Homework 8

This homework is due on Friday, October 21, 2022, at 11:59PM. Self-grades and HW Resubmissions are due on Friday, October 28, 2022, at 11:59PM.

1. Stability Criterion

Consider the complex plane below, which is broken into non-overlapping regions A through H. The circle drawn on the figure is the unit circle $|\lambda| = 1$.

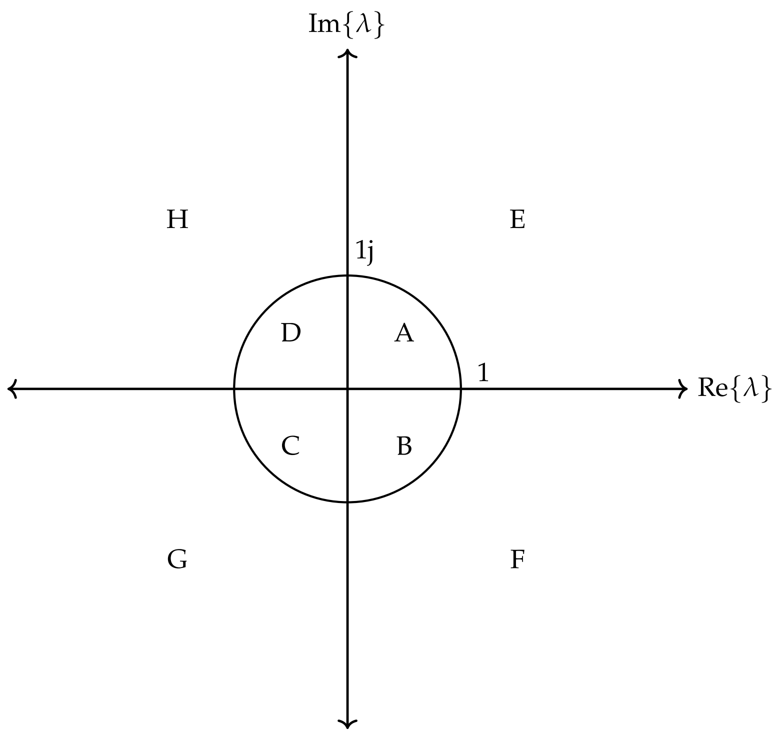


Figure 1: Complex plane divided into regions.

Consider the continuous-time system $\frac{d}{dt}x(t) = \lambda x(t) + v(t)$
and the discrete-time system $y[i+1] = \lambda y[i] + w[i]$. Here $v(t)$ and $w[i]$ are both disturbances to their respective systems.

In which regions can the eigenvalue λ be for the system to be *stable*? Fill out the table below to indicate *stable* regions. Assume that the eigenvalue λ does not fall directly on the boundary between two regions.

	A	B	C	D	E	F	G	H
Continuous Time System $x(t)$	◯	◯	◯	◯	◯	◯	◯	◯
Discrete Time System $y[i]$	◯	◯	◯	◯	◯	◯	◯	◯

Solution: For the continuous time system to be stable, we need the real part of λ to be less than zero. Hence, C, D, G, H satisfy this condition.

On the other hand, for the discrete time system to be stable, we need the norm of λ to be less than one. Hence, A, B, C, D satisfy this condition.

2. Bounded-Input Bounded-State (BIBS) Stability

BIBS stability is a system property where bounded inputs lead to bounded outputs. It's important because we want to certify that, provided our system inputs are bounded, the outputs will not "blow up". In this problem, we gain a better understanding of BIBS stability by considering some simple continuous and discrete systems, and showing whether they are BIBS stable or not.

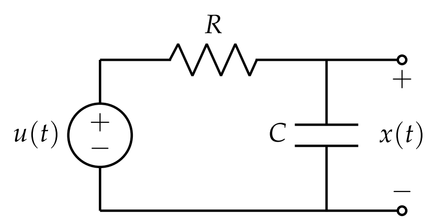
Recall that for the following simple scalar differential equation, we have the corresponding solution:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = ax(t) + bu(t) \qquad x(t) = \mathrm{e}^{at}x(0) + \int_0^t \mathrm{e}^{a(t-\tau)}bu(\tau)\,\mathrm{d}\tau\,. \tag{1}$$

And for the following discrete system, we have the corresponding solution:

$$x[i+1] = ax[i] + bu[i] \qquad x[i] = a^i x[0] + \sum_{k=0}^{i-1} a^k bu[i-1-k] \tag{2}$$

(a) Consider the circuit below with $R = 1\Omega$, $C = 0.5\text{F}$. Let $x(t)$ be the voltage over the capacitor.



This circuit can be modeled by the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -2x(t) + 2u(t) \tag{3}$$

Intuitively, we know that the voltage on the capacitor can never exceed the (bounded) voltage from the voltage source, so this system is BIBS stable. **Show that this system is BIBS stable, meaning that $x(t)$ remains bounded for all time if the input $u(t)$ is bounded. Equivalently, show that if we assume $|u(t)| < \epsilon$, $\forall t \geq 0$ and $|x(0)| < \epsilon$, then $|x(t)| < M$, $\forall t \geq 0$ for some positive constant M.** Thinking about this helps you understand what bounded-input-bounded-output stability means in a physical circuit.

(HINT: eq. (1) may be useful. You may want to write the expression for $x(t)$ in terms of $u(t)$ and $x(0)$ and then take the norms of both sides to show a bound on $|x(t)|$. Remember that norm in 1D is absolute value. Some helpful formulas are $|ab| = |a||b|$, the triangle inequality $|a+b| \leq |a| + |b|$, and the integral version of the triangle inequality $\left|\int_a^b f(\tau)\,\mathrm{d}\tau\right| \leq \int_a^b |f(\tau)|\,\mathrm{d}\tau$, which just extends the standard triangle inequality to an infinite sum of terms.)

Solution:

Using eq. (1), we get the solution to the scalar differential equation as

$$x(t) = \mathrm{e}^{-2t}x(0) + \int_0^t \mathrm{e}^{-2(t-\tau)}2u(\tau)\,\mathrm{d}\tau\,. \tag{4}$$

Then we can try to bound $x(t)$ for $t \geq 0$. We first use the triangle inequality ($|a+b| \leq |a| + |b|$) to get

$$|x(t)| = \left|\mathrm{e}^{-2t}x(0) + \int_0^t \mathrm{e}^{-2(t-\tau)}2u(\tau)\,\mathrm{d}\tau\right| \tag{5}$$

$$|x(t)| \leq \left|\mathrm{e}^{-2t}x(0)\right| + \left|\int_0^t \mathrm{e}^{-2(t-\tau)}2u(\tau)\,\mathrm{d}\tau\right| \tag{6}$$

We then use the property that the integral of absolute value will always be greater than the absolute value of the integral (equation (6) to (7)), and that an exponential is always positive (equation (7) to (8)):

$$|x(t)| \leq \left|\mathrm{e}^{-2t}x(0)\right| + \int_0^t \left|\mathrm{e}^{-2(t-\tau)}2u(\tau)\right|\mathrm{d}\tau \tag{7}$$

$$= \mathrm{e}^{-2t}|x(0)| + \int_0^t \mathrm{e}^{-2(t-\tau)}2|u(\tau)|\,\mathrm{d}\tau \tag{8}$$

Finally, plugging in our bounds for $|u(\tau)|$ and $|x(0)|$ and doing the integral:

$$|x(t)| \leq \mathrm{e}^{-2t}\epsilon + \int_0^t \mathrm{e}^{-2(t-\tau)}2\epsilon\,\mathrm{d}\tau \tag{9}$$

$$= \mathrm{e}^{-2t}\epsilon + 2\epsilon\mathrm{e}^{-2t}\int_0^t \mathrm{e}^{2\tau}\,\mathrm{d}\tau \tag{10}$$

$$= \mathrm{e}^{-2t}\epsilon + 2\epsilon\mathrm{e}^{-2t}\frac{1}{2}\left(\mathrm{e}^{2t}-1\right) \tag{11}$$

$$= \mathrm{e}^{-2t}\epsilon + \epsilon\left(1-\mathrm{e}^{-2t}\right) \tag{12}$$

$$= \epsilon,\ \forall t \geq 0 \tag{13}$$

So we see that our state's magnitude is bounded for all time. Note that the negative exponent of the exponential is what makes this system stay bounded.

(b) Assume $x(0) = 0$. **Show that the system eq. (1) is BIBS unstable when $a = \mathrm{j}2\pi$ by constructing a bounded input that leads to an unbounded $x(t)$.**

It can be shown that the system eq. (1) is unstable for any purely imaginary a by a similar construction of a bounded input.

Solution: Recall the solution of $x(t)$ with the initial condition at zero

$$x(t) = \int_0^t \mathrm{e}^{a(t-\tau)}bu(\tau)\,\mathrm{d}\tau\,. \tag{14}$$

Remember, the style of argumentation here is the "counterexample" style. The question asks you to show that *some* bounded input exists that will make the state grow without bound.

Because we know we can get an integral to diverge if we are just integrating a nonzero constant, we decide to try the bounded input $u(t) = \epsilon\mathrm{e}^{\mathrm{j}2\pi t}$, whose magnitude is equal to ϵ for all t.

Plugging this input and a value in, we see

$$x(t) = \int_0^t \mathrm{e}^{\mathrm{j}2\pi(t-\tau)}b\epsilon\mathrm{e}^{\mathrm{j}2\pi\tau}\,\mathrm{d}\tau = \int_0^t \mathrm{e}^{\mathrm{j}2\pi t}b\epsilon\,\mathrm{d}\tau\,. \tag{15}$$

Factoring out the terms that do not depend on τ, we are left with

$$x(t) = b\epsilon\mathrm{e}^{\mathrm{j}2\pi t}\int_0^t \mathrm{d}\tau\,. \tag{16}$$

Solving this integral, we get

$$x(t) = b\epsilon t\mathrm{e}^{\mathrm{j}2\pi t}. \tag{17}$$

Now taking the magnitude of $x(t)$ using the fact that $|e^{j\omega t}| = 1$ for all ω, we get $|x(t)| = \epsilon|b|t$ which clearly diverges as $t \to \infty$.

(c) Consider the discrete-time system and its solution in eq. (2). **Show that if $|a| > 1$, then even if $x[0] = 0$, a bounded input can result in an unbounded output, i.e. the system is BIBS unstable.** *(HINT: The formula for the sum of a geometric sequence may be helpful.)*

Solution: Consider when $x[0] = 0$ and $u[i] = 1\ \forall i$. This gives

$$x[i] = a^i x[0] + \sum_{k=0}^{i-1} a^k b u[i-1-k] \tag{18}$$

$$= \sum_{k=0}^{i-1} a^k b \tag{19}$$

$$= b\frac{a^i - 1}{a - 1} \quad \text{(as this is the sum of a geometric series)} \tag{20}$$

When $|a| > 1$, then a^i has magnitude that grows without bound, and thus $|x[i]|$ does as well. We also know this from the convergence criteria for geometric series; when the common ratio $a > 1$, the series does not converge to a finite number as $i \to \infty$.

(d) Consider the discrete-time system

$$x[i+1] = -3x[i] + u[i]. \tag{21}$$

Is this system stable or unstable? Give an initial condition $x(0)$ and a sequence of non-zero inputs for which the state $x[i]$ will always stay bounded. *(HINT: See if you can find any input pattern that results in an oscillatory behavior.))*

Solution:

The system is unstable since the eigenvalue -3 has magnitude ≥ 1. To see this more explicitly, any non-zero $x[0]$ and (bounded) $u[i] = 0\ \forall i \in \mathbb{N}$ will lead to unbounded x.

Consider $x[0] = 0$ and the input $u[i] = 1, 3, 1, 3, 1, 3, \ldots$.

t	0	1	2	3	$\cdots$
$x[i]$	0	1	0	1	$\cdots$
$u[i]$	1	3	1	3	$\cdots$
$-3x[i] + u[i]$	1	0	1	0	$\cdots$

In this case, we get $x[i] = 0$ when t is even, and $x[i] = 1$ when i is odd. In fact, there are an infinite number of input sequences that would result in bounded outputs.

3. Eigenvalue Placement through State Feedback

Consider the following discrete-time linear system:

$$\vec{x}[i+1] = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i]. \tag{22}$$

In standard language, we have $A = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in the form: $\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]$.

(a) **Is this system controllable?**

Solution: We calculate the controllability matrix

$$\mathcal{C} = \begin{bmatrix} \vec{b} & A\vec{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{23}$$

Observe that the $\mathcal{C}$ matrix has linearly independent columns and hence our system is controllable.

(b) **Is this discrete-time linear system stable in open loop (without feedback control)?**

Solution: We have to calculate the eigenvalues of matrix A. Thus,

$$0 = \det(\lambda I - A) \tag{24}$$

$$= \det \begin{bmatrix} \lambda + 2 & -2 \\ 2 & \lambda - 3 \end{bmatrix} \tag{25}$$

$$= \lambda^2 - \lambda - 2 \tag{26}$$

$$\implies \lambda_1 = 2, \qquad \lambda_2 = -1 \tag{27}$$

Since at least one eigenvalue has a magnitude that is greater than or equal to 1, the discrete-time system is unstable. In this case, both of the eigenvalues are unstable.

(c) Suppose we use state feedback of the form $u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i] = F\vec{x}[i]$.

Find the appropriate state feedback constants, f_1, f_2 so that the state space representation of the resulting closed-loop system has eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.

Solution: The closed loop system using state feedback has the form

$$\vec{x}[i+1] = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i] \tag{28}$$

$$= \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i] \tag{29}$$

$$= \left(\begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} f_1 & f_2 \\ f_1 & f_2 \end{bmatrix} \right) \vec{x}[i] \tag{30}$$

Thus, the closed loop system has the form

$$\vec{x}[i+1] = \underbrace{\begin{bmatrix} -2 + f_1 & 2 + f_2 \\ -2 + f_1 & 3 + f_2 \end{bmatrix}}_{A_{\text{cl}}} \vec{x}[i] \tag{31}$$

Finding the characteristic polynomial of the above system, we have

$$\det\left(\lambda I - \begin{bmatrix} -2+f_1 & 2+f_2 \\ -2+f_1 & 3+f_2 \end{bmatrix}\right) = (\lambda+2-f_1)(\lambda-3-f_2) - (-2-f_2)(2-f_1) \tag{32}$$

$$= \lambda^2 - f_1\lambda - f_2\lambda - \lambda + f_1 f_2 - 6 - 2f_2 + 3f_1 \tag{33}$$

$$- (-4 + f_1 f_2 + 2f_1 - 2f_2) \tag{34}$$

$$= \lambda^2 - (1+f_1+f_2)\lambda + f_1 - 2 \tag{35}$$

However, we want to place the eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$. That means we want

$$\lambda^2 - (1+f_1+f_2)\lambda + f_1 - 2 = \left(\lambda + \frac{1}{2}\right)\left(\lambda - \frac{1}{2}\right) \tag{36}$$

or equivalently:

$$\lambda^2 - (1+f_1+f_2)\lambda + f_1 - 2 = \lambda^2 - \frac{1}{4} \tag{37}$$

Equating the coefficients of the different powers of λ on both sides of the equation, we get,

$$1 + f_1 + f_2 = 0 \tag{38}$$

$$f_1 - 2 = -\frac{1}{4} \tag{39}$$

Solving the above system of equations gives us $f_1 = \frac{7}{4}, f_2 = -\frac{11}{4}$.

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