Homework 5

This homework is due on Friday, September 30, 2022, at 11:59PM. Self-grades and HW Resubmissions are due on the following Friday, October 7, 2022, at 11:59PM.

1. Hambley P6.55

Consider the first-order highpass filter shown in Figure 1. The input signal is given by

$$v_{\text{in}}(t) = 5 + 5\cos(2000\pi t) \tag{1}$$

Find an expression for the output $v_{\text{out}}(t)$ in steady-state conditions.

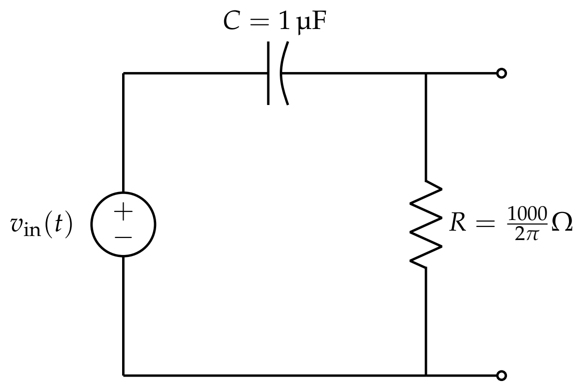


Figure 1: P6.55

Solution: The input signal is given by

$$v_{\text{in}}(t) = 5 + 5\cos(2000\pi t) \tag{2}$$

The signal has components at $f = 0\,\text{Hz}$ and $f = 1000\,\text{Hz}$. The transfer function values at these frequencies are

$$H(0) = \frac{\text{j}0}{1 + \text{j}0} = 0 \tag{3}$$

$$H(1000) = \frac{\text{j}1}{1 + \text{j}1} = 0.7071\angle 45^\circ \tag{4}$$

Applying these transfer function values to the respective components yields:

$$v_{\text{out}}(t) = 3.536\cos(2000\pi t + 45^\circ) \tag{5}$$

2. Hambley P6.71

Consider the series resonant circuit shown in Figure 2, with $L = 20\,\mu\text{H}$, $R = 14.14\,\Omega$, and $C = 1000\,\text{pF}$. Compute the resonant frequency, the bandwidth, and the half-power frequencies. Assuming that the frequency of the source is the same as the resonant frequency, find the phasor voltages across the elements and sketch a phasor diagram.

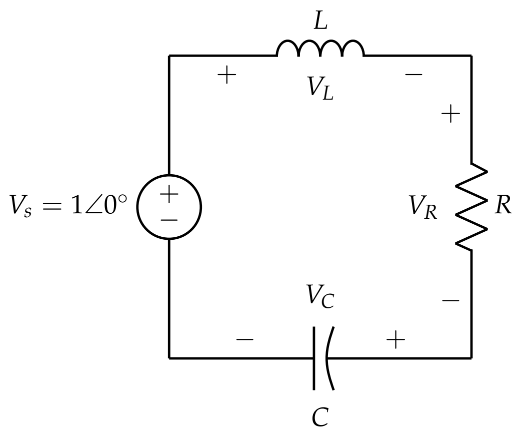


Figure 2: P6.71

Solution: The resonant frequency is given by the formula

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = 1.125\,\text{MHz} \tag{6}$$

To find the bandwidth, we apply the formula

$$B = \frac{f_0}{Q_s} = \frac{f_0}{\frac{2\pi f_0 L}{R}} = 112.5\,\text{kHz} \tag{7}$$

Lastly, to find the half power frequencies, we find

$$f_H = f_0 + \frac{B}{2} = 1.181\,\text{MHz} \tag{8}$$

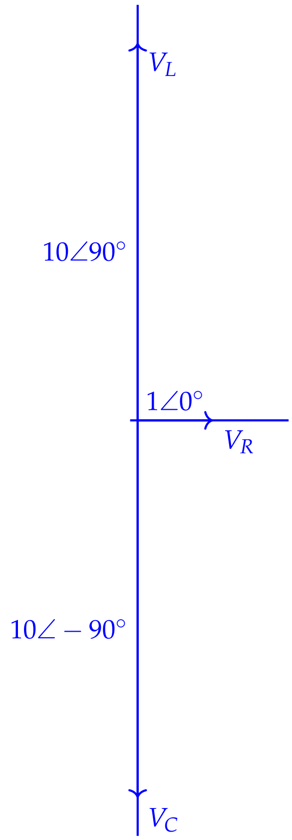
$$f_L = f_0 - \frac{B}{2} = 1.069\,\text{MHz} \tag{9}$$

The total phasor impedance at the resonant frequency is

$$Z = R - \frac{\mathrm{j}}{\frac{1}{\sqrt{LC}}C} + \mathrm{j}\frac{1}{\sqrt{LC}}L \tag{10}$$

$$= R - \mathrm{j}\sqrt{\frac{L}{C}} + \mathrm{j}\sqrt{\frac{L}{C}} = R \tag{11}$$

so the phasor current is $I = \frac{1}{14.14}\text{A}$. Hence, $V_R = 1\angle 0°$, $V_C = I \times \left(-\mathrm{j}\sqrt{\frac{L}{C}}\right) = -\mathrm{j}\frac{1}{14.14}141.4 = -10\mathrm{j} = 10\angle -90°$ and $V_L = I \times \left(\mathrm{j}\sqrt{\frac{L}{C}}\right) = 10\angle 90°$. The phasor diagram is shown below:


V_L
$10\angle 90^\circ$
$1\angle 0^\circ$
V_R
$10\angle -90^\circ$
V_C

3. Bandpass Half Power Derivation

For a series resonance circuit bandpass filter, prove that the two half-power frequencies can be written as

$$f_H = f_0 + \frac{B}{2} \tag{12}$$
$$f_L = f_0 - \frac{B}{2} \tag{13}$$

when $Q_s \gg 1$, where $B = \frac{f_0}{Q_s}$ and $Q_s = 2\pi f_0 \frac{L}{R}$.

Solution: Recall the the transfer function for series bandpass is

$$H(f) = \frac{1}{1 + \mathrm{j}Q_s\left(\frac{f}{f_0} - \frac{f_0}{f}\right)} \tag{14}$$

To find the half power frequency, we need to find the values of f where $|H(f)| = \frac{1}{\sqrt{2}}$, or equivalently, $\left|1 + \mathrm{j}Q_s\left(\frac{f}{f_0} - \frac{f_0}{f}\right)\right|^2 = 2$. Computing the magnitude and squaring, we have

$$1 + Q_s^2\left(\frac{f^2 - f_0^2}{f f_0}\right)^2 = 2 \tag{15}$$

Applying the quadratic formula with f^2, we obtain

$$f = \frac{\pm f_0 + f_0\sqrt{4Q_s^2 + 1}}{2Q_s} \tag{16}$$

For $Q_s \gg 1$, we can approximate this as

$$f = \frac{\pm f_0 + f_0\sqrt{4Q_s^2}}{2Q_s} \tag{17}$$

which we can simplify to

$$f = f_0 \pm \frac{f_0}{2Q_s} = f_0 \pm \frac{B}{2} \tag{18}$$

4. Hambley P6.73

Suppose we have a series resonant circuit for which $B = 15\,\text{kHz}$, $f_0 = 300\,\text{kHz}$, and $R = 40\,\Omega$. Determine the values of L and C.

Solution: We can first find $Q_s = \frac{f_0}{B} = \frac{300\,\text{kHz}}{15\,\text{kHz}} = 20$. Then, plugging in to $Q_s = 2\pi f_0 \frac{L}{R}$ and $Q_s = \frac{1}{2\pi f_0 RC}$ and solving for L and C respectively, we have $L = 42.44\,\mu\text{H}$ and $C = 663.1\,\text{pF}$.

5. Hambley P6.74

Derive an expression for the resonant frequency of the circuit shown in Figure 3. (Recall that we have defined the resonant frequency to be the frequency for which the impedance is purely resistive).

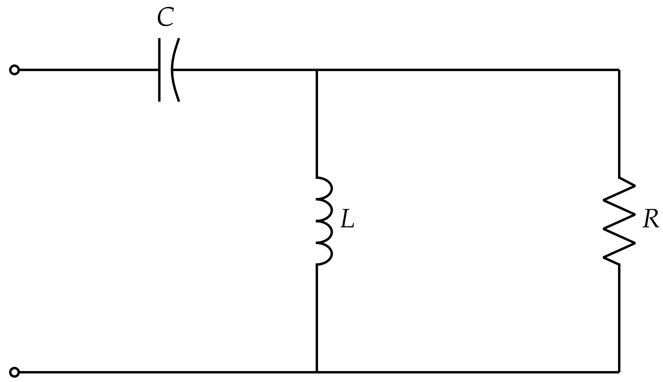


Figure 3: P6.74

Solution: The combined impedance is

$$Z = \frac{1}{j\omega C} + \frac{1}{\frac{1}{R} + \frac{1}{j\omega L}} \tag{19}$$

$$= -\frac{j}{\omega C} + \frac{\frac{1}{R} + j\frac{1}{\omega L}}{\left(\frac{1}{R}\right)^2 + \left(\frac{1}{\omega L}\right)^2} \tag{20}$$

At the resonance frequency, the imaginary part must go to zero, so

$$\frac{\frac{1}{\omega L}}{\left(\frac{1}{R}\right)^2 + \left(\frac{1}{\omega L}\right)^2} - \frac{1}{\omega C} = 0 \tag{21}$$

Solving for ω yields $\omega = \frac{1}{\sqrt{LC - \left(\frac{L}{R}\right)^2}}$.

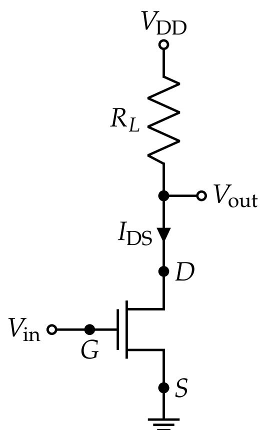
6. Using a Nonlinear NMOS Transistor for Amplification

Consider the following schematic where $V_{\text{DD}} = 1.5\,\text{V}$, $R_L = 400\,\Omega$ and the NMOS transistor has threshold voltage $V_T = 0.2\,\text{V}$. We are interested in analyzing the response of this circuit to input voltages of the form $V_{\text{in}}(t) = V_{\text{in,DC}} + v_{\text{in,AC}}(t)$, where $V_{\text{in,DC}}$ is some constant voltage and $v_{\text{in,AC}}(t) = 0.001\cos(\omega t)\text{V}$ is a sinusoidal signal whose magnitude is much smaller than $V_{\text{in,DC}}$.

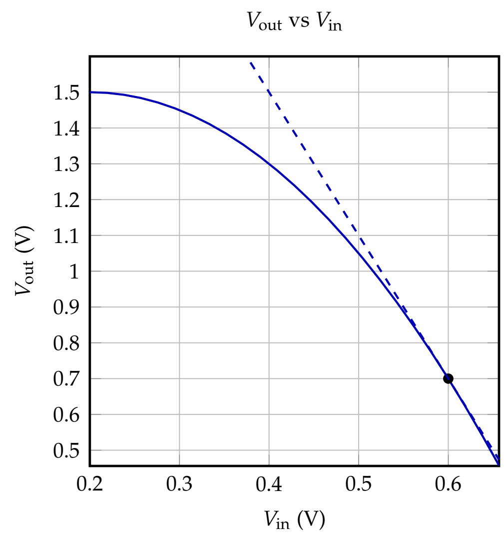
The I-V relationship of an NMOS can be modeled as non-linear functions over different regions of operation. For simplicity, let's just focus on the case when $0 \le V_{\text{GS}} - V_T < V_{\text{DS}}$. In this regime of interest, the relevant I-V relationship is given by

$$I_{\text{DS}}(V_{\text{GS}}) = \frac{K}{2}(V_{\text{GS}} - V_T)^2 \tag{22}$$

where K is a constant that depends on the NMOS transistor size and properties.



(a) NMOS Transistor circuit



(b) V_{out} vs V_{in} in the regime of interest. Tangent is drawn at the operating point $V_{\text{in,DC}} = 0.6\,\text{V}$, $V_{\text{out,DC}} = 0.7\,\text{V}$

Figure 4: NMOS figures.

From Ohm's law and KCL, we know that

$$V_{\text{out}}(t) = V_{\text{DD}} - R_L I_{\text{DS}}(t). \tag{23}$$

Note from Figure 4a that $V_{\text{in}} = V_{\text{GS}}$ and $V_{\text{out}} = V_{\text{DS}}$. In Figure 4b, we can see the curve of V_{out} vs V_{in} in the transistor operating regime of interest.

(a) **Using eq. (22) and eq. (23), express $V_{\text{out}}(t)$ as a function of $V_{\text{in}}(t)$ symbolically.** (You can use $V_{\text{DD}}, R_L, V_{\text{in}}, K, V_T$ in your answer.)

Solution:

$$V_{\text{out}}(t) = V_{\text{DD}} - R_L I_{\text{DS}}(t)\Big|_{V_{\text{GS}}=V_{\text{in}}(t)} \tag{24}$$

$$= V_{\text{DD}} - R_L \frac{K}{2}(V_{\text{in}}(t) - V_T)^2 \tag{25}$$

A plot of eq. (25) is shown in Figure 4b.

(b) We can decompose the input into constant (i.e., DC) and time-varying (i.e., AC) components to obtain $V_{\text{in}}(t) = V_{\text{in,DC}} + v_{\text{in,AC}}(t)$. Furthermore, we can consider $V_{\text{out}}(t)$ as a function of $V_{\text{in}}(t)$ and approximate it as a linear equation, as in

$$\widehat{V_{\text{out}}}(V_{\text{in}}) = V_{\text{out}}(V_{\text{in,DC}}) + \frac{\mathrm{d}V_{\text{out}}}{\mathrm{d}V_{\text{in}}}\Big|_{V_{\text{in}}=V_{\text{in,DC}}}(V_{\text{in}} - V_{\text{in,DC}}) \tag{26}$$

Solve for this approximation, i.e., find $\widehat{V_{\text{out}}}(V_{\text{in}})$. Write your answer in terms of the transconductance gain (g_m), $V_{\text{in,DC}}$, $v_{\text{in,AC}}$, and other constants provided in the problem. *This technique is called linearization, which we will cover later in this course.*

Solution: Using the equation above, we have

$$\widehat{V_{\text{out}}}(V_{\text{in}}) = V_{\text{DD}} - R_L\frac{K}{2}(V_{\text{in,DC}} - V_T)^2 + \frac{\mathrm{d}V_{\text{out}}}{\mathrm{d}V_{\text{in}}}\Big|_{V_{\text{in}}=V_{\text{in,DC}}}(V_{\text{in}} - V_{\text{in,DC}}) \tag{27}$$

where

$$\frac{\mathrm{d}V_{\text{out}}}{\mathrm{d}V_{\text{in}}}\Big|_{V_{\text{in}}=V_{\text{in,DC}}} = -R_L K(V_{\text{in,DC}} - V_T) \tag{28}$$

This is our slope. We can rewrite this as $-R_L g_m$ where $g_m := K(V_{\text{in,DC}} - V_T)$ is defined to be the *transconductance gain*. Altogether, we have

$$\widehat{V_{\text{out}}}(V_{\text{in}}) = V_{\text{DD}} - R_L\frac{K}{2}(V_{\text{in,DC}} - V_T)^2 - R_L g_m v_{\text{in,AC}} \tag{29}$$

where we substitute $V_{\text{in}} - V_{\text{in,DC}} = v_{\text{in,AC}}$

(c) Next, we can also decompose the output V_{out} into DC and AC components to obtain $V_{\text{out}} = V_{\text{out,DC}} + v_{\text{out,AC}}(t)$. **What is $V_{\text{out,DC}}$ from the linear approximation in part 6.b? Simplify the linear approximation to be in terms of $v_{\text{out,AC}}(t)$ and $v_{\text{in,AC}}(t)$, for very small $v_{\text{in,AC}}(t)$. Then, find the AC input-output gain (i.e., find $\frac{v_{\text{out,AC}}(t)}{v_{\text{in,AC}}(t)}$)**

(HINT: If $v_{\text{in,AC}}(t)$ is small, then $V_{\text{in,DC}} - V_T$ is small, which means our approximation $\widehat{V_{\text{out}}}(V_{\text{in}})$ is very close to the true V_{out}.)

Solution: The linearized equation is

$$\widehat{V_{\text{out}}}(V_{\text{in}}) = V_{\text{DD}} - R_L\frac{K}{2}(V_{\text{in,DC}} - V_T)^2 - R_L g_m v_{\text{in,AC}} \tag{30}$$

Note that the term $-R_L g_m v_{\text{in,AC}}$ is time-varying due to the $v_{\text{in,AC}}$ term being time-varying. Hence, we may define $V_{\text{out,DC}} := V_{\text{DD}} - R_L\frac{K}{2}(V_{\text{in,DC}} - V_T)^2$.

For very small $v_{\text{in,AC}}(t)$, we may write

$$V_{\text{out}} \approx V_{\text{out,DC}} - R_L g_m v_{\text{in,AC}} \tag{31}$$

since the linear approximation is close to the true value of V_{out}. Simplifying this, we obtain

$$v_{\text{out,AC}}(t) \approx -R_L g_m v_{\text{in,AC}}(t) \tag{32}$$

Thus,

$$\frac{v_{\text{out,AC}}(t)}{v_{\text{in,AC}}(t)} = -R_L g_m \tag{33}$$

