Homework 2

This homework is due on Friday, September 9, 2022 at 11:59PM. Self-grades and HW Resubmissions are due the following Friday, September 16, 2022 at 11:59PM.

1. Capacitor Physics

The two metallic plates of a parallel plate capacitor is separated by glass (permittivity $\varepsilon = 3.9 \times 8.854 \times 10^{-12}\,\frac{\mathrm{F}}{\mathrm{m}}$) whose thickness is $1\,\mu\mathrm{m}$.

(a) What is the capacitance per unit area?

Solution: We can assume that $A = 1$ (since we are looking at capacitance per unit area). Hence, we have

$$C = \frac{\varepsilon}{d} = 34.5\frac{\mu\mathrm{F}}{\mathrm{m}^2} \tag{1}$$

(b) It is known that the resistance of a material can be expressed as $R = \frac{\rho L}{A}$ where ρ is called resistivity (a material dependent constant), L is the length and A is the area of the resistor. Show the time constant in a R-C circuit with the same area for a resistor and capacitor is independent of the area.

Solution: We know that $\tau = RC$, so we can plug in formulas for R and C that we know:

$$\tau = RC = \left(\frac{\rho L}{A}\right)\left(\frac{A\varepsilon}{d}\right) = \frac{\rho L \epsilon}{d} \tag{2}$$

(c) If $\rho = 1.72 \times 10^{-8}\,\Omega\,\mathrm{m}$ (this is the value for copper), and the length of the resistor used is 1 meter, what is the time constant of this R-C circuit?

Solution: Plugging in the result from the previous part, we have

$$\tau = 1.72 \times 10^{-8} \times 3.9 \times 8.854 \times 10^{-12} \times 10^{6} \tag{3}$$

$$\approx 0.6 \times 10^{-12} = 0.6\,\mathrm{ps} \tag{4}$$

2. Hambley P4.3 and P4.4

(a) The initial voltage across the capacitor shown in Figure 1 is $v_c(0+) = 0$. Find an expression for the voltage across the capacitor as a function of time, and sketch it to scale versus time.

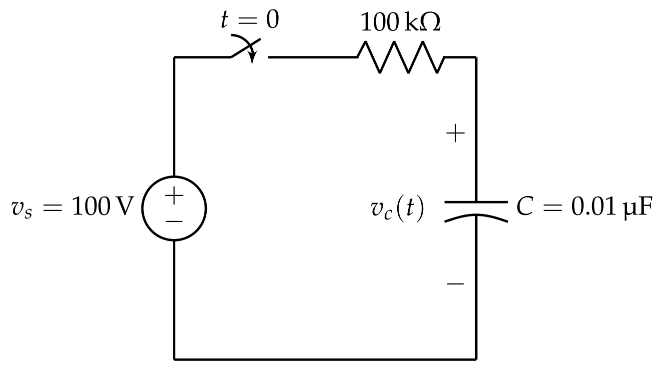


Figure 1: P4.3

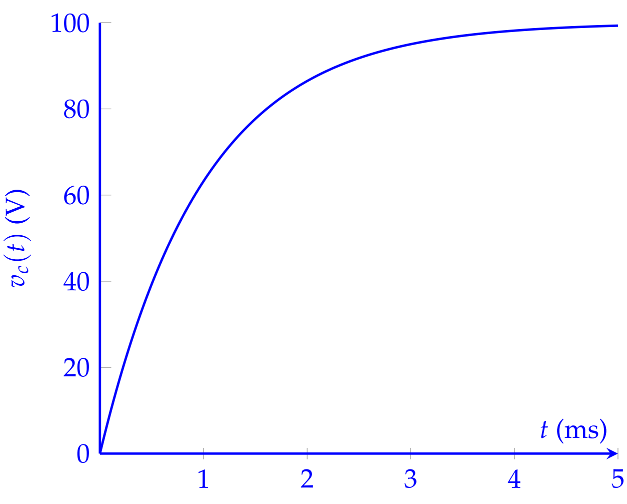
Solution: Recall the derivation performed in lecture:

$$v_c(t) = v_s\left(1 - e^{-\frac{t}{\tau}}\right) + v_c(0+)e^{-\frac{t}{\tau}} \tag{5}$$

where $\tau = RC = 1\,\mathrm{ms}$. In this specific case, we have $v_c(0+) = 0$ and $v_s = 100$, so we are left with

$$v_c(t) = 100\left(1 - e^{-\frac{t}{10^{-3}}}\right) \tag{6}$$

The graph of this is plotted below:



(b) Repeat part (a) for an initial voltage $v_c(0+) = -50\,\mathrm{V}$.

Solution: Using the same solution as above but setting $v_c(0+) = -50\,\mathrm{V}$, we end up with

$$v_c(t) = 100\left(1 - e^{-\frac{t}{10^{-3}}}\right) - 50e^{-\frac{t}{10^{-3}}} \tag{7}$$

which is plotted below:

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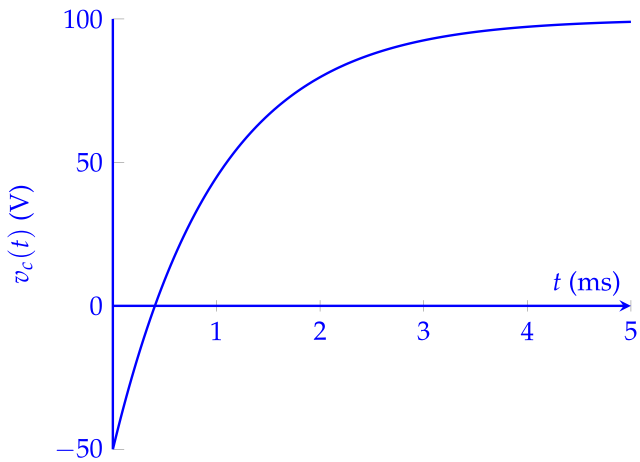
100
50
0
−50
$v_c(t)$ (V)
t (ms)
1
2
3
4
5

3. Hambley P4.7

The capacitor shown in Figure 2 is charged to a voltage of 50 V prior to $t = 0$.

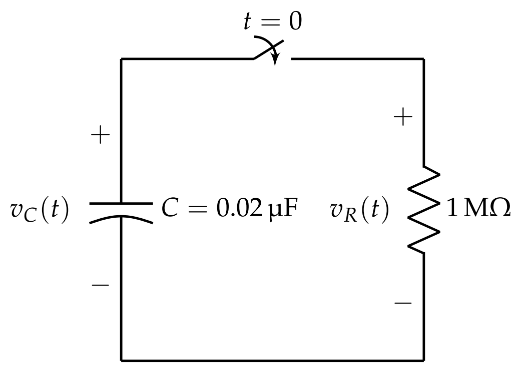

Figure 2: P4.7

(a) Find expressions for the voltage across the capacitor $v_C(t)$ and the voltage across the resistor $v_R(t)$.

Solution: We have that $RC = 0.02\,\text{s}$. Hence, using the derivation from lecture,

$$v_C(t) = \begin{cases} 50 & t < 0 \\ 50e^{-\frac{t}{0.02}} = 50e^{-50t} & t > 0 \end{cases} \tag{8}$$

and

$$v_R(t) = \begin{cases} 0 & t < 0 \\ 50e^{-\frac{t}{0.02}} = 50e^{-50t} & t > 0 \end{cases} \tag{9}$$

(b) Find an expression for the power delivered to the resistor.

Solution: We have that

$$p_R(t) = \frac{v_R(t)^2}{R} = \frac{2500}{10^6} e^{-100t} = 2.5 \times 10^{-3} e^{-100t}\,\text{W} \tag{10}$$

for $t > 0$, and $p_R(t) = 0\,\text{W}$ for $t < 0$

(c) Integrate the power from $t = 0$ to $t = \infty$ to find the energy delivered.

Solution: We have that

$$\begin{aligned} W &= \int_0^\infty p_R(t)\,dt && (11) \\ &= 2.5 \times 10^{-3} \int_0^\infty e^{-100t}\,dt && (12) \\ &= -2.5 \times 10^{-5} \left(e^{-100t}\big|_0^\infty \right) && (13) \\ &= 25\,\mu\text{J} && (14) \end{aligned}$$

(d) Show that the energy delivered to the resistor is equal to the energy stored in the capacitor prior to $t = 0$.

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Solution: The initial energy stored in the capacitance is

$$W = \frac{1}{2}C(v_C(0))^2 \tag{15}$$
$$= \frac{1}{2} \times 0.02 \times 10^{-6} \times 50^2 \tag{16}$$
$$= 25\,\mu\text{J} \tag{17}$$

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4. Hambley P4.15

A capacitance C is charged to an initial voltage V_i. At $t = 0$, a resistance R is connected across the capacitance. Write an expression for the current. Then, integrate the current from $t = 0$ to $t = \infty$, and show that the result is equal to the initial charge stored on the capacitance.

Solution: The voltage across the resistance and capacitance is

$$v_C(t) = V_i \mathrm{e}^{-\frac{t}{RC}} \tag{18}$$

The initial charge stored on the capacitance is

$$Q_i = CV_i \tag{19}$$

The current through the resistance is

$$i_R(t) = \frac{V_i}{R}\mathrm{e}^{-\frac{t}{RC}} \tag{20}$$

The total charge passing through the resistance is

$$\begin{aligned} Q &= \int_0^\infty i_R(t)\,\mathrm{d}t & (21)\\ &= \int_0^\infty \frac{V_i}{R}\mathrm{e}^{-\frac{t}{RC}}\,\mathrm{d}t & (22)\\ &= \frac{V_i}{R}\left(-RC\mathrm{e}^{-\frac{t}{RC}}\right)_0^\infty & (23)\\ &= CV_i & (24)\end{aligned}$$

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5. Hambley P4.18

Consider the circuit shown in Figure 3. Prior to $t = 0$, $v_1 = 100\,\text{V}$ and $v_2 = 0$.

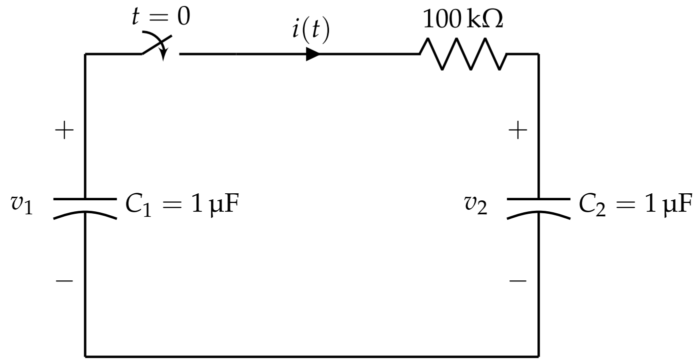

Figure 3: P4.18

(a) Immediately after the switch is closed, what is the value of the current (i.e., what is the value of $i(0+)$)?

Solution: The voltage across the capacitors cannot change instantaneously. Therefore,

$$i(0+) = \frac{v_1 - v_2}{100\,\text{k}\Omega} = \frac{100}{100 \times 10^3} = 1\,\text{mA} \tag{25}$$

(b) Write the KVL equation for the circuit in terms of the current and initial voltages. Take the derivative to obtain a differential equation.

Solution: Applying KVL, we have

$$-v_1(t) + Ri(t) + v_2(t) = 0 \tag{26}$$

$$-\left(\frac{1}{C_1}\int_0^t -i(t')\,\mathrm{d}t' - v_1(0)\right) + Ri(t) + \left(\frac{1}{C_2}\int_0^t i(t')\,\mathrm{d}t' + v_2(0)\right) = 0 \tag{27}$$

Taking a derivative with respect to time and rearranging, we obtain

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{1}{R}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)i(t) = 0 \tag{28}$$

(c) What is the value of the time constant in this circuit?

Solution: The time constant is $\tau = R\frac{C_1 C_2}{C_1 + C_2} = 50\,\text{ms}$.

(d) Find an expression for the current as a function of time.

Solution: From part (b),

$$i(t) = i(0)\mathrm{e}^{-\frac{t}{\tau}} \tag{29}$$

$$= \mathrm{e}^{-\frac{t}{50 \times 10^{-3}}}\,\text{mA} \tag{30}$$

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(e) Find the value that v_2 approaches as t becomes very large.

Solution: We have

$$
\begin{aligned}
v_2(\infty) &= \frac{1}{C_2}\int_0^\infty i(t)\,\mathrm{d}t + v_2(0) && (31)\\
&= 10^6 \times 10^{-3}\int_0^\infty \mathrm{e}^{-\frac{t}{50\times 10^{-3}}}\,\mathrm{d}t + 0 && (32)\\
&= 10^3 \times \left(-50\times 10^{-3}\right)\left(\mathrm{e}^{-\frac{t}{50\times 10^{-3}}}\right)_0^\infty && (33)\\
&= 50\,\mathrm{V} && (34)
\end{aligned}
$$

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6. Capacitor Energy

Say a series R-C circuit is supplied by a constant voltage V. At $t = 0$, voltage across the capacitor was 0. We know how to find the expression for capacitor voltage as a function of time. Using this expression,

(a) Find the expression for total stored energy at $t = \infty$, $w_s = \int_0^\infty v(t)i(t)\,\mathrm{d}t$ (i.e., at steady state).

Solution: Applying the given formula, we have

$$w_s = \int_0^\infty v(t)i(t)\,\mathrm{d}t \tag{35}$$

$$= C\int_0^\infty v\frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}t \tag{36}$$

$$= \frac{CV^2}{RC}\int_0^\infty \left(1 - \mathrm{e}^{-\frac{t}{RC}}\right)\mathrm{e}^{-\frac{t}{RC}}\,\mathrm{d}t \tag{37}$$

$$= \frac{CV^2}{RC}(-RC)\left(-1 + \frac{1}{2}\right) \tag{38}$$

$$= \frac{1}{2}CV^2 \tag{39}$$

(b) We know that when a current flows through a resistor, we dissipate energy at a rate of i^2R. Using this relation, find the total dissipated energy at $t = \infty$, $w_d = \int_0^\infty i^2R\,\mathrm{d}t$ (i.e., at steady state).

Solution: First we can solve for the current, i.e.,

$$i = C\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{CV}{RC}\mathrm{e}^{-\frac{t}{RC}} \tag{40}$$

Next, we can apply it to the given function for energy dissipated to obtain

$$w_d = \int_0^\infty i^2R\,\mathrm{d}t \tag{41}$$

$$= R\frac{C^2V^2}{R^2C^2}\int_0^\infty \mathrm{e}^{-\frac{2t}{RC}}\,\mathrm{d}t \tag{42}$$

$$= R\frac{C^2V^2}{R^2C^2}\left(\frac{RC}{2}\right) \tag{43}$$

$$= \frac{1}{2}CV^2 \tag{44}$$

(c) Find the total energy taken from the source, noting that some of it was stored and some of it was dissipated.

Solution: Adding the results from the previous two parts, we have that the total energy from the source is CV^2.

(d) Does the result in part (b) vary if $R = 100\,\Omega$ vs $R = 1\,\mathrm{k}\Omega$? What about $R = 0\,\Omega$? *OPTIONAL: Can you explain this result?*

Solution: The total energy dissipated is independent of R. This means that even if $R = 0$, the dissipation is $\frac{1}{2}CV^2$.

Essentially, when a circuit is suddenly turned ON, the large $\frac{\mathrm{d}\xi}{\mathrm{d}t}$, where ξ is an electric field, leads to radiation through power which is lost. Understanding this more deeply requires knowledge of electromagnetcis, which is the topic of EE118. This is also at the heart of how antennas are able to radiate energy into the atmosphere.

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7. Hambley P4.21

Solve for the steady-state values of i_1, i_2, and i_3 in the circuit shown in Figure 4.

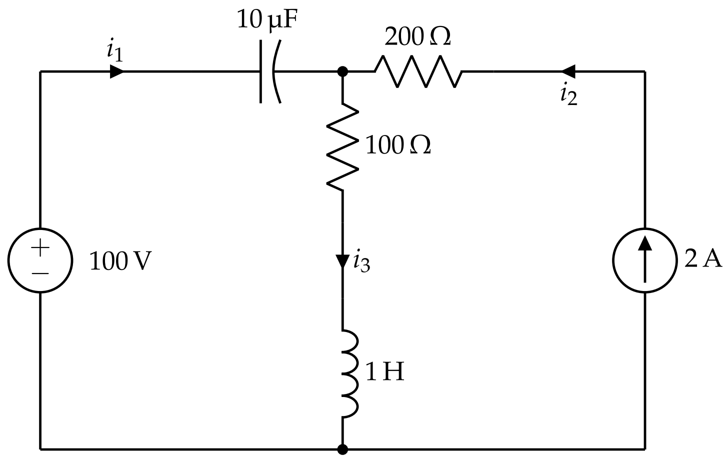
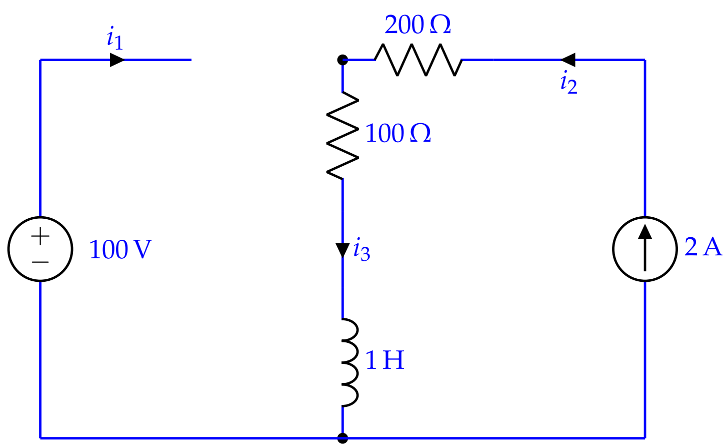


Figure 4: P4.21

Solution: At steady state, the equivalent circuit is



so $i_1 = 0$ and $i_2 = i_3 = 2\,\text{A}$.

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8. Hambley P4.41

Determine expressions for and sketch $v_R(t)$ to scale versus time for the circuit of Figure 5. The circuit is operating in steady state with the switch closed prior to $t = 0$. Consider the time interval $-0.2 \leq t \leq 1\,\text{ms}$.

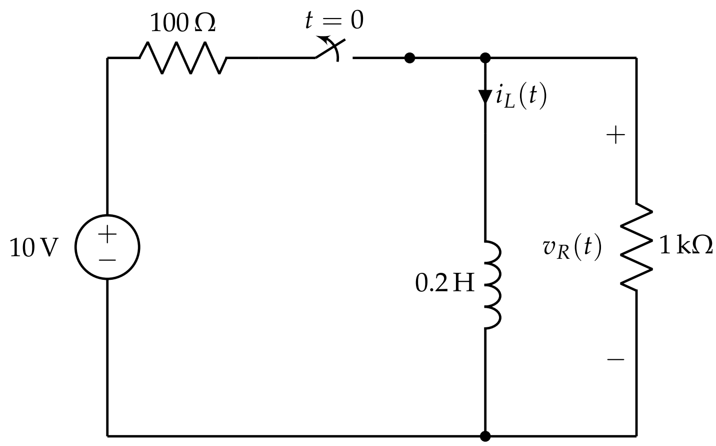
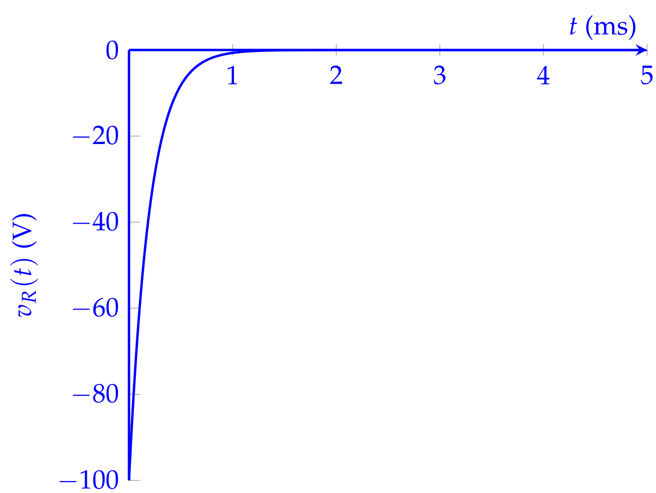


Figure 5: P4.41

Solution: In the steady state the inductor acts as a short circuit (essentially shorting the 1 kΩ resistance on the right). Then $i_L = \frac{10}{100} = 0.1\,\text{A}$.

As the switch is opened, the circuit essentially becomes a R-L circuit with an initial voltage of $v_R(0) = -1000 \times 0.1 = -100\,\text{V}$. Then from $L\frac{\text{d}i}{\text{d}t} + iR = 0$, we get $i = i(0)\text{e}^{-\frac{R}{L}t}$. Thus, $v_R(t) = -100\text{e}^{-5000t}$. A plot of this is shown below:



9. Hambley P4.46

Consider the circuit shown in Figure 6. The voltage source is known as a **ramp function**, which is defined by

$$v(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases} \tag{45}$$

Assume that $v_C(0) = 0$. Derive an expression for $v_C(0)$ for $t \geq 0$. Sketch $v_C(t)$ to scale versus time.

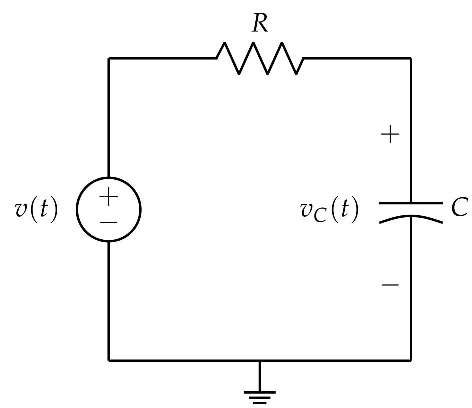


Figure 6: P4.46

Solution: From KCL, we have the following equations:

$$\frac{v(t) - v_C(t)}{R} = C\frac{\mathrm{d}v_C}{\mathrm{d}t} \tag{46}$$

$$\frac{\mathrm{d}v_C}{\mathrm{d}t} + \frac{1}{RC}v_C(t) = \frac{1}{RC}v(t) \tag{47}$$

We can define our integrating factor as $f(t) = \mathrm{e}^{\int \frac{1}{RC}\mathrm{d}t} = \mathrm{e}^{\frac{t}{RC}}$ which leaves us with

$$\mathrm{e}^{\frac{t}{RC}}\left(\frac{\mathrm{d}v_C}{\mathrm{d}t} + \frac{1}{RC}v_C(t)\right) = \mathrm{e}^{\frac{t}{RC}}\left(\frac{1}{RC}v(t)\right) \tag{48}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\frac{t}{RC}}v_C(t)\right) = \mathrm{e}^{\frac{t}{RC}}\left(\frac{1}{RC}v(t)\right) \tag{49}$$

Integrating both sides, we obtain

$$\int \frac{\mathrm{d}}{\mathrm{d}t'}\left(\mathrm{e}^{\frac{t'}{RC}}v_C(t')\right)\mathrm{d}t' = \int \mathrm{e}^{\frac{t'}{RC}}\left(\frac{1}{RC}v(t')\right)\mathrm{d}t' \tag{50}$$

$$\mathrm{e}^{\frac{t}{RC}}v_C(t) - C_1 = \int \mathrm{e}^{\frac{t'}{RC}}\left(\frac{1}{RC}v(t')\right)\mathrm{d}t' \tag{51}$$

$$v_C(t) = C_1\mathrm{e}^{-\frac{t}{RC}} + \mathrm{e}^{-\frac{t}{RC}}\int \mathrm{e}^{\frac{t'}{RC}}\left(\frac{1}{RC}v(t')\right)\mathrm{d}t' \tag{52}$$

for some arbitrary constant C_1 (that we will eventually solve for). Assuming $v(t) = t$ and writing $\tau = RC$, we obtain

$$\mathrm{e}^{-\frac{t}{\tau}}\int \mathrm{e}^{\frac{t'}{\tau}}\left(\frac{1}{\tau}v(t')\right)\mathrm{d}t' = \mathrm{e}^{-\frac{t}{\tau}}\int \mathrm{e}^{\frac{t'}{\tau}}\left(\frac{t'}{\tau}\right)\mathrm{d}t' = t - \tau + \tau\mathrm{e}^{-\frac{t}{\tau}} \tag{53}$$

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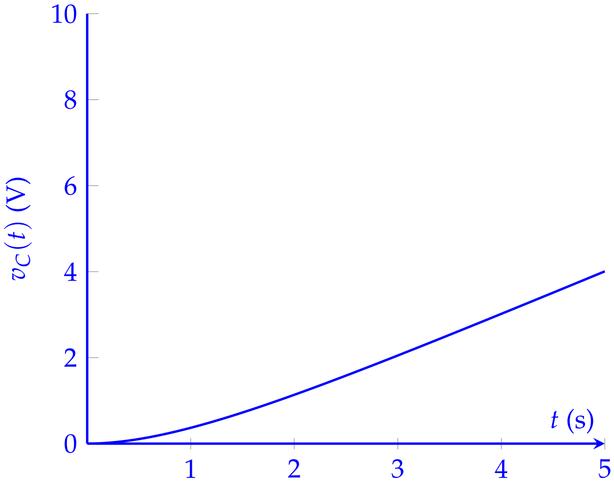
Then,

$$v_C(t) = (C_1 + \tau)e^{-\frac{t}{\tau}} + t - \tau \tag{54}$$

To find C_1, we plug in $t = 0$ to the above equation and obtain $C_1 = v_C(0) = 0$, so the final expression is

$$v_C(t) = \tau e^{-\frac{t}{\tau}} + t - \tau \tag{55}$$

A sketch of this function might look like the following (when $\tau = 1$):



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