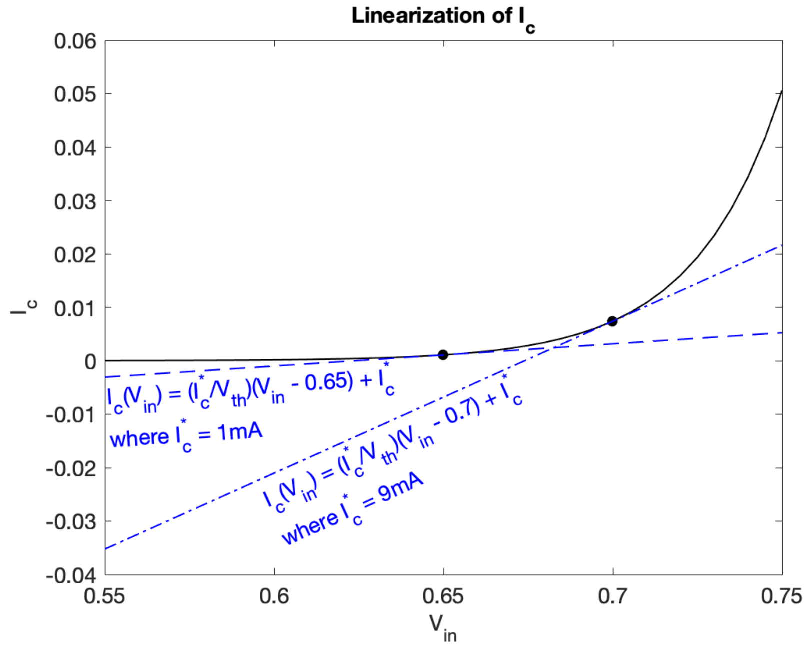

Figure 2: Linearization of the non linear I_C (black curve)

For understanding this linearization graphically, we can choose to plot I_C vs. V_{in} and look at how the slope (i.e. g_m) changes with different values of $(V_{\text{in}}^{\star}, I_C^{\star})$. In Fig. 2, we see the linearizations around $V_{\text{in}}^{\star} = 0.65\,\text{V}$ ($I_C^{\star} = 1\,\text{mA}$) and $V_{\text{in}}^{\star} = 0.7\,\text{V}$ ($I_C^{\star} = 9\,\text{mA}$) given in part (d) below.

(c) We now have a linear relationship between small changes in current and voltage, $\delta I_C = g_m\,\delta V_{\text{in}}$ around a known solution $(V_{\text{in}}^{\star}, I_C^{\star})$.

As a reminder, the goal of this problem is to pick a particular point $(V_{\text{in}}^{\star}, V_{\text{out}}^{\star})$ so that any small variation δV_{in} in the input voltage V_{in} can be amplified to a relatively larger variation δV_{out} in the output voltage V_{out}. In other words, if $V_{\text{in}} = V_{\text{in}}^{\star} + \delta V_{\text{in}}$ and $V_{\text{out}} = V_{\text{out}}^{\star} + \delta V_{\text{out}}$, then we want the magnitude of the "amplification gain" given by $\left|\frac{\delta V_{\text{out}}}{\delta V_{\text{in}}}\right|$ to be large.

Plug in your linearized equation for I_C in the answer from part (a). It may help to define the output voltage operating point as $V_{\text{out}}^{\star}$, where

$$V_{\text{out}}^{\star} = V_{\text{DD}} - RI_C^{\star} \tag{8}$$

so that we can view $V_{\text{out}} = V_{\text{out}}^{\star} + \delta V_{\text{out}}$ when we have $V_{\text{in}} = V_{\text{in}}^{\star} + \delta V_{\text{in}}$.

Find the linearized relationship between δV_{out} and δV_{in}. The ratio $\frac{\delta V_{\text{out}}}{\delta V_{\text{in}}}$ is called the "small-signal voltage gain" of this amplifier around this operating point.

Solution: We have two equations for V_{out}:

$$V_{\text{out}} = V_{\text{out}}^{\star} + \delta V_{\text{out}} \tag{9}$$

and

$$V_{\text{out}} = V_{\text{DD}} - RI_C \tag{10}$$

We know from equation (3) that $I_C = I_C^{\star} + g_m \delta V_{\text{in}}$, so we can re-write the above two equations as:

$$V_{\text{out}}^{\star} + \delta V_{\text{out}} = V_{\text{DD}} - R(I_C^{\star} + g_m \delta V_{\text{in}}) \tag{11}$$

We also know the output voltage operating point $V_{\text{out}}^{\star}$ is related to the current operating point $I_C^{\star}$ as $V_{\text{out}}^{\star} = V_{\text{DD}} - RI_C^{\star}$, hence:

$$V_{\text{DD}} - RI_C^{\star} + \delta V_{\text{out}} = V_{\text{DD}} - R(I_C^{\star} + g_m \delta V_{\text{in}}) \tag{12}$$

$$\Rightarrow \delta V_{\text{out}} = -Rg_m \delta V_{\text{in}} \tag{13}$$

We re-arrange and solve for the small-signal voltage gain:

$$\frac{\delta V_{\text{out}}}{\delta V_{\text{in}}} = -Rg_m = -\frac{RI_C^{\star}}{V_{\text{TH}}} \tag{14}$$

You are not required to simplify it beyond this point. However, recognize that $I_C^{\star}R = V_{\text{DD}} - V_{\text{out}}^{\star}$, so we can relate the small-signal voltage gain directly to the output voltage operating point:

$$\frac{\delta V_{\text{out}}}{\delta V_{\text{in}}} = -\frac{V_{\text{DD}} - V_{\text{out}}^{\star}}{V_{\text{TH}}} \tag{15}$$

This suggests that we want the voltage "gap" between the supply voltage V_{DD} and the output voltage bias (i.e. DC) point $V_{\text{out}}^{\star}$ to be large if we want a large voltage gain. For a fixed V_{DD}, this means a lower $V_{\text{out}}^{\star}$. However, when you learn about BJT devices properly, you will see the output bias voltage can only go so low before our models fails. We also notice from equations (14) and (15) that to get a higher voltage gain, we need a larger bias (DC) current $I_C^{\star}$ (to get a lower $V_{\text{out}}^{\star}$ means we need a larger $I_C^{\star}$ through the resistor). In other words, to get higher voltage gain, we need to burn more power.

(d) Assuming that $V_{\text{DD}} = 10\,\text{V}$, $R = 1\,\text{k}\Omega$, and $I_C^{\star} = 1\,\text{mA}$ when $V_{\text{in}}^{\star} = 0.65\,\text{V}$, **verify that the magnitude of the small-signal voltage gain** $\left|\frac{\delta V_{\text{out}}}{\delta V_{\text{in}}}\right|$ **is approximately** 38.

Next, if $I_C^{\star} = 9\,\text{mA}$ when $V_{\text{in}}^{\star} = 0.7\,\text{V}$ with all other parameters remaining fixed, **verify that the magnitude of the small-signal voltage gain** $\left|\frac{\delta V_{\text{out}}}{\delta V_{\text{in}}}\right|$ **between the input and the output around this operating point is approximately** 346.

(HINT: Remember $V_{\text{TH}} = 26\,\text{mV}$.

)

Solution: Just plugging in to equation (14):

$$\left|\frac{\delta V_{\text{out}}}{\delta V_{\text{in}}}\right| = \frac{I_C^{\star}R}{V_{\text{TH}}} = \left(\frac{1\,\text{mA} \times 1\,\text{k}\Omega}{26\,\text{mV}}\right) = \frac{1\,\text{V}}{26\,\text{mV}} \approx 38 \tag{16}$$

Now, if $I_C^{\star} = 9\,\text{mA}$ when $V_{\text{in}}^{\star} = 0.7\,\text{V}$, we have

$$\left|\frac{\delta V_{\text{out}}}{\delta V_{\text{in}}}\right| = \frac{I_C^{\star}R}{V_{\text{TH}}} = \left(\frac{9\,\text{mA} \times 1\,\text{k}\Omega}{26\,\text{mV}}\right) = \frac{9\,\text{V}}{26\,\text{mV}} \approx 346 \tag{17}$$

As an aside, notice here that $V_{\text{out}}^{\star}$ has already been pulled down to around $1\,\text{V}$ ($= V_{\text{DD}} - RI_C^{\star} = 10\,\text{V} - 1\,\text{k}\Omega \times 9\,\text{mA}$). Realistically, this is close to as low as $V_{\text{out}}^{\star}$ can get for this device; the small-signal voltage gain is close to its upper limit. When you first saw the BJT circuit, it may not have been obvious that V_{DD} and V_{TH} provide the fundamental limit on the small-signal gain for such circuits - you may have been tempted to say the upper bound was related to the resistor value. But the simple linearization analysis in part (c) reveals V_{DD} and V_{TH} are setting the true limit. Courses like EE105 and EE140 further develop these insights in circuit design along with feedback control in interesting and very practical ways.

(e) If you wished to make an amplifier with as large of a small signal gain as possible, **which operating (bias) point would you choose among $V_{\text{in}}^{\star} = 0.65\,\text{V}$ and $V_{\text{in}}^{\star} = 0.7\,\text{V}$?**

Solution: We would choose $V_{\text{in}}^{\star} = 0.7\,\text{V}$ since the magnitude of the small signal gain in this operating point is much higher than that at $V_{\text{in}}^{\star} = 0.65\,\text{V}$.

Note that since $I_C^{\star}$ is related to $V_{\text{in}}^{\star}$ by (1), and $V_{\text{out}}^{\star}$ is related to $I_C^{\star}$ by (2), just choosing $V_{\text{in}}^{\star}$ fixes the small signal gain of the circuit in Fig. 1.

This shows you that by appropriately biasing (choosing an operating point), we can adjust what our gain is for small signals. While we just wanted to show you a simple application of linearization here, these ideas are developed a lot further in EE105, EE140, and other courses to create things like op-amps and other analog information-processing systems. Simple voltage amplifier circuits like these are used in everyday circuits like the sensors in your smartwatch, wireless transceivers in your phone, and communication circuits in CPUs and GPUs.

2. Linearization of a scalar system

In this question, we linearize the scalar differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \sin(x(t)) + u(t) \tag{18}$$

around equilibria, discretize it, and apply feedback control to stabilize the resulting system.

(a) The first step is to find the equilibria that we will linearize around. Recall that equilibria are the values of (x, u) such that $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = 0$. Suppose we want to linearize around equilibria $(x^\star, u^\star)$ where $u^\star = 0$. **Sketch** $\sin(x)$ **for** $-4\pi \leq x \leq 4\pi$ **and intersect it with the horizontal line at** 0**. Then, show that** $x_m^\star = m\pi$ **and** $u^\star = 0$ **are equilibria of system (18).**

Solution:

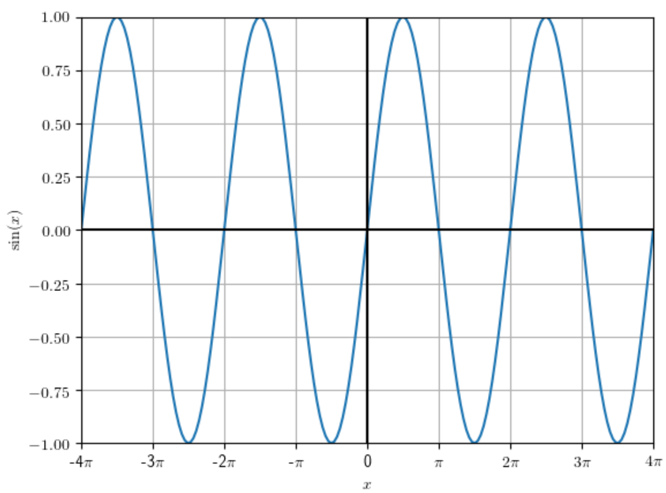

Figure 3: Plot of $\sin(x)$

We can see that all the multiples of π are where the line intersects the sine wave.

At $(x_m^\star, u^\star)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \sin(x_m^\star) + u^\star \tag{19}$$

$$= \sin(m\pi) + 0 \tag{20}$$

$$= 0 \tag{21}$$

$$\tag{22}$$

so $(x_m^\star, u^\star)$ are equilibria of the system.

(b) We will linearize around $x_{-1}^\star = -\pi$ and $x_0^\star = 0$. Looking at the sketch we made, these seem representative of the two types of equilibria where $u^\star = 0$. Linearize system (18) around the equilibrium $(x_0^\star, u^\star) = (0, 0)$. **What is the resulting linearized scalar differential equation for** $\delta x(t) = x(t) - x_0^\star = x(t) - 0$**, involving** $\delta u(t) = u(t) - u^\star = u(t) - 0$**?**

Solution: We have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x(t), u(t)) = \sin(x(t)) + u(t) \tag{23}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta x(t) \approx \frac{\partial f}{\partial x}(x^\star, u^\star)\delta x(t) + \frac{\partial f}{\partial u}(x^\star, u^\star)\delta u(t) \tag{24}$$
$$= \cos(0)\delta x(t) + (1)\delta u(t) \tag{25}$$
$$= \delta x(t) + \delta u(t). \tag{26}$$

(c) Given an arbitrary, continuous linear system as in

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + bu(t) \tag{27}$$

discretizing it into intervals of Δ gives the discrete-time system

$$x[i+1] = e^{\lambda\Delta}x[i] + \frac{b(e^{\lambda\Delta}-1)}{\lambda}u[i] \tag{28}$$

Using this result, discretize the approximate linear system. Is the (approximate) discrete-time system stable?

Solution: By pattern matching from eq. (26) to eq. (27), we have that $\lambda = 1$ and $b = 1$. Plugging into eq. (28), we get

$$\delta x[i+1] = e^{\Delta}\delta x[i] + \delta u[i](e^{\Delta} - 1) \tag{29}$$

For a linear scalar discrete time recurrence relation, stability is determined by the coefficient of the system's variable, in this case δx. We know that if the magnitude of this coefficient is between -1 and 1, our system is stable. But $e^{\Delta} > 1$ for all positive Δ (and Δ by definition has to be positive). Hence, our system is not stable.

(d) Now linearize the system (18) around the equilibrium $(x^\star_{-1}, u^\star) = (-\pi, 0)$. **What is the resulting scalar differential equation for $\delta x(t) = x(t) - (-\pi)$ involving $\delta u(t) = u(t) - 0$? As before, discretize the approximate linear system. Is the (approximate) discrete-time system stable?**

Solution: As before, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x(t), u(t)) = \sin(x(t)) + u(t) \tag{30}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\delta x(t) \approx \frac{\partial f}{\partial x}(x^\star_{-1}, u^\star)\delta x(t) + \frac{\partial f}{\partial u}(x^\star_{-1}, u^\star)\delta u(t) \tag{31}$$
$$= \cos(-\pi)\delta x(t) + (1)\delta u(t) \tag{32}$$
$$= -\delta x(t) + \delta u(t) \tag{33}$$

To discretize, we apply pattern matching again. By pattern matching from eq. (33) to eq. (27), we have that $\lambda = -1$ and $b = 1$. Plugging into eq. (28), we get

$$\delta x[i+1] = e^{-\Delta}\delta x[i] + \delta u[i](1 - e^{-\Delta}). \tag{34}$$

In this case, $0 < e^{-\Delta} < 1$ for all positive Δ, hence our system is stable.

(e) Suppose for the two linearized discrete-time systems that you found in the previous parts, we apply the feedback law

$$\delta u[i] = -k(\delta x[i] - x^\star).$$

For what range of k values would the resulting linearized discrete-time systems be stable? Your answer will depend on Δ.

Solution: Let's begin with the first case, $x^{\star} = 0$. Based on our defintion of δx, we have, $\delta u[n] = -k\delta x[n]$. Substituting and grouping the terms, we get

$$\delta x[i+1] = \delta x[i] \left(e^{\Delta} - k(e^{\Delta} - 1)\right) \tag{35}$$

Hence, we want the above coefficient to between -1 and 1.

$$-1 < e^{\Delta} - k(e^{\Delta} - 1) < 1 \tag{36}$$
$$-(1 + e^{\Delta}) < -k(e^{\Delta} - 1) < 1 - e^{\Delta} \tag{37}$$
$$\implies 1 < k < \frac{e^{\Delta} + 1}{e^{\Delta} - 1} \tag{38}$$

Looking at the second case, with $x^{\star} = -\pi$, we get

$$\delta x[i+1] = e^{-\Delta}\delta x[i] + k\delta x[i](e^{-\Delta} - 1) + k\pi(e^{-\Delta} - 1) \tag{39}$$

Grouping terms, and further simplifying

$$\delta x[i+1] = \delta x[i] \left(e^{-\Delta} + k(e^{-\Delta} - 1)\right) + k\pi(e^{-\Delta} - 1) \tag{40}$$

As before, we want this coefficient to be between -1 and 1, hence

$$-1 < e^{-\Delta} + k(e^{-\Delta} - 1) < 1 \tag{41}$$
$$-(1 + e^{-\Delta}) < k(e^{-\Delta} - 1) < 1 - e^{-\Delta} \tag{42}$$
$$\implies -1 < k < \frac{1 + e^{-\Delta}}{1 - e^{-\Delta}} \tag{43}$$

3. Tracking a Desired Trajectory in Continuous Time

The treatment in 16B so far has treated closed-loop control as being about holding a system steady at some desired operating point, by placing the eigenvalues of the state transition matrix. This control used something proportional to the actual present state to apply a control signal designed to bring the eigenvalues in the region of stability. Meanwhile, the idea of controllability itself was more general and allowed us to make an open-loop trajectory that went pretty much anywhere. This problem is about combining these two ideas together to make feedback control more practical — how we can get a system to more-or-less closely follow a desired trajectory, even though it might not start exactly where we wanted to start and in principle could be affected by small disturbances throughout.

In this question, we will also see that everything that you have learned to do closed-loop control in discrete-time can also be used to do closed-loop control in continuous time.

Consider the specific 2-dimensional system

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) + \vec{w}(t) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\vec{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u(t) + \vec{w}(t) \tag{44}$$

where $u(t)$ is a scalar valued continuous control input and $\vec{w}(t)$ is a bounded disturbance (noise).

(a) In an ideal noiseless scenario, the desired control signal $u^*(t)$ makes the system follow the desired trajectory $\vec{x^*}(t)$ that satisfies the following dynamics:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x^*}(t) = A\vec{x^*}(t) + \vec{b}u^*(t). \tag{45}$$

The presence of the bounded noise term $\vec{w}(t)$ makes the actual state $\vec{x}(t)$ deviate from the desired $\vec{x^*}(t)$ and follow (44) instead. In the following subparts, we will analyze how we can adjust the desired control signal $u^*(t)$ in (45) to the control input $u(t)$ in (44) so that the deviation in the state caused by $\vec{w}(t)$ remains bounded.

Represent the state as $\vec{x}(t) = \vec{x^*}(t) + \Delta\vec{x}(t)$ and $u(t) = u^*(t) + \Delta u(t)$. Using (44) and (45), **show that we can represent the evolution of the trajectory deviation $\Delta\vec{x}(t)$ as a function of the control deviation $\Delta u(t)$ and the bounded disturbance $\vec{w}(t)$ as:**

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta\vec{x}(t) = A\Delta\vec{x}(t) + \vec{b}\Delta u(t) + \vec{w}(t). \tag{46}$$

(HINT: Write out equation (44) in terms of $\vec{x^*}(t)$, $\Delta\vec{x}(t)$, $u^*(t)$ and $\Delta u(t)$.))

Solution: Using the change of variables $\vec{x}(t) = \vec{x^*}(t) + \Delta\vec{x}(t)$ and $u(t) = u^*(t) + \Delta u(t)$ in (44), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) + w(t) \tag{47}$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}t}\vec{x^*}(t) + \frac{\mathrm{d}}{\mathrm{d}t}\Delta\vec{x}(t) = A\vec{x^*}(t) + A\Delta\vec{x}(t) + \vec{b}u^*(t) + \vec{b}\Delta u(t) + \vec{w}(t) \tag{48}$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}t}\Delta\vec{x}(t) = A\Delta\vec{x}(t) + \vec{b}\Delta u(t) + \vec{w}(t) + \left(A\vec{x^*}(t) + \vec{b}u^*(t) - \frac{\mathrm{d}}{\mathrm{d}t}\vec{x^*}(t)\right) \tag{49}$$

Using (45) we know that the last term in parenthesis in zero, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta\vec{x}(t) = A\Delta\vec{x}(t) + \vec{b}\Delta u(t) + \vec{w}(t) \tag{50}$$

Note that this implies the disturbance $\vec{w}(t)$ is entirely something that must be dealt with in the $\Delta\vec{x}$ dynamics. It doesn't affect the desired trajectory at all.

(b) **Are the dynamics that you found for $\Delta\vec{x}(t)$ in part 3.a stable? Based on this, in the presence of bounded disturbance $\vec{w}(t)$, will $\vec{x}(t)$ in (44) follow the desired trajectory $\vec{x}^*(t)$ closely if we just apply the control $u(t) = u^*(t)$ to the original system in (44), i.e. $\Delta u(t) = 0$?**

(HINT: Use the numerical values of A and $\vec{b}$ from (44) in the solution from part (b) to determine stability of $\Delta\vec{x}(t)$.))

Solution: If we just set $\Delta u(t) = 0$, then our $v(t)$ dynamics becomes open-loop so

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta\vec{x}(t) = A\Delta\vec{x}(t) + \vec{w}(t). \tag{51}$$

Recall that the condition for stability in the continuous-time case is that the real part of the eigenvalues of the state transition matrix A must be less then zero. Note that since A is an upper-triangular matrix, its eigenvalues lie on the diagonal, so they are 2 and 2. In this case, they have real parts greater than zero so the open-loop $\Delta\vec{x}(t)$ system is unstable.

$\Delta\vec{x}(t)$ will then follow a growing exponential trajectory in the form of e^{2t}, and will thus amplify any disturbance $\vec{w}(t)$ to the state. Therefore, $\Delta\vec{x}(t)$ will not go to $\vec{0}$ and we will not end up following the intended trajectory $\vec{x}^*(t)$.

(c) Now, we want to apply state feedback control to the system using $\Delta u(t)$ to get our system to follow the desired trajectory $\vec{x}^*(t)$. For the $\Delta\vec{x}(t), \Delta u(t)$ system, **apply feedback control by letting $\Delta u(t) = F\Delta\vec{x}(t) = \begin{bmatrix} f_0 & f_1 \end{bmatrix} \Delta\vec{x}(t)$ that would place both the eigenvalues of the closed-loop $\Delta\vec{x}(t)$ system at -10. Find f_0 and f_1.**

Solution: With the new input, the system equation for $\Delta\vec{x}(t)$ is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta\vec{x}(t) = A_v\Delta\vec{x}(t) + \vec{b}\begin{bmatrix} f_0 & f_1 \end{bmatrix}\Delta\vec{x}(t) + \vec{w}(t) \tag{52}$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}t}\Delta\vec{x}(t) = \begin{bmatrix} 2+f_0 & 1+f_1 \\ f_0 & 2+f_1 \end{bmatrix}\Delta\vec{x}(t) + \vec{w}(t) \tag{53}$$

where we denote $A_{cl} = \begin{bmatrix} 2+f_0 & 1+f_1 \\ f_0 & 2+f_1 \end{bmatrix}$ as the state matrix for the closed loop system. The characteristic polynomial for finding the eigenvalues of A_{cl} is given by:

$$\det(\lambda I - A_{cl}) = \begin{bmatrix} \lambda-2-f_0 & -1-f_1 \\ -f_0 & \lambda-2-f_1 \end{bmatrix} \tag{54}$$

$$= \lambda^2 - (4+f_0+f_1)\lambda + f_0 + 2f_1 + 4 \tag{55}$$

To set the eigenvalues to be where we want, we set this equal to $(\lambda+10)(\lambda+10) = \lambda^2 + 20\lambda + 100$.

By comparing the coefficients, we have:

$$-(4+f_0+f_1) = 20 \tag{56}$$

$$f_0 + 2f_1 + 4 = 100 \tag{57}$$

Solving the above system of equations, we can find $f_0 = -144$, $f_1 = 120$. Therefore, we can design the state-feedback $\Delta u(t) = \begin{bmatrix} -144 & 120 \end{bmatrix} \Delta\vec{x}(t)$ which will place both the eigenvalues of the closed loop system at -10.

Why did we pick -10? So that our closed-loop system would converge faster and aggressively reject disturbances.

(d) Based on what you did in the previous parts, and given access to the desired trajectory $\vec{x^*}(t)$, the desired control $u^*(t)$, and the actual measurement of the state $\vec{x}(t)$, **come up with a way to do feedback control that will keep the trajectory staying close to the desired trajectory no matter what the small bounded disturbance $\vec{w}(t)$ does.** *(HINT: Express the control input $u(t)$ in terms of $u^*(t)$, $\vec{x^*}(t)$, and $\vec{x}(t)$.))*

Solution: From the previous parts, we have successfully found a feedback control law $\Delta u(t) = \begin{bmatrix} f_0 & f_1 \end{bmatrix} \Delta\vec{x}(t)$ such that the closed-loop system for $\Delta\vec{x}(t)$ is stable and convering to $\vec{0}$ as long as the disturbances are bounded. As a result, by changing variables $\vec{x}(t) = \vec{x^*}(t) + \Delta\vec{x}(t)$ and $u(t) = u^*(t) + \Delta u(t)$ that we performed in (b), we can infer that the state $\vec{x}(t)$ will stay close to the desired trajectory $\vec{x^*}(t)$ no matter what the bounded disturbance $\vec{w}(t)$ does.

From our initial change of variables, we want to set

$$u(t) = u^*(t) + \Delta u(t) = u^*(t) + \begin{bmatrix} -144 & 120 \end{bmatrix} \Delta\vec{x}(t) \tag{58}$$

$$= u^*(t) + \begin{bmatrix} -144 & 120 \end{bmatrix} (\vec{x}(t) - \vec{x^*}(t)) \tag{59}$$

as our overall system input to achieve this.

This lets us have our cake and eat it too! We can use the desired system dynamics from (45) to plan, and by using closed-loop feedback we can make sure that we mostly follow our plan even in the face of disturbances.

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