EE16B
Designing Information Devices and Systems II

Lecture 9A
Singular Value Decomposition (SVD)

Recap

- Last time
  - Finished outputs and system ID

- Today:
  - New module: the Singular Value Decomposition
  - Describe the SVD
  - Show example SVD is useful in applications
- Start
  - How to calculate the SVD

Rank 1 Matrix

Consider the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 \end{bmatrix}$$

Rank = 1

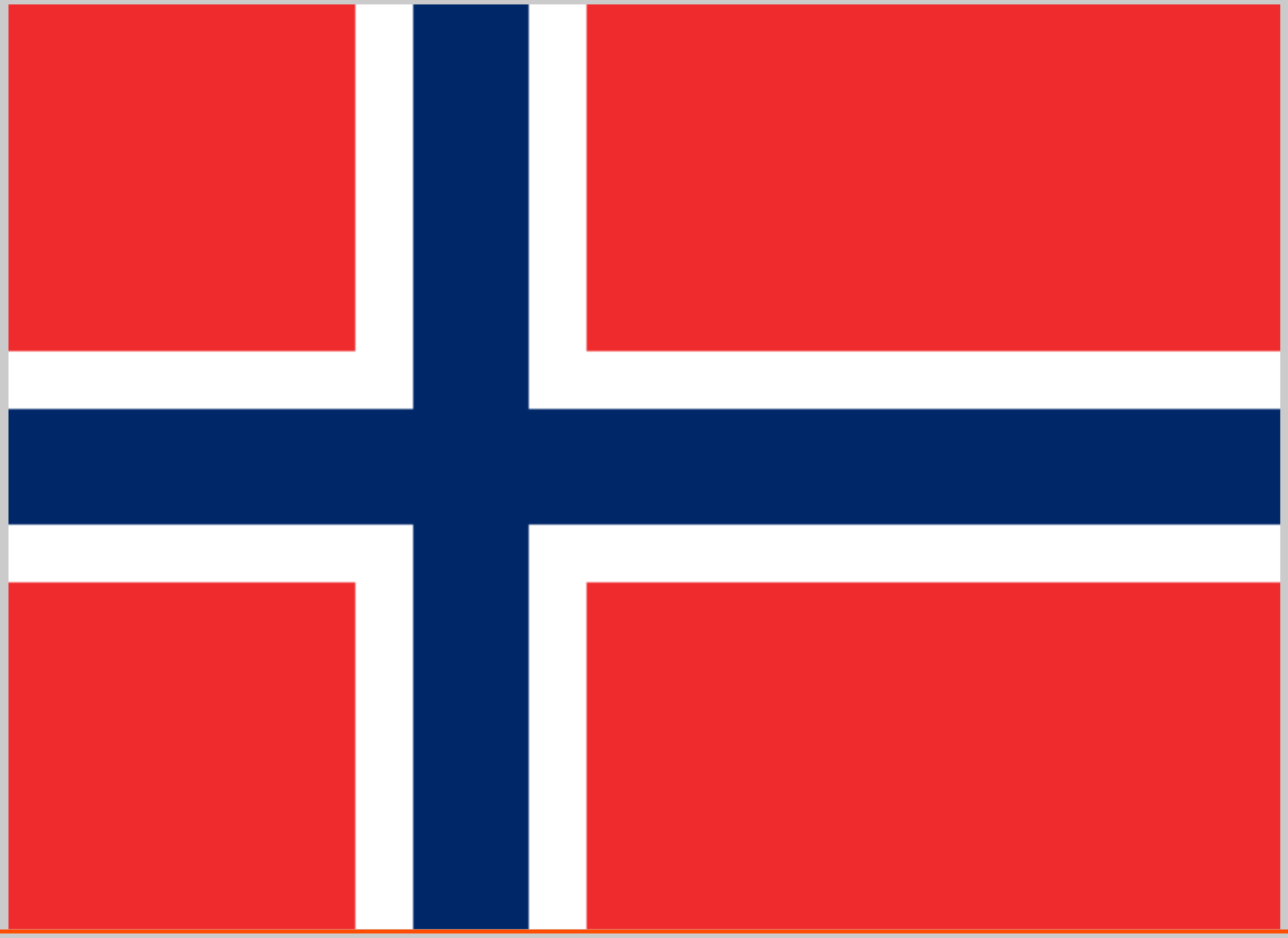
We can decompose a rank-1 matrix as an outer product:

$$\vec{u}\vec{v}^T \in \mathrm{R}^{m \times n}$$

$$\vec{u} \in \mathrm{R}^m$$
$$\vec{v} \in \mathrm{R}^n$$

$$\underset{m \times 1}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \underset{1 \times n}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}}$$

Flags as low-rank matrices



SVD

SVD decomposes a rank r matrix $A \in \mathrm{R}^{m \times n}$
into a sum of r rank-1 matrices:

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

1) $\vec{u}_i^T \vec{u}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad \Rightarrow ||\vec{u}_i|| = 1 \quad \vec{u}_i \perp \vec{u}_j$

2) $\vec{v}_i^T \vec{v}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad \Rightarrow ||\vec{v}_i|| = 1 \quad \vec{v}_i \perp \vec{v}_j$

3) $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

SVD

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

$m \times n$ $\quad m + n \quad m + n \quad m + n$

$r(m+n) \leq mn$ If m,n are large and r is small

Typically, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\hat{r}} >> \sigma_{\hat{r}+1} \geq \cdots \geq \sigma_r$

$10 \quad 8 \quad 5 \quad 0.1 \quad 0.001$

$$\begin{bmatrix} 1.02 & 0.99 & 0.98 & 1.03 & 1.01 & 1 \\ 2 & 1.98 & 2.01 & 2.03 & 1.99 & 1.97 \\ 3.01 & 2.98 & 3 & 2.99 & 3.03 & 3.02 \end{bmatrix}$$

$$A \approx \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_1 \vec{v}_1^T + \cdots + \sigma_{\hat{r}} \vec{u}_{\hat{r}} \vec{v}_{\hat{r}}^T$$

History

Known for a long time – but not considered important

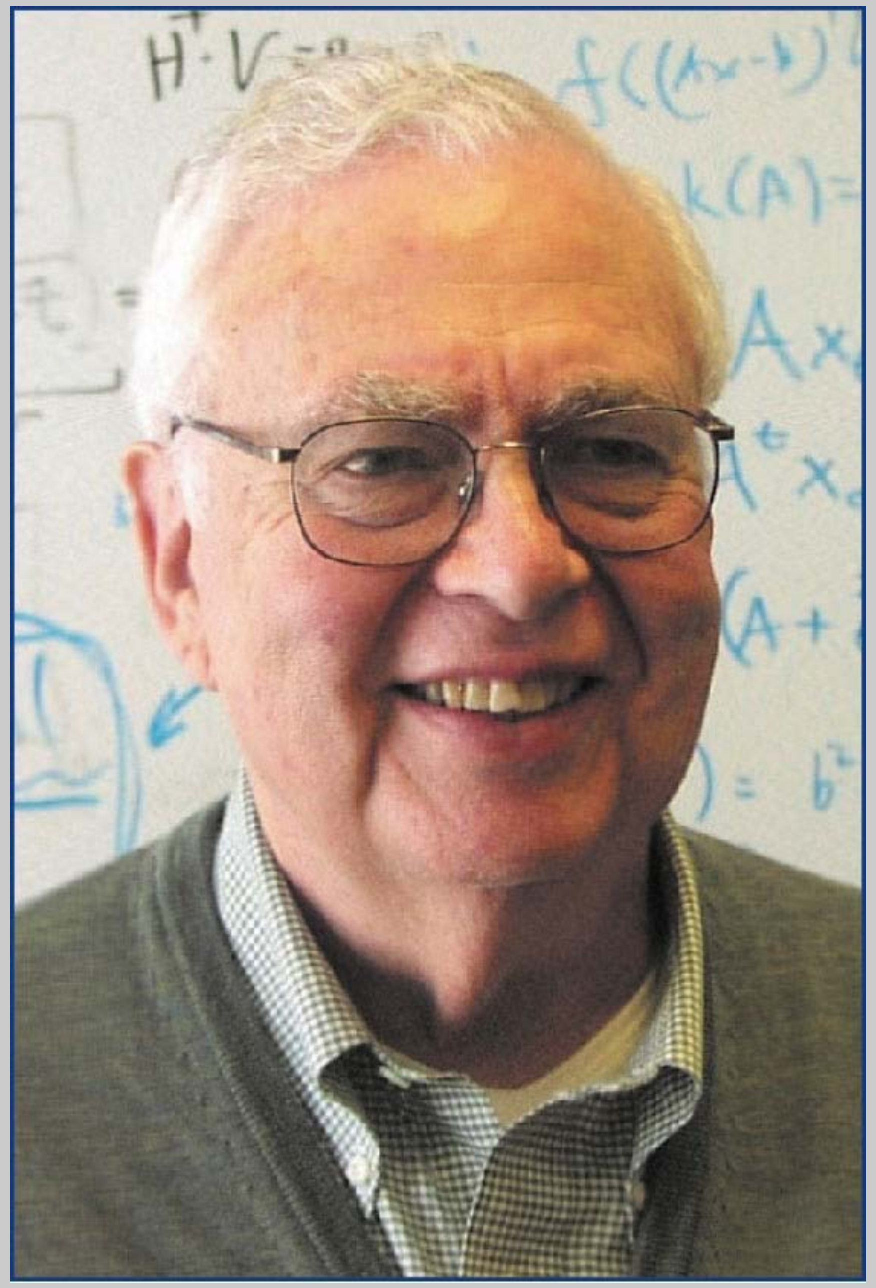
1965 – G. Golub and W. Kahan – practical way of computing the SVD – numerically stable

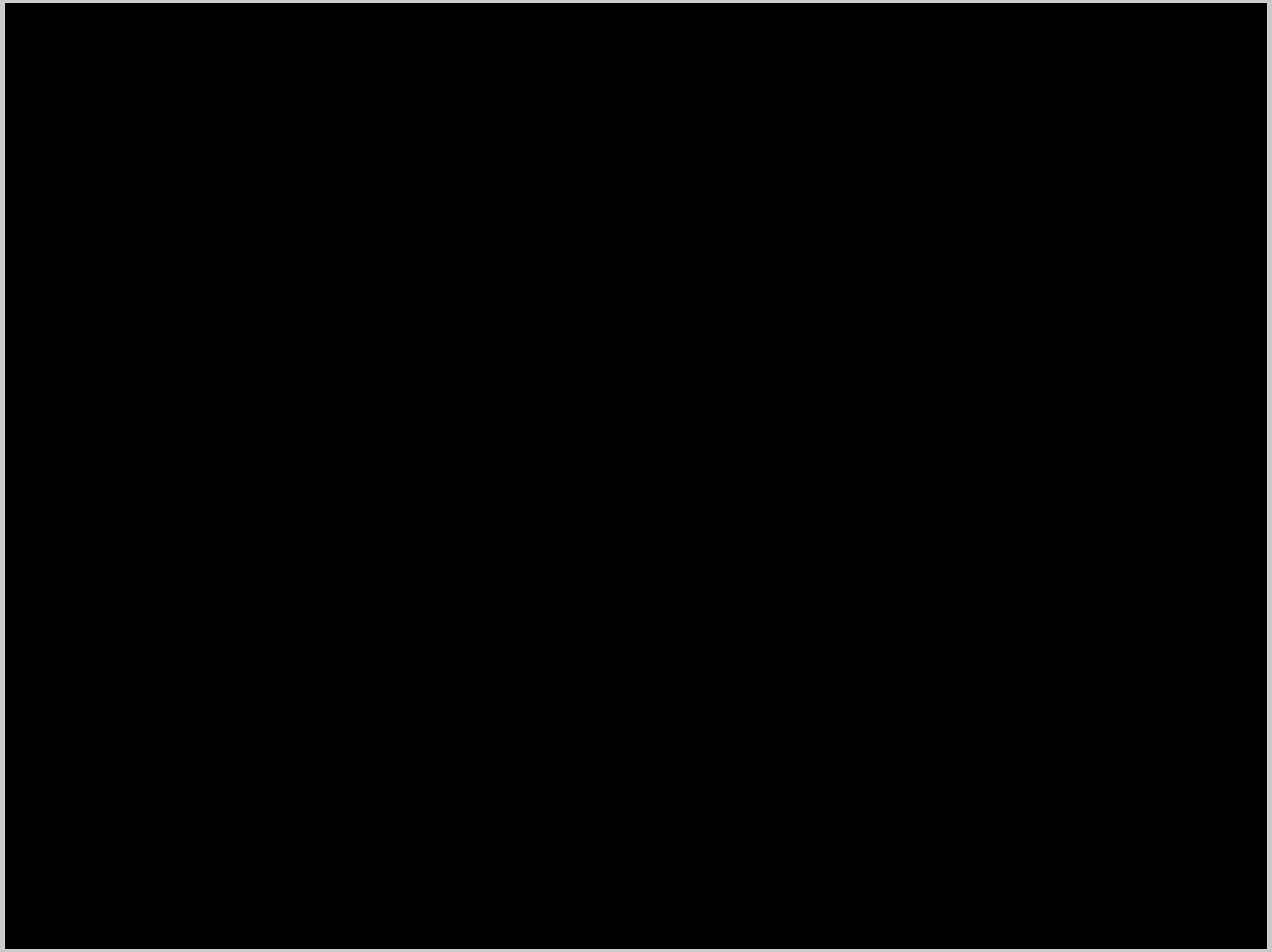
Today, used everywhere!

LAPack implementation –Demmel & Kahan 1990

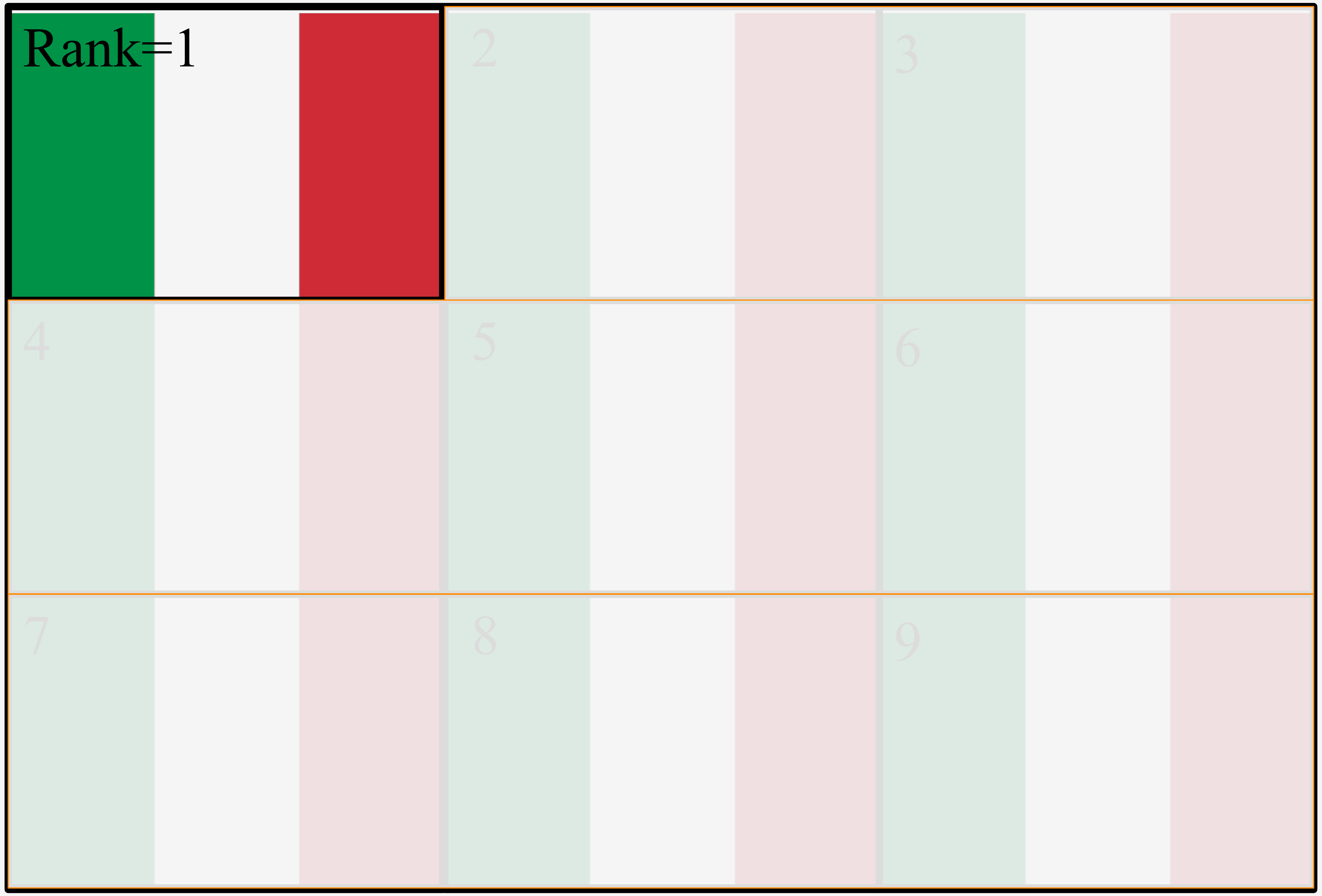
(numpy and matlab use LAPack)

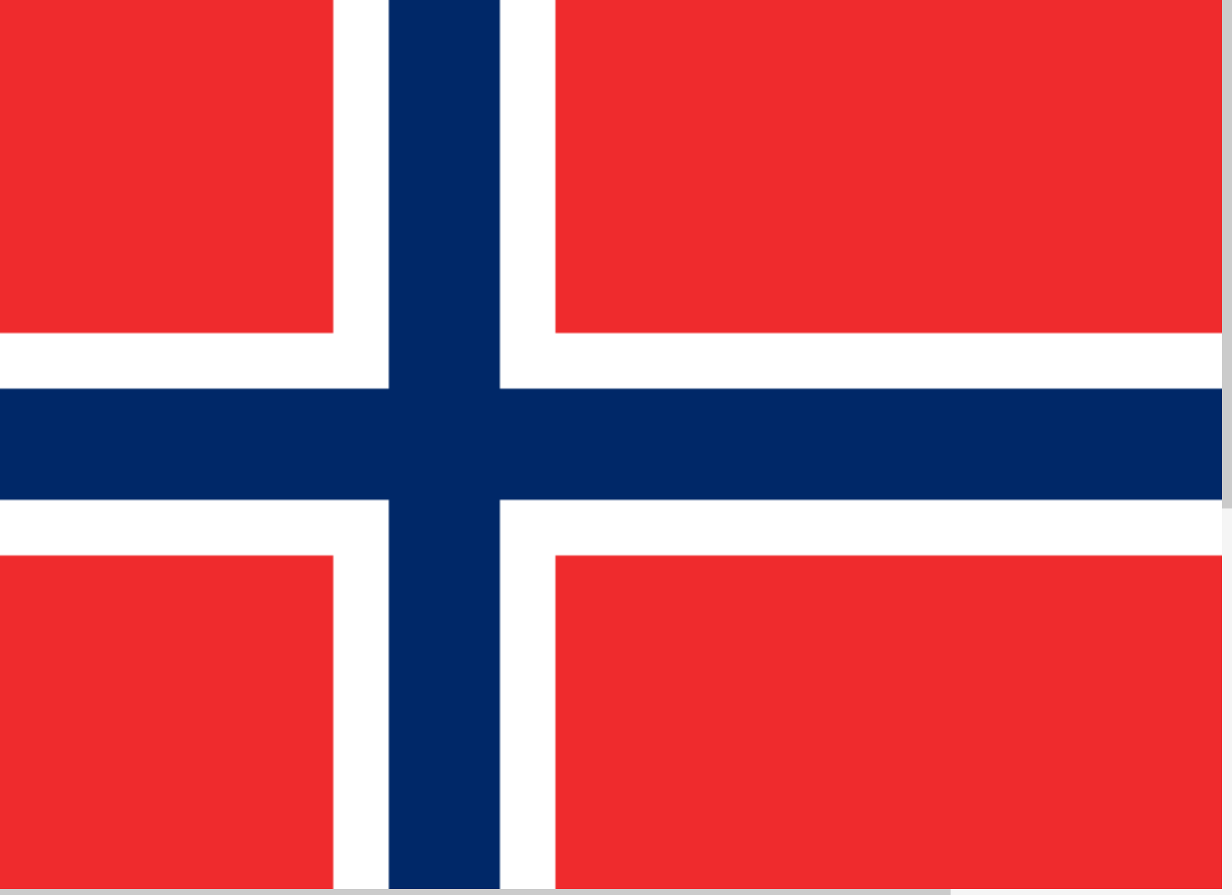
Gene Golub's license plate, photographed by Professor P. M. Kroonenberg of Leiden University.

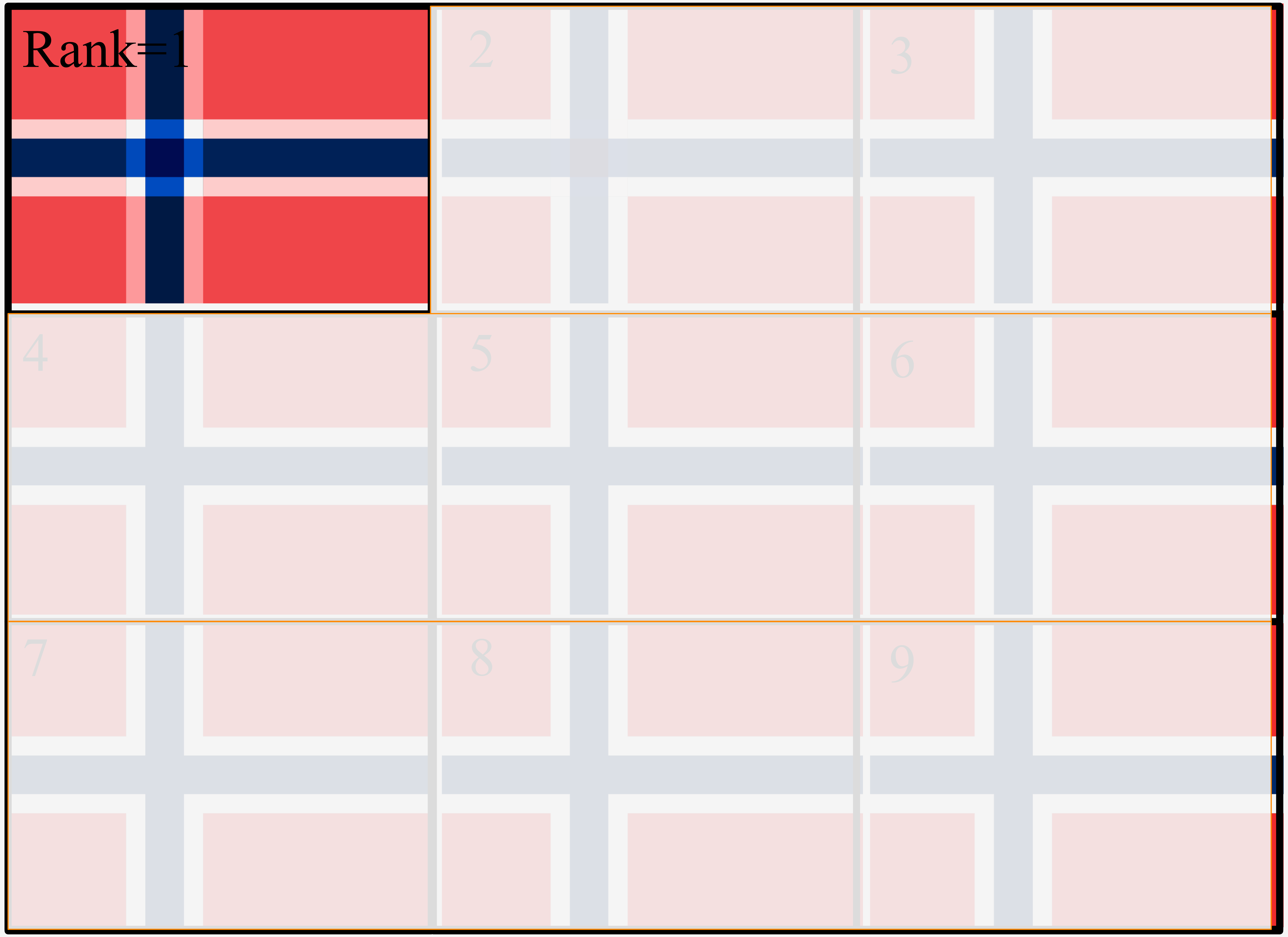


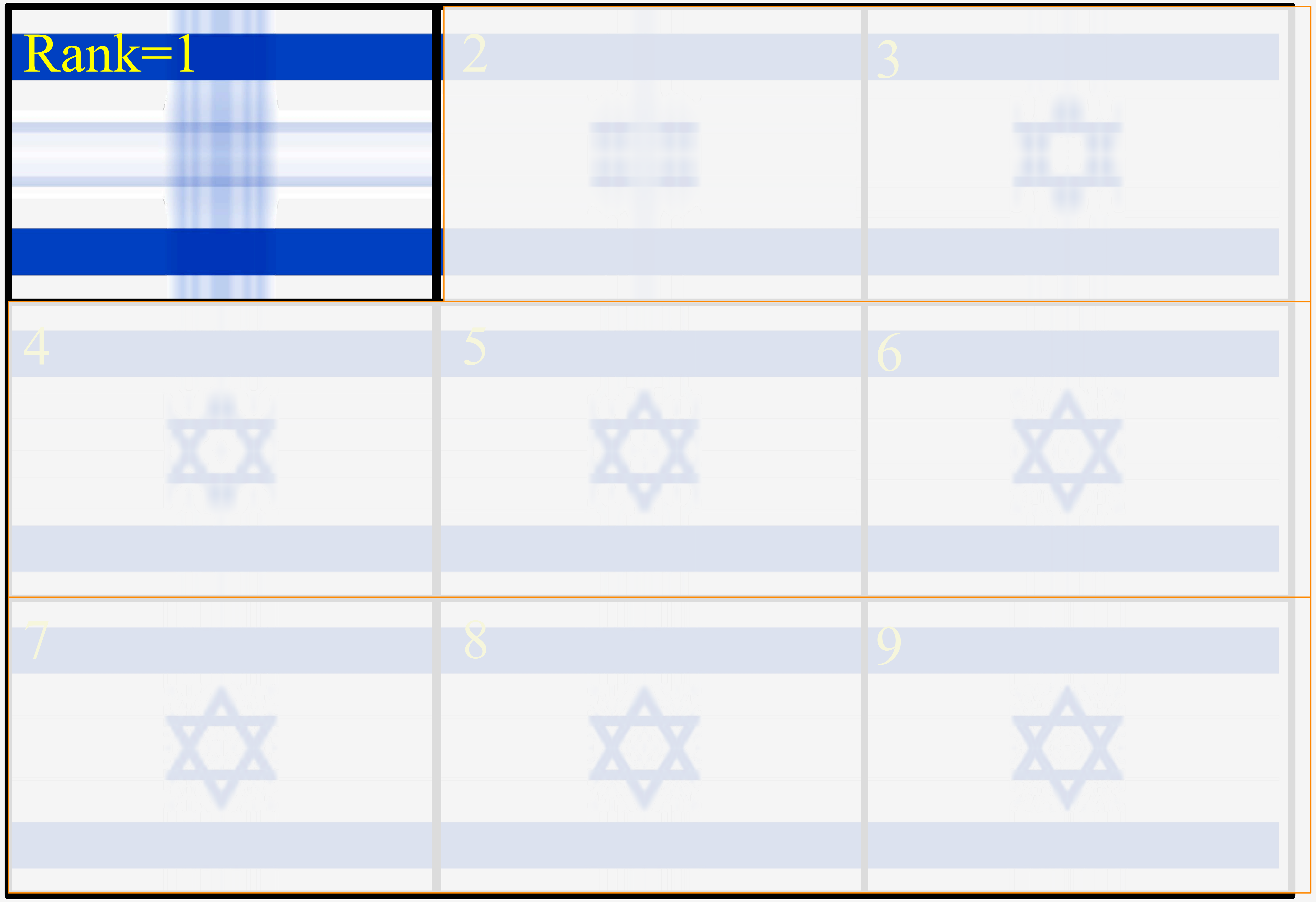


- Los Alamos video 1976 on the SVD. Then Relatively unknow, but today used everywhere. 3D computer graphics by Cleve Moler (Matlab)
- Lecture by Moler: “SVD Saves the Universe” https://www.mathworks.com/videos/the-singular-value-decomposition-saves-the-universe-1481294462044.html

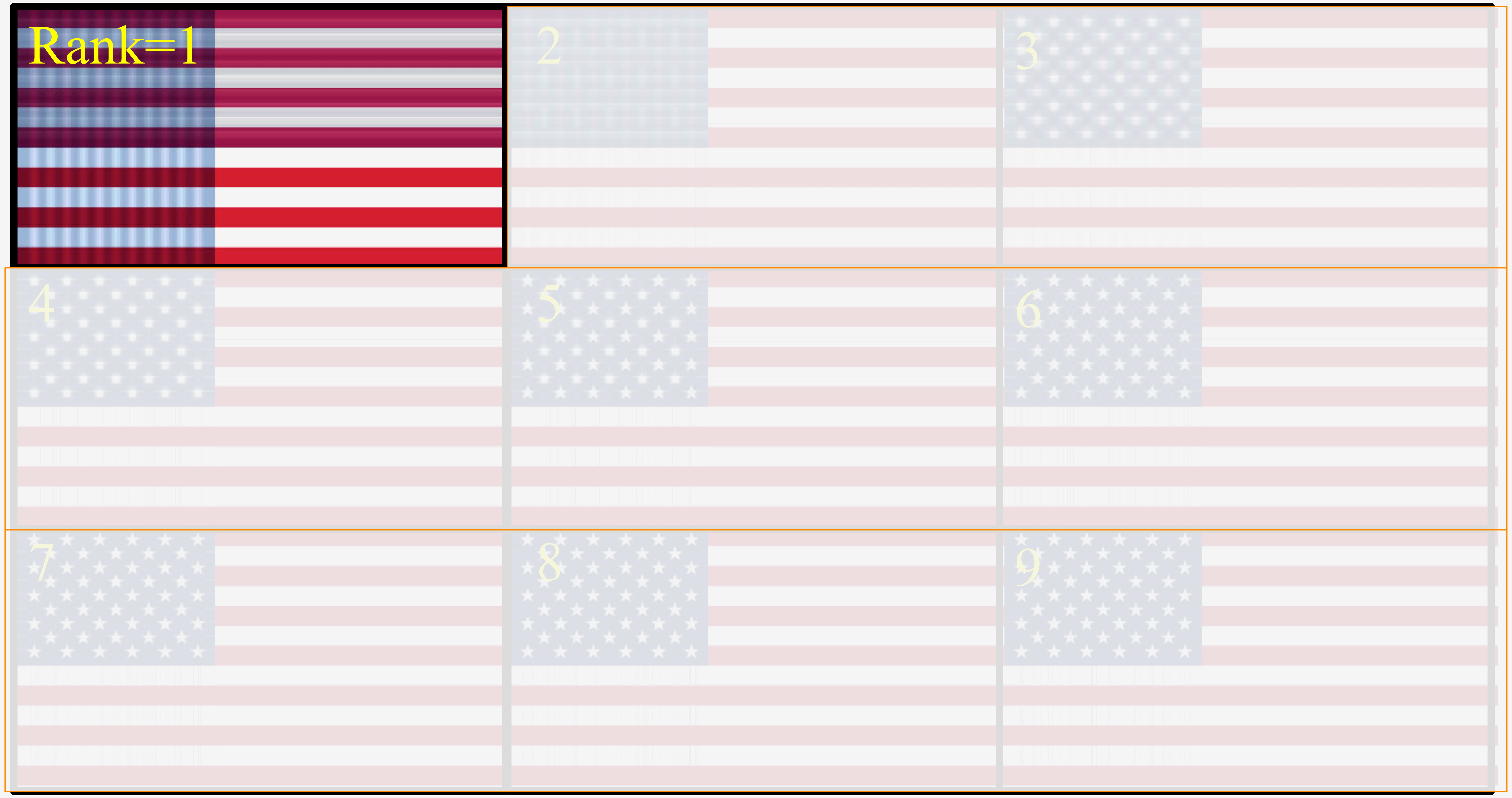

Rank=1
2
3
4
5
6
7
8
9

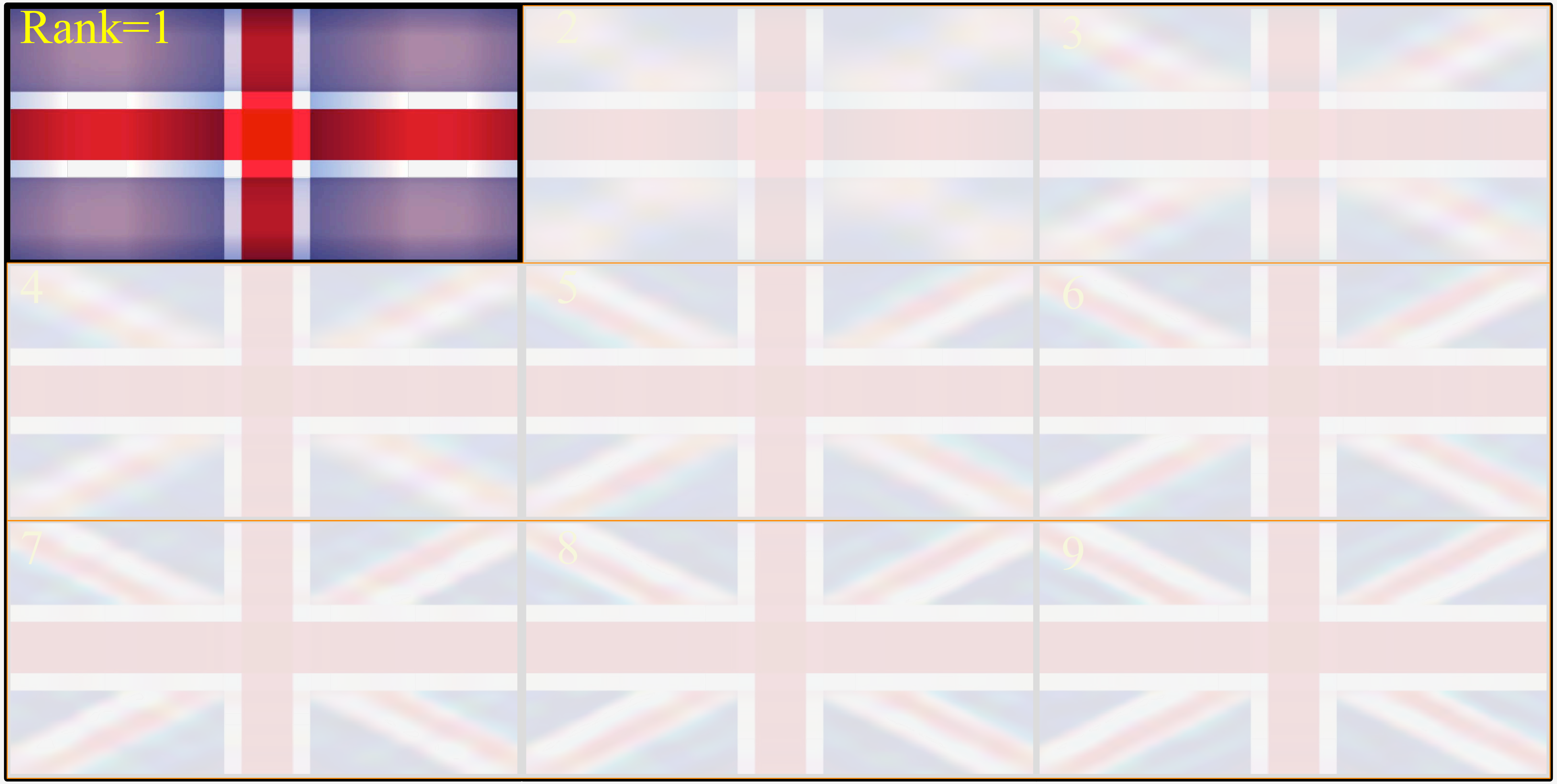



Rank=1
2
3
4
5
6
7
8
9



Rank=1
2
3
4
5
6
7
8
9

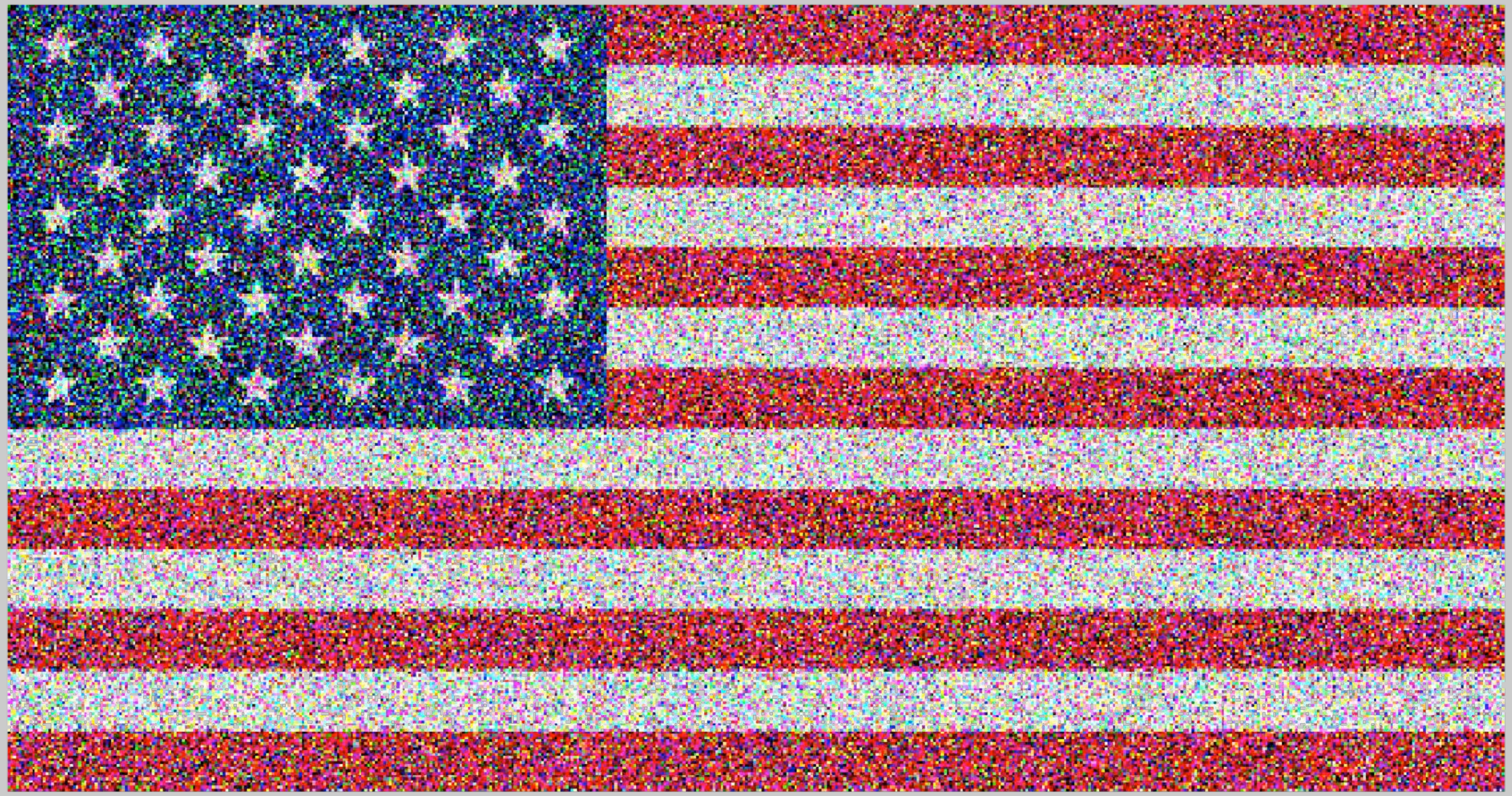

Rank=1
2
3
4
5
6
7
8
9

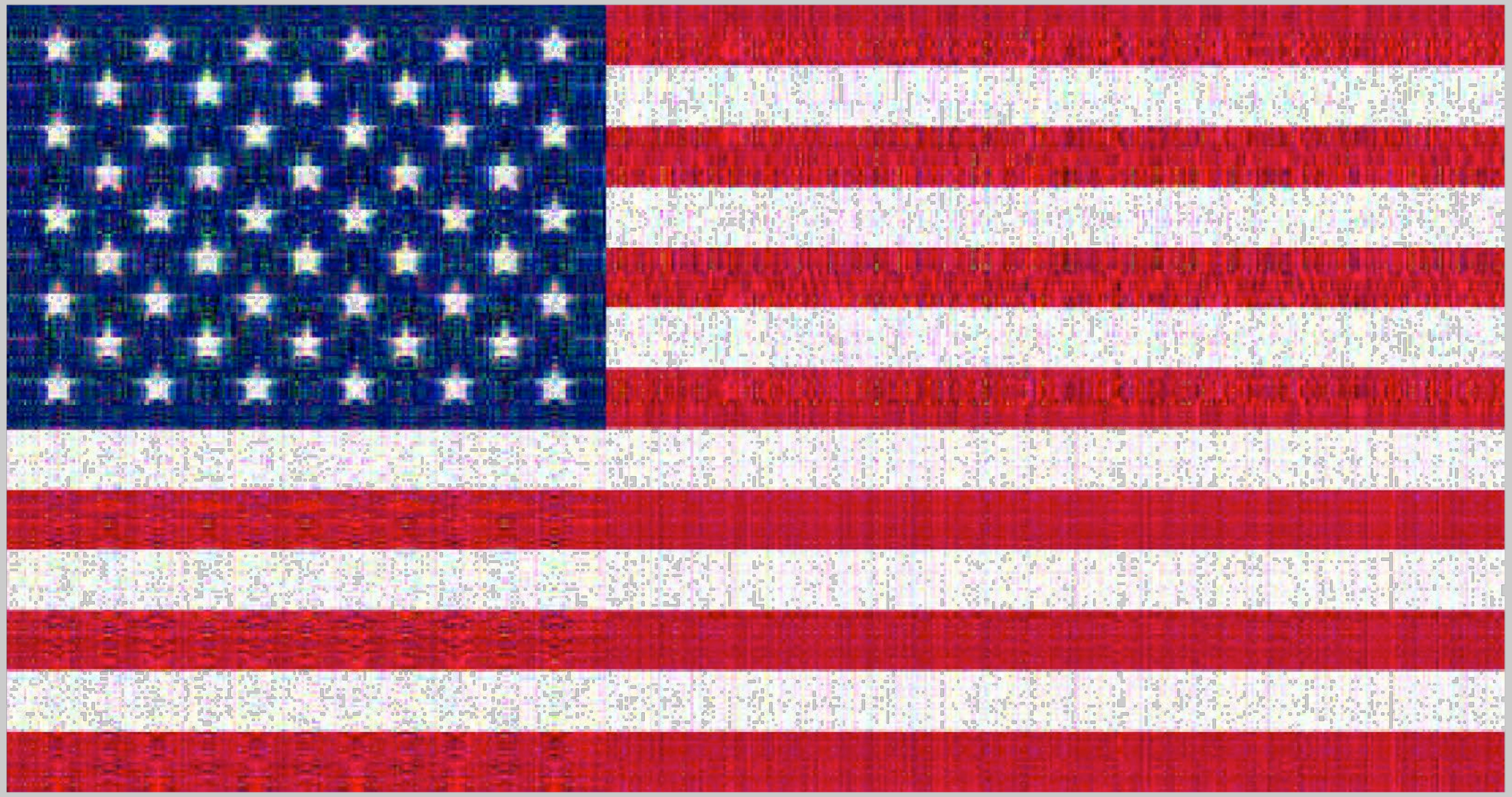

Rank=1
2
3
4
5
6
7
8
9

Noisy Flag

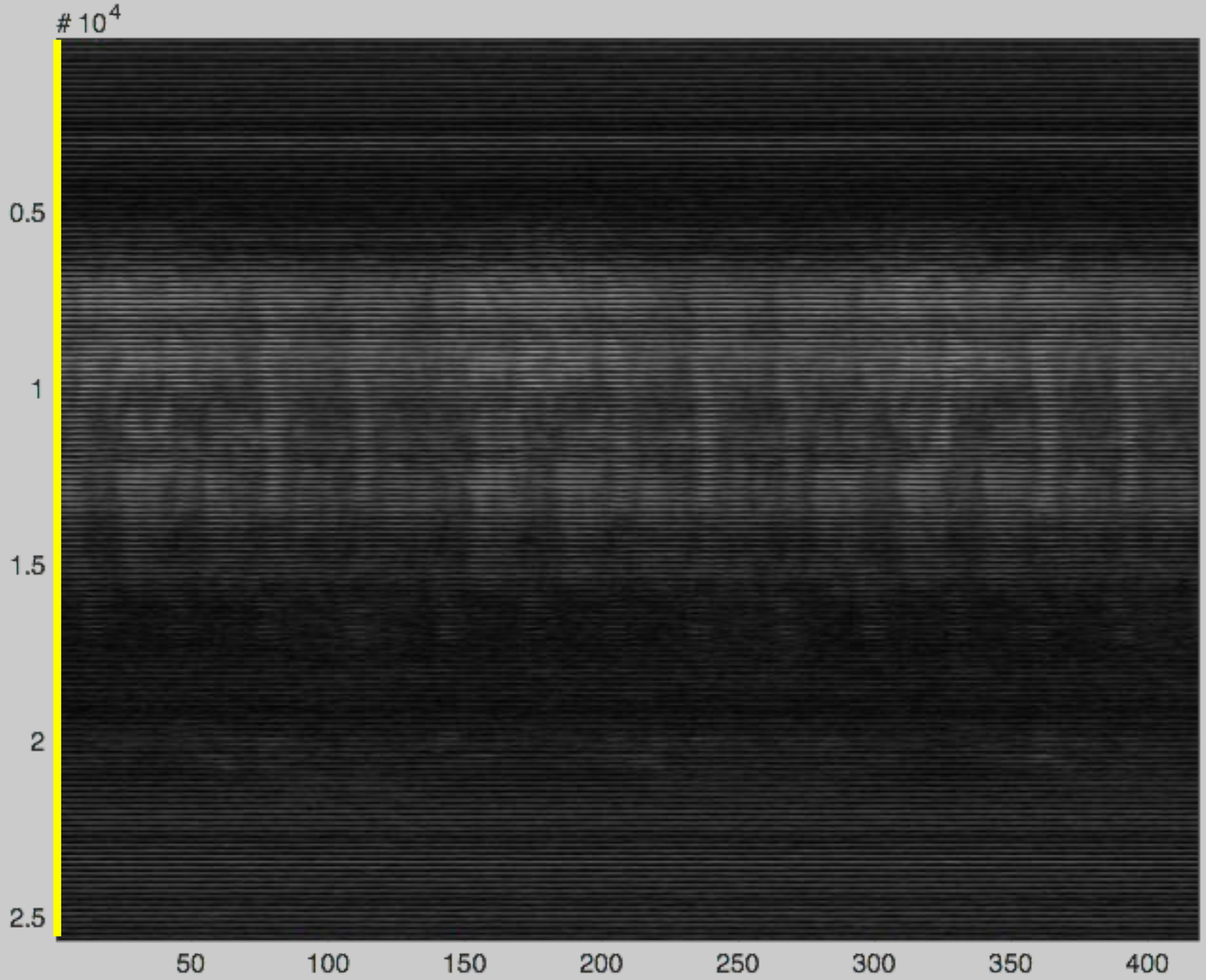
$$\begin{bmatrix} 1.02 & 0.99 & 0.98 & 1.03 & 1.01 & 1 \\ 2 & 1.98 & 2.01 & 2.03 & 1.99 & 1.97 \\ 3.01 & 2.98 & 3 & 2.99 & 3.03 & 3.02 \end{bmatrix}$$

$$\sum_{i=0}^{5} \sigma_i \vec{u}_i \vec{v}_i^T$$





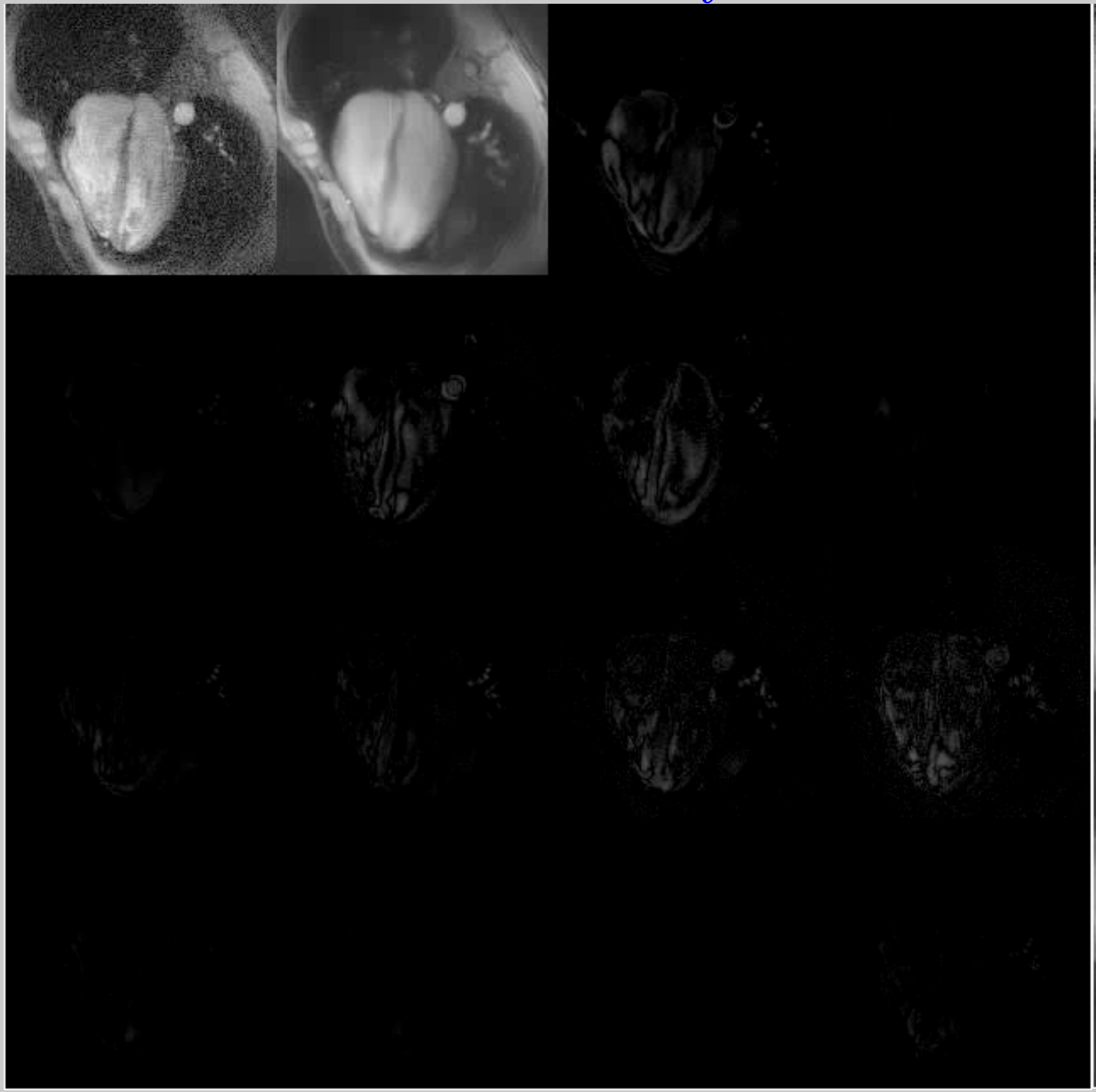
Video of a Heart

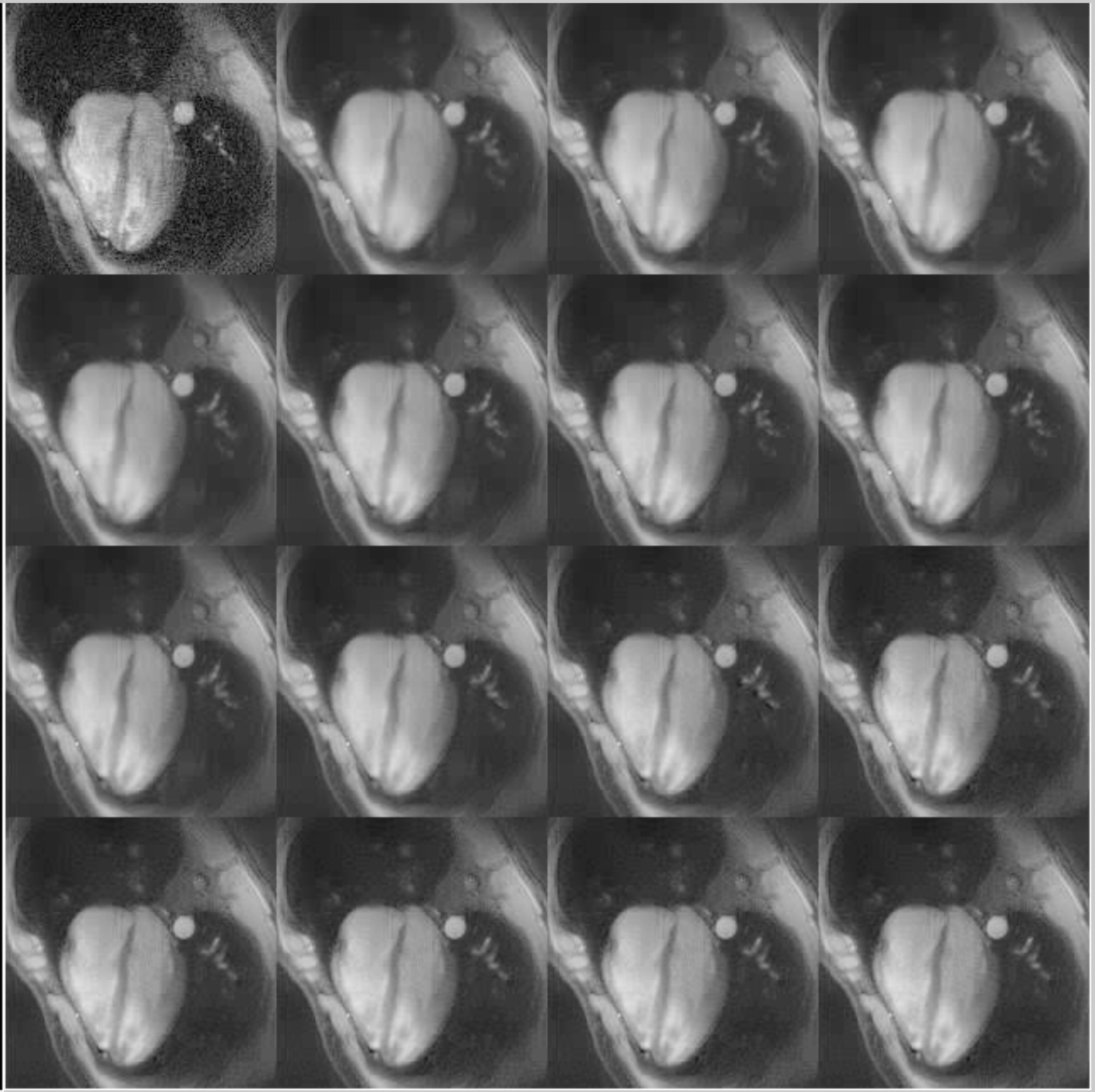


Heart Example

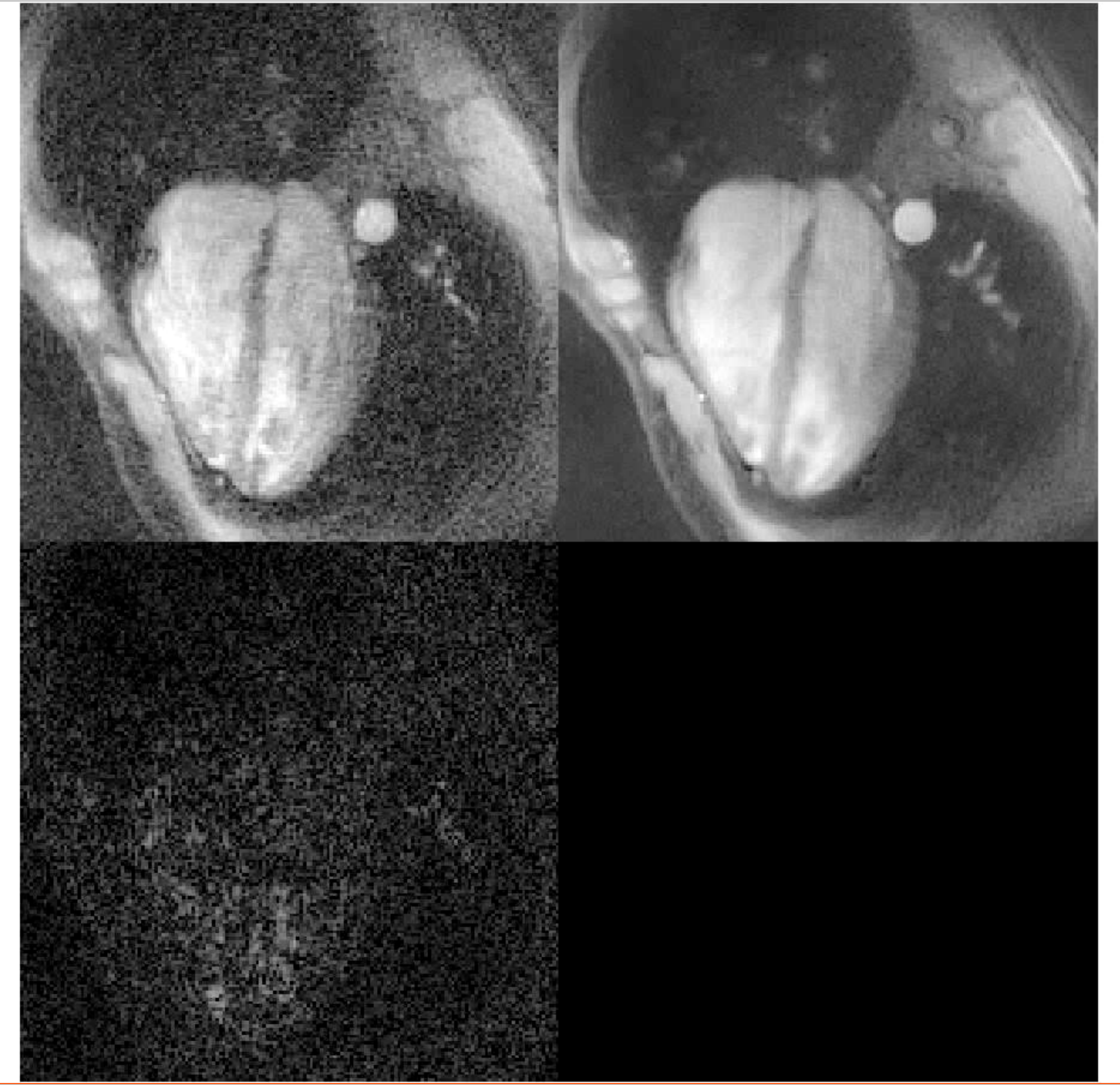
$\sigma_i \vec{u}_i \vec{v}_i^T$

$\sum_{i=0}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$





Heart Example



$$\sum_{i=0}^{15} \sigma_i \vec{u}_i \vec{v}_i^T$$

$$\sum_{i=16}^{417} \sigma_i \vec{u}_i \vec{v}_i^T$$

Data Analysis with SVD

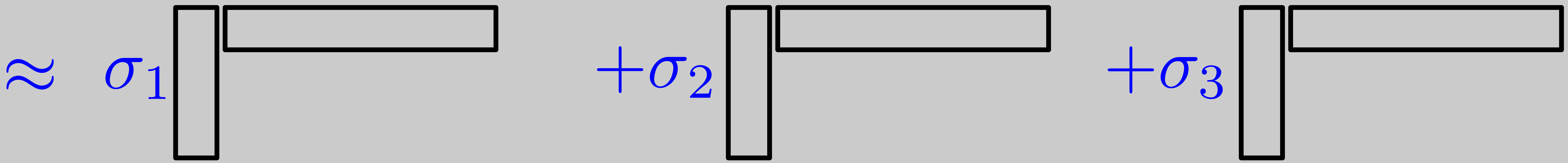
$$A \approx \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_1 \vec{v}_1^T + \cdots + \sigma_{\hat{r}} \vec{u}_{\hat{r}} \vec{v}_{\hat{r}}^T$$

effective

n movies

m views

1	2	5	5	1	3	3	3	2	5	5	4	4	3	2	2	5	1	1	
5	3	2	1	1	3	3	3	1	1	2	4	5	5	4	4	1	3	2	
5	1	2	1	1	2	3	3	1	1	2	1	5	5	3	5	1	1	2	
5	3	2	1	1	3	2	3	1	1	2	4	1	1	4	4	5	1	5	
1	2	3	2	1	3	2	3	2	1	2	1	1	1	4	4	5	1	5	
1	1	1	1	5	3	3	3	1	5	2	4	4	4	2	5	5	1	1	

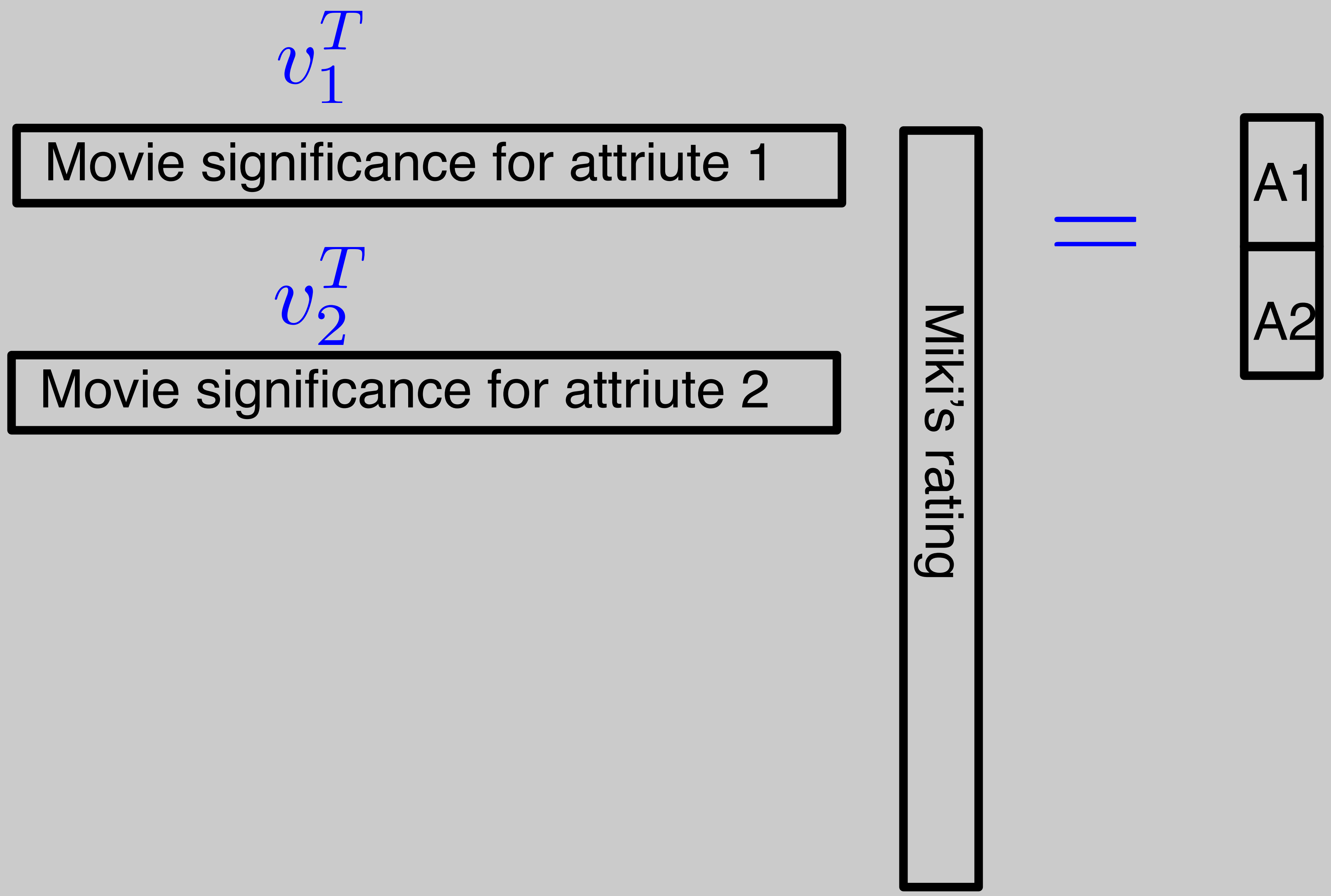


Attribute 1

Attribute 2

Attribute 3

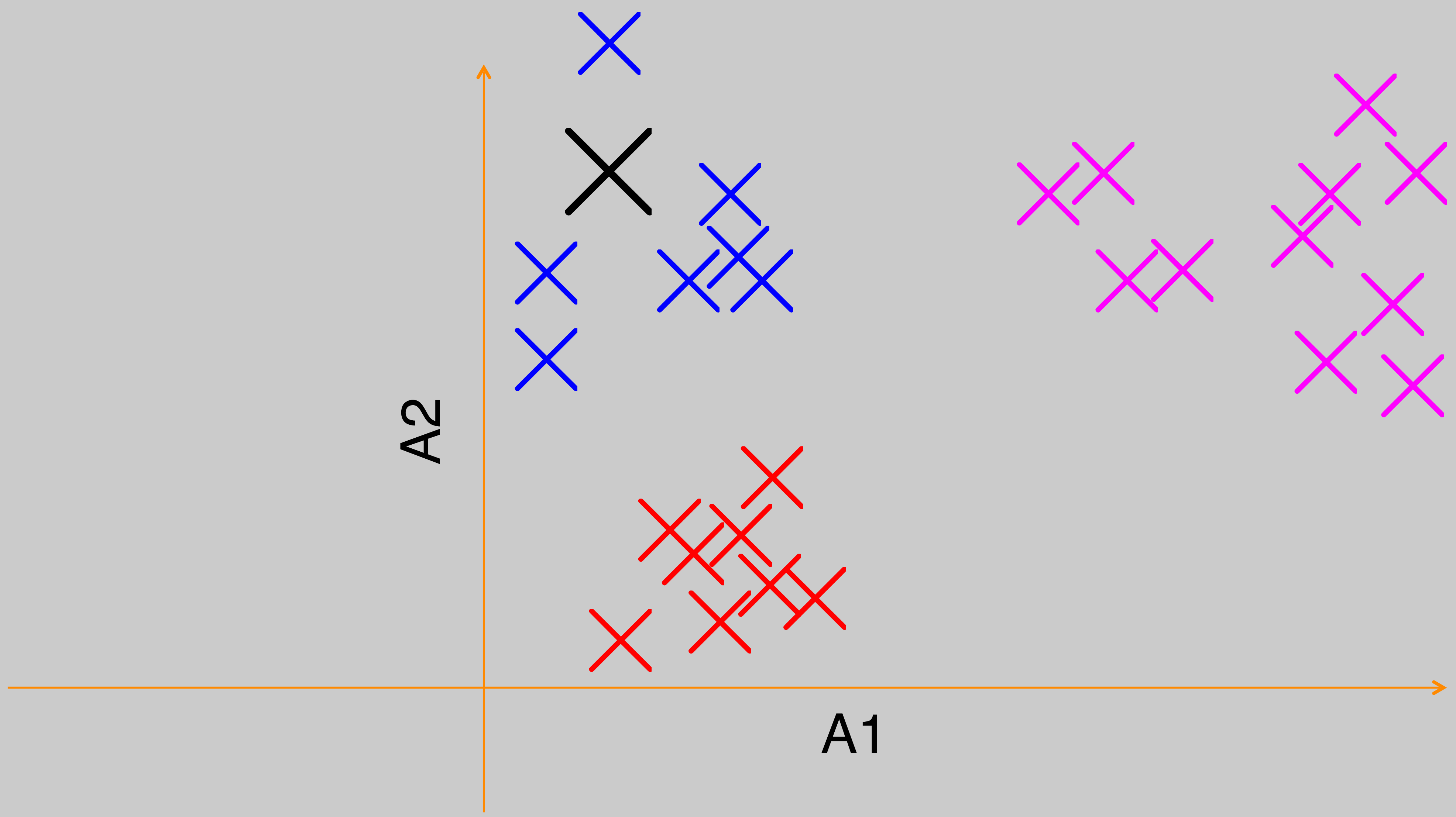
Classification with SVD



n movies

m views

1	2	5	5	1	3	3	3	2	5	5	4	4	3	2	2	5	1	1
5	3	2	1	1	3	3	3	1	1	2	4	5	5	4	4	1	3	2
5	1	2	1	1	2	3	3	1	1	2	1	5	5	3	5	1	1	2
5	3	2	1	1	3	2	3	1	1	2	4	1	1	4	4	5	1	5
1	2	3	2	1	3	2	3	2	1	2	1	1	1	4	4	5	1	5
1	1	1	1	5	3	3	3	1	5	2	4	4	4	2	5	5	1	1



Miki belongs to class: like A2 don't like A1

Prediction with SVD

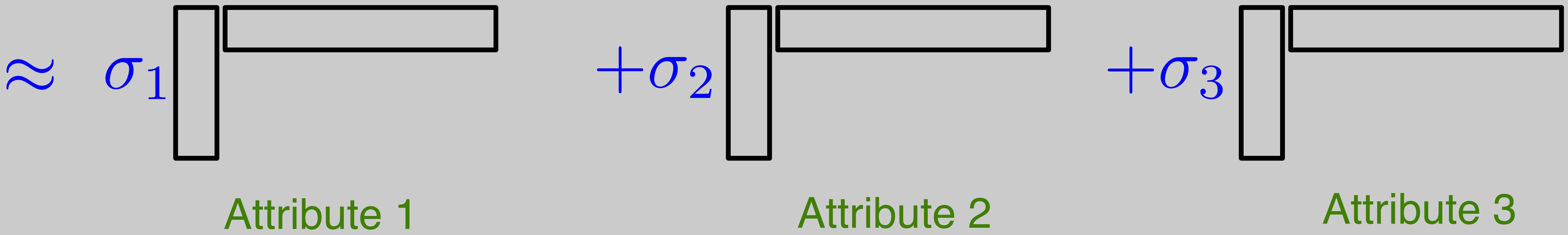
Can try to predict preferences of a new customer with few ratings

See homework!

n movies

m views

1	2	5	5	1	3	3	3	2	5	5	4	4	3	2	2	5	1	1
5	3	2	1	1	3	3	3	1	1	2	4	5	5	4	4	1	3	2
5	1	2	1	1	2	3	3	1	1	2	1	5	5	3	5	1	1	2
5	3	2	1	1	3	2	3	1	1	2	4	1	1	4	4	5	1	5
1	2	3	2	1	3	2	3	2	1	2	1	1	1	4	4	5	1	5
1	1	1	1	5	3	3	3	1	5	2	4	4	4	2	5	5	1	1
1	?	?	2	?	?	?	?	3	5	1	?	?	?	?	5	2	?	3



Low-rank Completion

What if my database is full of “holes”?

Should be still low-rank!

n movies

m views

1		5	5	1	3		3	2	5	5	4		3	2	2	5		1	
5	3		1	1	3	3		1	1	2		5		4	4	1	3	2	
	1	2	1	1		3	3		1	2	1	5	5	3	5		1	2	
5	3		1		3	2		1		2	4	1		4		5		5	
1		3	2	1	3	2	3	2	1		1		1	4		5	1	5	
1	1			5	3	3		5	2	4			2	5	5	1	1		

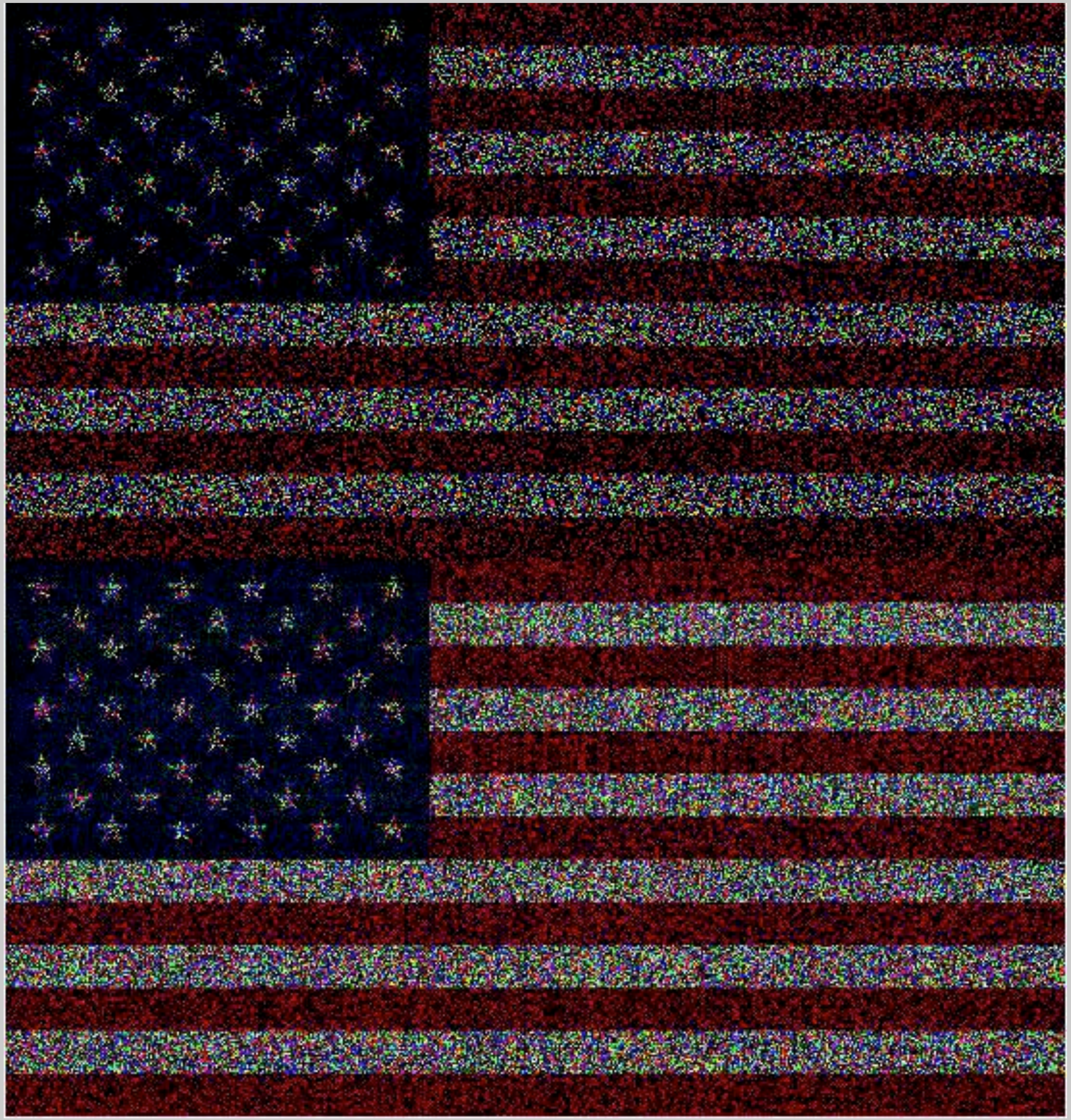
Q) Can we complete missing data?

A) Sometimes! Very recent mathematical and practical results show you can.
Keywords: Compressed Sensing, Low-rank completion, robust PCA

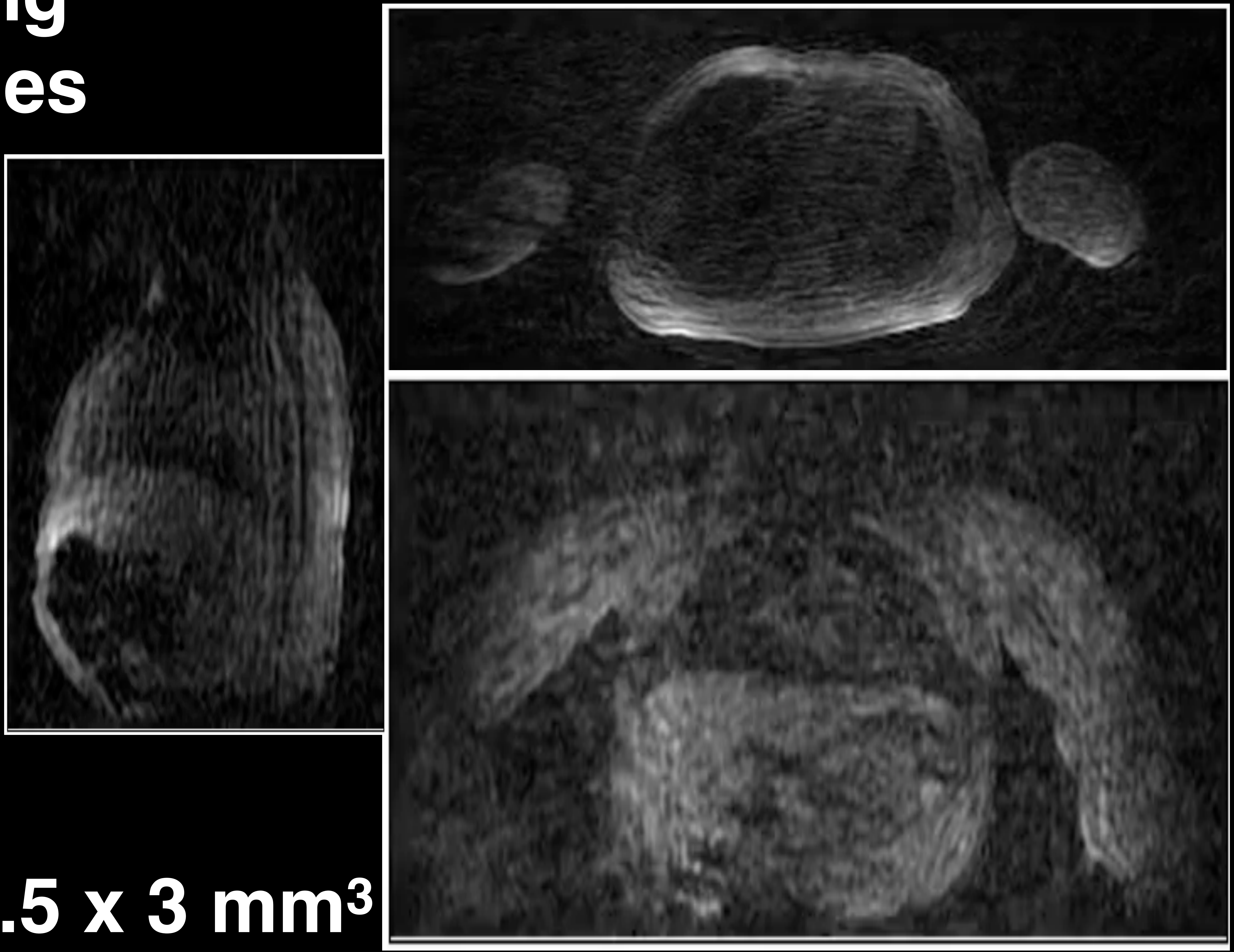
E. Candes and B. Recht, Foundation of Computational Mathematics, 2009;9:717

Low-Rank Recovery from 20% pixels

- Algorithm for low-rank completion:
  - flag_hat = flag
  - Compute [U,S,V] = svd(flag_hat)
  - $$\text{flag}_{\text{hat}} = \sum_{i=0}^{6} \sigma_i \vec{u}_i \vec{v}_i^T$$
  - update missing pixels in flag from flag_hat
  - repeat (250 times here)

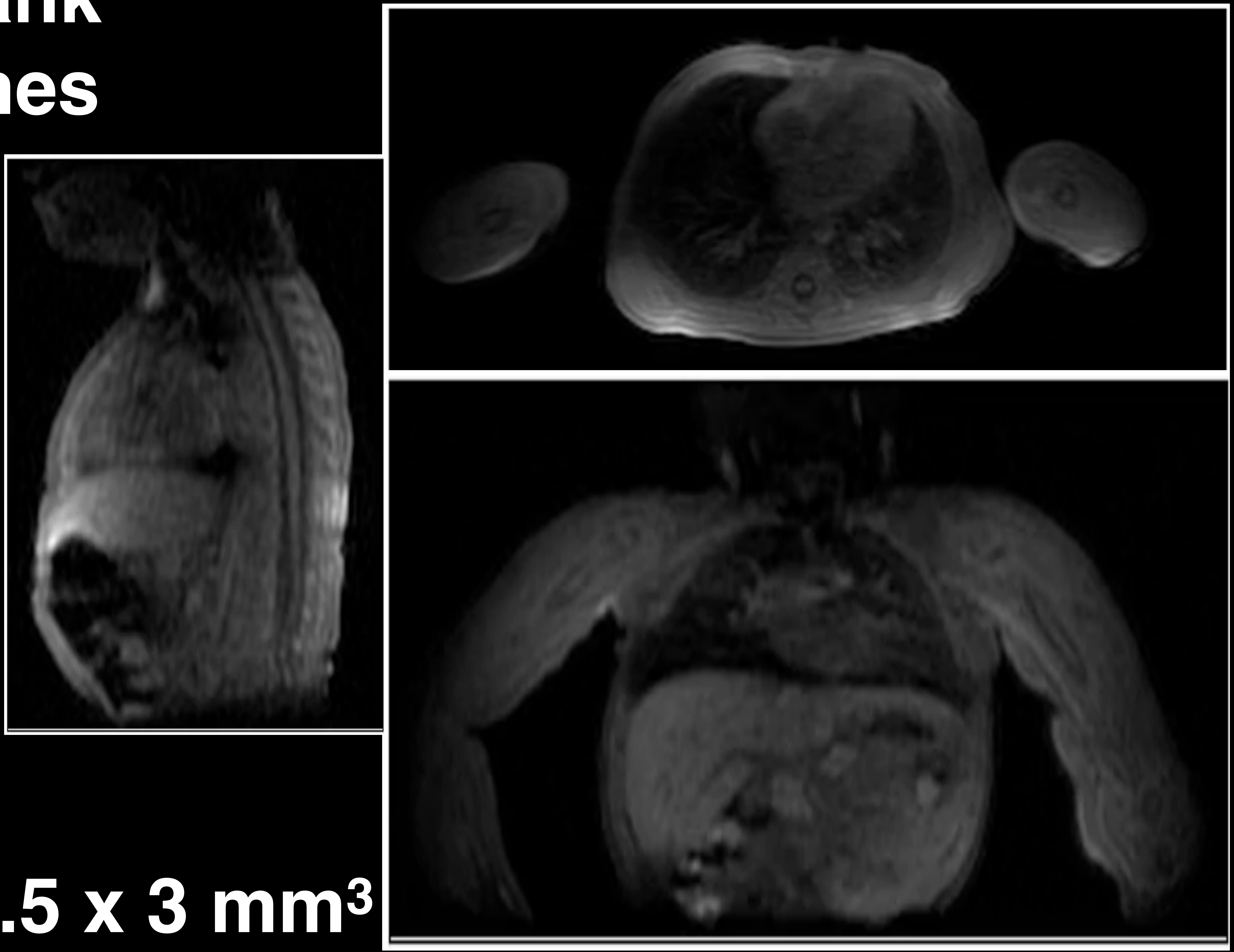


Gridding
30 frames



~1.5 x 1.5 x 3 mm^3

Low Rank
30 frames



~1.5 x 1.5 x 3 mm^3

SVD

SVD decomposes a rank r matrix $A \in \mathrm{R}^{m \times n}$
into a sum of r rank-1 matrices:

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

1) $\vec{u}_i^T \vec{u}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \Rightarrow ||\vec{u}_i|| = 1 \quad \vec{u}_i \perp \vec{u}_j$

2) $\vec{v}_i^T \vec{v}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \Rightarrow ||\vec{v}_i|| = 1 \quad \vec{v}_i \perp \vec{v}_j$

3) $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

Matrix Form of SVD

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

$$U_1 = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_r \end{bmatrix} \quad S = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix} \quad V_1 = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_r \end{bmatrix}$$

$m \times r$ $\quad$ $r \times r$ $\quad$ $n \times r$

$$A = U_1 S V_1^T$$

$$U_1^T U_1 = I_{r \times r}$$

$$V_1^T V_1 = I_{r \times r}$$

$$S \succ 0 \qquad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$$

Matrix Form of SVD

$$A = U_1 S V_1^T$$

Full Matrix Form of SVD

$$U_1 = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_r \end{bmatrix}$$

$m \times r$

$$S = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix}$$

$r \times r$

$$V_1 = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_r \end{bmatrix}$$

$n \times r$

$$U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$$

$m \times m$

Full Matrix Form of SVD

$$U_1 = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_r \end{bmatrix}$$
$m \times r$

$$S = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix}$$
$r \times r$

$$V_1 = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_r \end{bmatrix}$$
$n \times r$

$$U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$$
$m \times m$

$$\Sigma = \left[\begin{array}{c|c} S & 0 \\ \hline 0 & 0 \end{array}\right]$$
$m \times n$

$$V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$$
$n \times n$

$$A = U\Sigma V^T$$

$$U^T U = I_{m \times m}$$
$$V^T V = I_{n \times n}$$
$$\Sigma \succeq 0$$

Computing the SVD

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

1) $\vec{u}_i^T \vec{u}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

2) $\vec{v}_i^T \vec{v}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

3) $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

- What's the singular value of A:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Computing the SVD

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

1) $\vec{u}_i^T \vec{u}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ 2) $\vec{v}_i^T \vec{v}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

3) $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

- What's the singular value of A:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \underbrace{\sqrt{2}}_{\sigma_1} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}}_{\vec{u}_1} \overbrace{[1 \quad 0]}^{\vec{v}_1^T}$$