This homework is due on Thursday, September 24, 2020, at 10:59PM.
Self-grades are due on Thursday, October 1, 2020, at 10:59PM.

1 Eigenvalues, Eigenvectors and Diagonalization

a) Find the eigenvalues and eigenvectors for the following matrices. Are these matrices diagonalizable?

i.) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Solution

We can find the eigenvalues for A using its characteristic polynomial

$$\begin{aligned} \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}\right) &= 0 \\ (1-\lambda)^2 &= 0 \\ \lambda &= 1 \end{aligned}$$

Let us denote an eigenvector for A corresponding to the eigenvalue $\lambda = 1$ to be $\vec{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. An eigenvalue, eigenvector pair $(\lambda, \vec{v})$ must satisfy

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \\ (A - \lambda I)\,\vec{v} &= 0 \\ \begin{bmatrix} -\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

For A,

$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 0 \\ x_2 &= 0 \end{aligned}$$

Therefore, A has one linearly independent eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Since A is a 2×2 matrix but does not have 2 linearly independent eigenvectors, **A is not diagonalizable**.

ii.) $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Solution

The characteristic polynomial for B is

$$\begin{aligned} \det\left(\begin{bmatrix} -\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix}\right) &= 0 \\ \lambda(1-\lambda) &= 0 \\ \lambda &\in \{0, 1\} \end{aligned}$$

Solving for eigenvectors for the individual eigenvalues.
For $\lambda = 0$, we have

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x_2 = 0$$

This gives us an eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
Similarly, for $\lambda = 1$, we have

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x_1 = 0$$

This gives us an eigenvector $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Since the two eigenvectors for B, $\vec{v}_1$ and $\vec{v}_2$ are linearly independent, **B is diagonalizable**. Since B has 2 distinct eigenvalues, we can say that it is diagonalizable without having computed its eigenvectors explicitly.

iii.) $C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Solution

Characteristic polynomial for C is given by

$$\det\left(\begin{bmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix}\right) = 0$$
$$(1-\lambda)(\lambda^2 - 1) = 0$$
$$\lambda \in \{1, -1\}$$

For $\lambda = -1$, we have

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$x_2 = 0$$
$$x_1 + x_3 = 0$$

$\lambda = -1, \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is an eigenvalue, eigenvector pair.
For $\lambda = 1$, we have

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$x_1 + x_3 = 0$$

There, with $\lambda = 1$, we have multiple possible eigenvectors. $\vec{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}1\\1\\1\end{bmatrix}$ are two example eigenvectors of C with an eigenvalue 1. Also note that any linear combination of $\vec{v}_1$ and $\vec{v}_2$ is also an eigenvector of C with eigenvalue 1. These linear combinations constitute an eigenspace for C corresponding to the eigenvalue $\lambda = 1$. Since we can find 3 linearly independent eigenvectors for C, **C is diagonalizable**.

b) Let A be an $n \times n$ matrix with n linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Define P to be a matrix with $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ as its columns, $P = \left[\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\right]$.

i. Show that $AP = P\Lambda$, where $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, a diagonal matrix with the eigenvalues of A as its diagonal entries.

Solution

$$\begin{aligned}
AP &= A[\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n] \\
&= [A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_n] \\
&= [\lambda_1\vec{v}_1, \lambda_2\vec{v}_2, \ldots, \lambda_n\vec{v}_n] \\
&= [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]\begin{bmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{bmatrix} \\
&= P\Lambda
\end{aligned}$$

ii. Argue that P is invertible, and therefore, $A = P\Lambda P^{-1}$.

Solution

Columns of P are eigenvectors of A which are known to be linearly independent. Since P has linearly independent columns, it has full column rank, and therefore, is invertible.

$$\begin{aligned}
AP &= P\Lambda \\
P^{-1}AP &= P^{-1}P\Lambda \\
P^{-1}AP &= \Lambda
\end{aligned}$$

c) For a matrix A and a positive integer k, we define the exponent to be

$$A^k = \underbrace{A * A * \cdots * A * A}_{k \text{ times}} \tag{1}$$

Let's assume that matrix A is diagonalizable with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

i. Show that A^k has eigenvalues $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$.

ii. Show that A^k has eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and conclude that A^k is diagonalizable.

Solution

Consider the i^{th} eigenvector of A, $\vec{v}_i$ and the corresponding eigenvalue λ_i. Here is an intuition for how the proof works out.

$$\begin{aligned}
A^k\vec{v}_i &= A^{k-1} * A\vec{v}_i \\
&= A^{k-1}\lambda_i\vec{v}_i \\
&= \lambda_i A^{k-2} * A\vec{v}_i \\
&= \lambda_i^2 A^{k-3} * A\vec{v}_i \\
&\vdots \\
&= \lambda_i^k\vec{v}_i
\end{aligned}$$

We now look at a more formal proof using induction.
Claim: Let $\vec{v}_i$ be an eigenvector of A^{k-1} with an eigenvalue λ_i^{k-1}, then $\vec{v}_i$ is an eigenvector of A^k with eigenvalue λ_i^k.

$$A^k\vec{v}_i = A * A^{k-1}\vec{v}_i$$

Since $\vec{v}_i$ is an eigenvector of A^{k-1} with eigenvalue λ_i^{k-1}, we have

$$\begin{aligned}
A^k\vec{v}_i &= A * \lambda_i^{k-1}\vec{v}_i \\
&= \lambda_i^{k-1} * A\vec{v}_i \\
&= \lambda_i^{k-1}\lambda_i\vec{v}_i \\
&= \lambda_i^k\vec{v}_i
\end{aligned}$$

Our assumption holds for $k = 1$ since $(\lambda_i, \vec{v}_i)$ is an eigenvalue, eigenvector pair for A, by induction, for every positive integers k, $(\lambda_i{}^k, \vec{v}_i)$ is an eigenvalue, eigenvector pair.
We have shown that A^k has n linearly independent vectors which is sufficient to claim that A^k is diagonalizable.
Alternate method Since A is diagonalizable, we can express A as

$$A = P\Lambda P^{-1} \tag{2}$$

Substituting A as shown in Equation 2 in 1, we get

$$\begin{aligned}
A^k &= \underbrace{A * A * \cdots * A * A}_{k \text{ times}} \\
&= \underbrace{P\Lambda P^{-1} * P\Lambda P^{-1} * \cdots * P\Lambda P^{-1} * P\Lambda P^{-1}}_{k \text{ times}} \\
&= \underbrace{P\Lambda\left(P^{-1} * P\right)\Lambda P^{-1} * \cdots * P\Lambda\left(P^{-1} * P\right)\Lambda P^{-1}}_{k \text{ times}} \\
&= P\underbrace{\Lambda * \Lambda * \cdots * \Lambda * \Lambda}_{k \text{ times}}P^{-1} \\
&= P\Lambda^k P^{-1}
\end{aligned}$$

Since Λ is a diagonal matrix,

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix}$$

2 Circuits puzzle

Prof. Sanders enjoys designing circuits. Having finished his circuits, he decided to take a break and tasked you with analyzing what these circuits were doing. Analyze the following circuits and answer the questions attached to them.

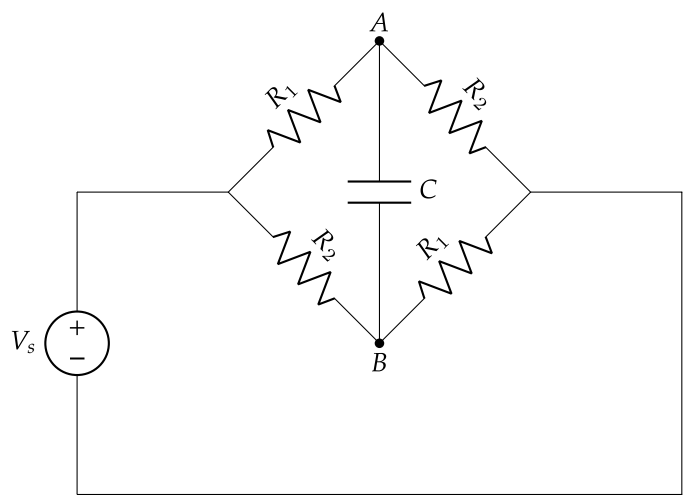

Figure 1: Modified Wheatstone bridge circuit.

a) In Figure 1, we have a modified Wheatstone bridge circuit with a capacitor in the bridge. Find the steady state voltage across the capacitor (V_{AB}). If the capacitor starts with no charge, how long will it take for it to reach 95% of its steady state value with $C = 100\text{nF}$, $R_1 = 10\Omega$, and $R_2 = 100\Omega$?

Hint: Use nodal analysis at nodes A and B.

Solution

In steady state, no current flows through the capacitor and it acts as an open circuit. The circuit shown in Figure 1 can therefore be analyzed using the circuit shown in Figure 2.

Voltages at nodes A and B can be analyzed using a voltage divider formula. $v_A = V_s \frac{R_2}{R_1+R_2}$ and

$v_B = V_s \frac{R_1}{R_1+R_2}$. The voltage across the capacitor is given by

$$\begin{aligned} V_C &= V_A - V_B \\ &= V_s \frac{R_2}{R_1+R_2} - V_s \frac{R_1}{R_1+R_2} \\ &= V_s \frac{R_2 - R_1}{R_1+R_2} \end{aligned}$$

In order to evaluate the time-constant for the charging of the capacitor in our circuit, we write KCL equations at nodes A and B.

KCL at node A gives us

$$\frac{V_A}{R_2} + \frac{V_A - V_s}{R_1} + C\frac{d}{dt}(V_A - V_B) = 0 \tag{3}$$

KCL at node B gives us

$$\frac{V_B}{R_1} + \frac{V_B - V_s}{R_2} - C\frac{d}{dt}(V_A - V_B) = 0 \tag{4}$$

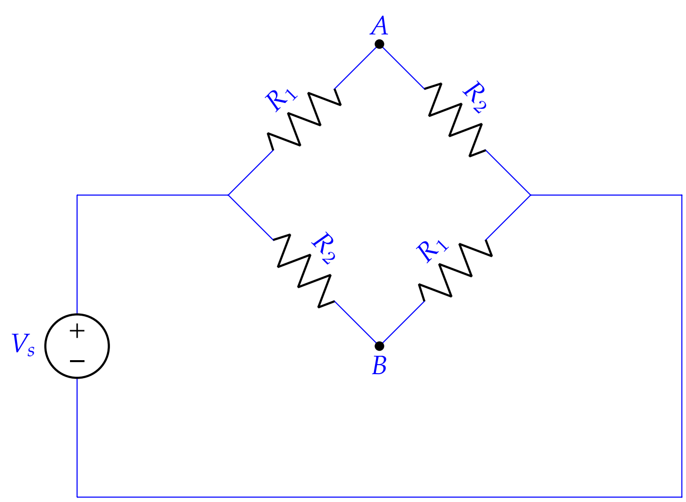

Figure 2: Modified Wheatstone bridge circuit in steady-state.

Subtracting Equation 4 from 3, we get

$$\frac{V_A}{R_2} + \frac{V_A - V_s}{R_1} - \frac{V_B}{R_1} - \frac{V_B - V_s}{R_2} + 2C\frac{d}{dt}(V_A - V_B) = 0$$

$$(V_A - V_B)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + V_s\left(\frac{1}{R_2} - \frac{1}{R_1}\right) + 2C\frac{d}{dt}(V_A - V_B) = 0$$

$$\frac{d}{dt}V_C = -\frac{V_C}{2C}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - \frac{V_s}{2C}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$$

The time constant for capacitor charging in this case is $\tau = 2C\left(\frac{R_1 R_2}{R_1+R_2}\right) = 2 \cdot \frac{100e^{-9} * (10*100)}{110} = 1.82us$. In order to charge the capacitor to 95% of its final value, we need $t = 3\tau = 5.46us$.

Observe that the steady state value of V_C can be computed by setting $\frac{d}{dt}V_C = 0$ in the differential equation above. This gives us

$$\begin{aligned}\frac{V_C}{2C}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) &= -\frac{V_s}{2C}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\\ \frac{V_C}{2C}\left(\frac{R_1+R_2}{R_1R_2}\right) &= -\frac{V_s}{2C}\left(\frac{R_1-R_2}{R_2R_1}\right)\\ V_C &= V_s\left(\frac{R_2-R_1}{R_2+R_1}\right)\end{aligned}$$

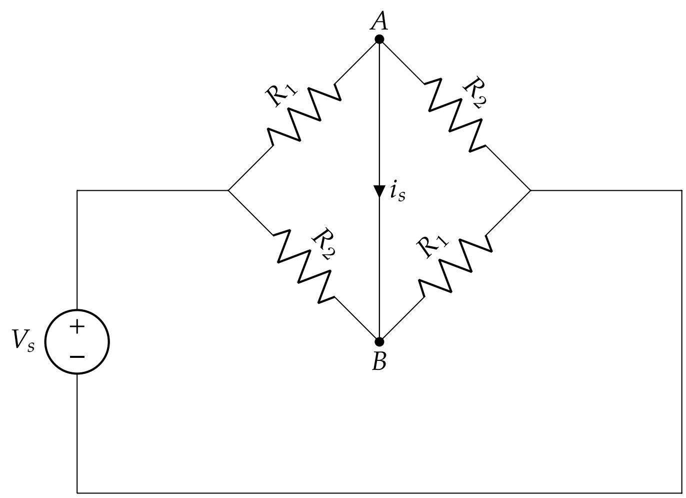

Figure 3: Modified Wheatstone bridge circuit.

b) Someone was tinkering with the Wheatstone bridge circuit and accidentally shorted the nodes A and B. In Figure 3, we have the short-circuited Wheatstone bridge. Find the steady state current through the short (i_s).

Hint: Use the current i_s in nodal analysis at nodes A and B.

Solution

Writing out KCL at node A gives us

$$\frac{V_A - V_s}{R_1} + \frac{V_A}{R_2} + i_s = 0 \tag{5}$$

Similarly, KCL at node B gives us

$$\frac{V_B - V_s}{R_2} + \frac{V_B}{R_1} - i_s = 0 \tag{6}$$

Since nodes A and B are connected through a wire, $V_A = V_B$. Subtracting Equation 6 from 6 gives us

$$\begin{aligned} \frac{V_A - V_s}{R_1} + \frac{V_A}{R_2} + i_s - \frac{V_B - V_s}{R_2} - \frac{V_B}{R_1} + i_s &= 0 \\ (V_A - V_B)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + V_s\left(\frac{1}{R_2} - \frac{1}{R_1}\right) + 2i_s &= 0 \\ i_s &= \frac{V_s}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \end{aligned}$$

c) In Figure 4, we have an operational amplifier in negative feedback. However, it seems that one of the resistors has been replaced with a capacitor. Find the relationship between the output voltage V_{out} and V_{in}. What mathematical operation does this circuit perform?

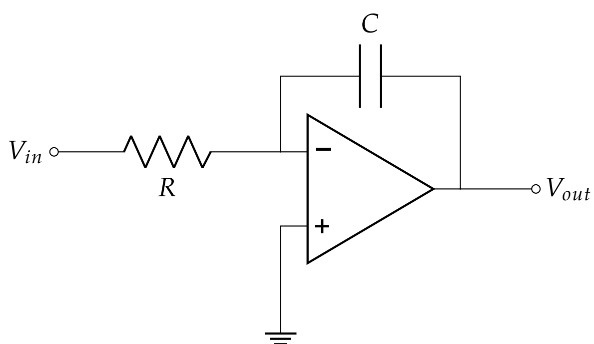

Figure 4: Operational Amplifier with RC elements.

Solution

Since the operational amplifier is in negative feedback, we can apply our two golden rules.

$$\begin{aligned} V_+ &= V_- = 0\,, \\ I_+ &= I_- = 0\,, \end{aligned}$$

where V_+ and V_- are the potentials at the positive and negative terminal of the operational amplifier, and I_+ and I_- are the currents entering the positive and negative terminal respectively.

Applying KCL at the negative terminal of the operational amplifier, we get

$$I_C + I_R = 0 \tag{7}$$

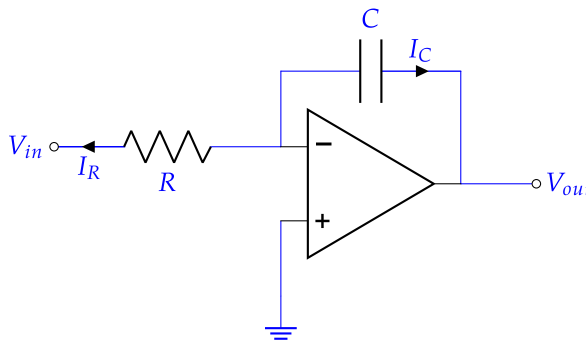

Figure 5: Operational Amplifier with RC elements.

We know that $I_R = \frac{V_- - V_{in}}{R} = -\frac{V_{in}}{R}$, $I_C = C\frac{d}{dt}(V_- - V_{out}) = -C\frac{d}{dt}V_{out}$, and $I_- = 0$. Plugging these currents into Equation 7, we get

$$-\frac{V_{in}}{R} - C\frac{d}{dt}V_{out} = 0 \tag{8}$$

$$\frac{d}{dt}V_{out} = -\frac{V_{in}}{RC} \tag{9}$$

$$V_{out} = \frac{-1}{RC}\int_0^t V_{in}\,dt \tag{10}$$

The circuit shown in Figure 4 performs integration.

d) In Figure 6, the positions of the resistor and the capacitor seem to have been swapped. Find the relationship between the output voltage V_{out} and V_{in}. What mathematical operation does this circuit perform?

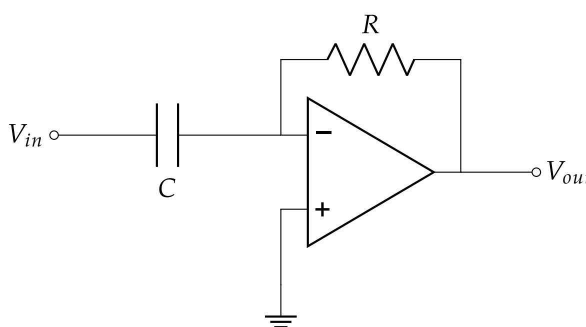

Figure 6: Operational Amplifier with RC elements.

Solution

Proceeding similarly to the previous part, the operational amplifier is in negative feedback and we can apply our two golden rules.

$$V_+ = V_- = 0\,,$$
$$I_+ = I_- = 0\,,$$

where V_+ and V_- are the potentials at the positive and negative terminal of the operational amplifier, and I_+ and I_- are the currents entering the positive and negative terminal respectively.

Applying KCL at the negative terminal of the operational amplifier, we get

$$I_- + I_C + I_R = 0 \tag{11}$$

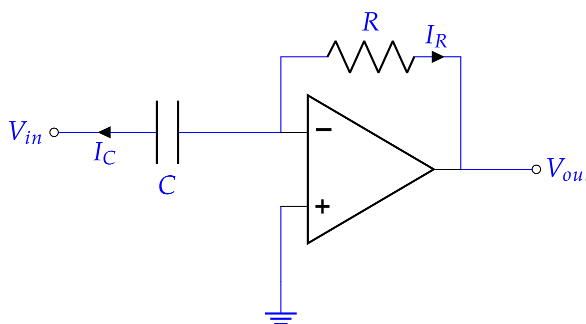

Figure 7: Operational Amplifier with RC elements.

We know that $I_R = \frac{V_- - V_{out}}{R} = -\frac{V_{out}}{R}$, $I_C = C\frac{d}{dt}(V_- - V_{in}) = -C\frac{d}{dt}V_{in}$, and $I_- = 0$. Plugging these currents into Equation 11, we get

$$-\frac{V_{out}}{R} - C\frac{d}{dt}V_{in} = 0 \tag{12}$$

$$V_{out} = -RC\frac{d}{dt}V_{in} \tag{13}$$

The circuit shown in Figure 6 performs differentiation.

3 Matrix Differential Equations

In this problem, we consider ordinary differential equations which can be written in the following form

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \tag{14}$$

where x, y are variables depending on time t, and A is a 2×2 matrix with constant coefficients. We call (14) a matrix differential equation.

a) Suppose we have a system of ordinary differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 7x - 8y \tag{15}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 4x - 5y \tag{16}$$

Write this in the form of (14).

Solution

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & -8 \\ 4 & -5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}, \tag{17}$$

b) Compute the eigenvalues of the matrix A from the previous part.

Solution

The characteristic polynomial of A is

$$\det\left(\begin{bmatrix} 7-\lambda & -8 \\ 4 & -5-\lambda \end{bmatrix}\right) = (7-\lambda)(-5-\lambda)+32$$
$$= \lambda^2 - 7\lambda + 5\lambda - 35 + 32$$
$$= \lambda^2 - 2\lambda - 3$$
$$= (\lambda+1)(\lambda-3)\,.$$

Thus the eigenvalues of A are $\lambda = -1, 3$.

c) We claim that the solution for $x(t), y(t)$ is of the form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_0 t} + c_3 e^{\lambda_1 t} \end{bmatrix},$$

where c_0, c_1, c_2, c_3 are constants, and λ_0, λ_1 are the eigenvalues of A. Suppose that the initial conditions are $x(0) = 1, y(0) = -1$. Solve for the constants c_0, c_1, c_2, c_3.

Hint: What are $\frac{dx}{dt}(0)$ and $\frac{dy}{dt}(0)$?

Solution

Substituting the eigenvalues computed in the previous part, we have

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_0 e^{-t} + c_1 e^{3t} \\ c_2 e^{-t} + c_3 e^{3t} \end{bmatrix}.$$

At $t = 0$, we have $x(0) = 1, y(0) = -1$ so we can compute its derivatives' initial conditions as

$$\begin{bmatrix} \frac{dx}{dt}(0) \\ \frac{dy}{dt}(0) \end{bmatrix} = \begin{bmatrix} 7x(0) - 8y(0) \\ 4x(0) - 5y(0) \end{bmatrix} = \begin{bmatrix} 15 \\ 9 \end{bmatrix}.$$

$$1 = c_0 + c_1 \tag{18}$$
$$15 = -c_0 + 3c_1 \tag{19}$$
$$-1 = c_2 + c_3 \tag{20}$$
$$9 = -c_2 + 3c_3 \tag{21}$$

This gives $c_0 = -3$, $c_1 = 4$, $c_2 = -3$, and $c_3 = 2$. Thus we have

$$x(t) = -3e^{-t} + 4e^{3t} \tag{22}$$
$$y(t) = -3e^{-t} + 2e^{3t} \tag{23}$$

d) Verify that the solution for $x(t), y(t)$ found in the previous part satisfies the original system of differential equations (15), (16).

Solution

We compute the derivative of x with respect to t in (22) to get

$$\frac{d}{dt}x(t) = 3e^{-t} + 12e^{3t}\,.$$

The right hand side of (15) is

$$7x - 8y = -21e^{-t} + 28e^{3t} + 24e^{-t} - 16e^{3t} = 3e^{-t} + 12e^{3t}$$

hence our solution for $x(t)$ satisfies (15).

Similarly, we compute the derivative of y with respect to t in (23) to get

$$\frac{d}{dt}y(t) = 3e^{-t} + 6e^{3t}\,.$$

The right hand side of (16) is

$$4x - 5y = -12e^{-t} + 16e^{3t} + 15e^{-t} - 10e^{3t} = 3e^{-t} + 6e^{3t}$$

hence our solution for $y(t)$ satisfies (16).

e) We now apply the method above to solve another second-order ordinary differential equation. Suppose we have the system

$$\frac{\mathrm{d}^2 z(t)}{\mathrm{d}t^2} - 5\frac{\mathrm{d}z(t)}{\mathrm{d}t} + 6z(t) = 0, \tag{24}$$

Write this in the form of (14), by choosing appropriate variables $x(t)$ and $y(t)$.

Solution

If we set $x(t) = z(t)$, $y(t) = \frac{\mathrm{d}z(t)}{\mathrm{d}t}$, then we have

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{\mathrm{d}z(t)}{\mathrm{d}t} = y(t) \tag{25}$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2 z(t)}{\mathrm{d}t^2} = 5\frac{\mathrm{d}z(t)}{\mathrm{d}t} - 6z(t) = 5y(t) - 6x(t) \tag{26}$$

We can write this in the form of (14) as follows

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}, \tag{27}$$

f) Solve the system in (24) with the initial conditions $z(0) = 1$, $\frac{\mathrm{d}z}{\mathrm{d}t}(0) = 1$, using the method developed in parts (b) and (c).

Solution

We first compute the eigenvalues of the matrix from the previous part. The characteristic polynomial is

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -6 & 5-\lambda \end{bmatrix}\right) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)\,.$$

Thus the eigenvalues are $\lambda = 2, 3$.

From part (c), the solution for $x(t), y(t)$ is of the form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_0 e^{2t} + c_1 e^{3t} \\ c_2 e^{2t} + c_3 e^{3t} \end{bmatrix}.$$

At $t = 0$, we have $x(0) = z(0) = 1$ and $\frac{dx}{dt}(0) = \frac{dz}{dt}(0) = 1$:

$$1 = c_0 + c_1 \tag{28}$$
$$1 = 2c_0 + 3c_1 \tag{29}$$

This gives $c_0 = 2$ and $c_1 = -1$. Thus we have

$$x(t) = 2e^{2t} - e^{3t} \tag{30}$$
$$y(t) = \frac{dx(t)}{dt} = 4e^{2t} - 3e^{3t} \tag{31}$$

Hence we have the solution

$$z(t) = 2e^{2t} - e^{3t}.$$

4 Multi-Capacitor Circuit

Consider the circuit below

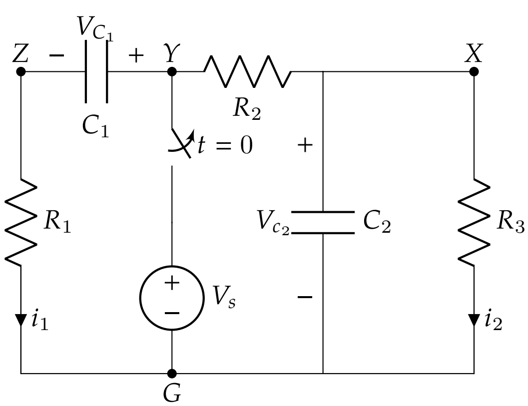

Figure 8: Circuit with multiple capacitors.

The resistors shown in the circuit have the same value $R_1 = R_2 = R_3 = R$. Capacitors C_1 and C_2 have the same capacitance $C_1 = C_2 = C$. Further, $RC = 1s$.

a) Assume that the switch shown in Figure 8 was held in the closed position for a long time before $t = 0$. At $t = 0$, immediately after the switch is opened, what are the capacitor voltages $V_{C_1}(0)$ and $V_{C_2}(0)$?

Solution

In steady state, capacitors act as open circuits.

Immediately before $t = 0$,

$$\begin{aligned} V_Y &= V_s\,, \\ i_2 &= \frac{V_s}{R_2 + R_3} = \frac{V_s}{2R}\,, \\ V_X &= i_2 R = \frac{V_s}{2}\,, \\ i_1 &= 0\,, \\ V_Z &= i_1 R_1 = 0. \end{aligned}$$

Since the voltage across the capacitors cannot change immediately in this circuit, the capacitor voltages immediately after the switch is opened at $t = 0$ will be

$$\begin{aligned} V_{C_1}(0) &= V_s\,, \\ V_{C_2}(0) &= \frac{V_s}{2}. \end{aligned}$$

b) How are the current i_2 and the capacitor voltage V_{C_2} related?

Solution

Since the resistor R_3 and capacitor C_2 are connected in parallel, they will have the same voltage drop across them. We can write the voltage drop across the resistor as $V_R = i_2 R$. This gives us $i_2 R = V_{C_2}$.

c) Using KCL on node X and the relationship above, write an equation relating V_{C_2} and i_1.

Solution

KCL at node X gives us

$$I_{C_2} + i_2 + i_1 = 0$$

$$C_2 \frac{d}{dt} V_{C_2} + \frac{V_{C_2}}{R} + i_1 = 0 \tag{32}$$

d) Using KVL on the loop comprising of both capacitors C_1 and C_2, find a relationship between V_{C_1}, V_{C_2} and i_1.

Solution

KVL on the mentioned loop gives us

$$V_{C_2} - i_1 R_2 - V_{C_1} - i_1 R_1 = 0$$

$$V_{C_2} - V_{C_1} - 2 i_1 R = 0 \tag{33}$$

e) Rewrite the equations derived above, eliminating the current i_1 to obtain a system of differential equations involving V_{C_1} and V_{C_2}. Write this system of equations in a matrix form

$$\frac{d}{dt}\begin{bmatrix} V_{C_1} \\ V_{C_2} \end{bmatrix} = A \begin{bmatrix} V_{C_1} \\ V_{C_2} \end{bmatrix}$$

What is the matrix A and what are its eigenvalues?

Hint: You can use the relation $i_1 = C_1 \frac{d}{dt} V_{C_1}$ in addition to the relations we have derived so far.

Solution

We can use KCL at node Z to write $i_1 = C_1 \frac{d}{dt} V_{C_1}$ and use it to eliminate i_1. Substituting this in Equations 32 and 33, we get

$$\begin{aligned} V_{C_2} - V_{C_1} - 2C_1 R \frac{d}{dt} V_{C_1} &= 0 \\ \frac{d}{dt} V_{C_1} &= \frac{-V_{C_1} + V_{C_2}}{2RC}, \end{aligned}$$

and substituting $i_1 = \frac{V_{C_2} - V_{C_1}}{2R}$ from Equation 33 in 32, we get

$$\begin{aligned} C_2 \frac{d}{dt} V_{C_2} + \frac{V_{C_2}}{R} + \frac{V_{C_2} - V_{C_1}}{2R} &= 0 \\ \frac{d}{dt} V_{C_2} + \frac{3V_{C_2} - V_{C_1}}{2RC} &= 0 \\ \frac{d}{dt} V_{C_2} &= \frac{V_{C_1} - 3V_{C_2}}{2RC} \end{aligned}$$

Combining the relations above, we can write

$$\frac{d}{dt}\begin{bmatrix} V_{C_1} \\ V_{C_2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2RC} & \frac{1}{2RC} \\ \frac{1}{2RC} & -\frac{3}{2RC} \end{bmatrix} \cdot \begin{bmatrix} V_{C_1} \\ V_{C_2} \end{bmatrix} \tag{34}$$

Plugging in $RC = 1$, this simplifies to

$$\frac{d}{dt}\begin{bmatrix} V_{C_1} \\ V_{C_2} \end{bmatrix} = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & -1.5 \end{bmatrix} \cdot \begin{bmatrix} V_{C_1} \\ V_{C_2} \end{bmatrix} \tag{35}$$

The characteristic polynomial for A is

$$\begin{aligned} \det\left(\begin{bmatrix} \lambda + 0.5 & -0.5 \\ -0.5 & \lambda + 1.5 \end{bmatrix}\right) &= 0 \\ (\lambda + 0.5) \cdot (\lambda + 1.5) - 0.25 &= 0 \\ \lambda^2 + 2\lambda + 0.75 - 0.25 &= 0 \\ \lambda^2 + 2\lambda + 0.5 &= 0 \\ \lambda &= \frac{-2 \pm \sqrt{2}}{2} \end{aligned}$$

Eigenvalues for the matrix A are $\lambda_1 = \frac{-2+\sqrt{2}}{2}$, and $\lambda_2 = \frac{-2-\sqrt{2}}{2}$.

f) In order to solve for the capacitor voltages, we finally need the initial values of the voltage derivatives. Immediately after the switch is opened, what are the voltage derivatives for the two capacitors, $\frac{\mathrm{d}V_{C_1}}{\mathrm{d}t}(0)$ and $\frac{\mathrm{d}V_{C_2}}{\mathrm{d}t}(0)$?

Hint: Calculate the currents i_1 and i_2 immediately after the switch if opened at $t = 0$. While the capacitor voltages do not change immediately, the current through them will change.

Solution

After the switch is opened, we can use the KVL equation we previously wrote in Equation 33

$$\begin{aligned} V_{C_2} - V_{C_1} - 2i_1R &= 0 \\ \frac{V_s}{2} - V_s - 2i_1(0)R &= 0 \\ i_1(0) &= -\frac{V_s}{4R} \end{aligned}$$

Since the capacitor voltage and current for C_1 are related as

$$I_{C_1} = C_1\frac{d}{dt}V_{C_1} \tag{36}$$

Since $I_{C_1} = i_1$, we have

$$\begin{aligned} \frac{d}{dt}V_{C_1}(0) &= \frac{i_1(0)}{C_1} \\ &= -\frac{V_s}{4RC} \end{aligned}$$

The capacitor current I_{C_2} can be found using the KCL relationship we wrote earlier in Equation 32.

$$\begin{aligned} i_1 + i_2 + I_{C_2} &= 0 \\ -\frac{V_s}{4R} + \frac{V_s}{2R} + I_{C_2}(0) &= 0 \\ I_{C_2}(0) &= -\frac{V_s}{4R} \end{aligned}$$

We can now use the charge-voltage relationship for the capacitor C_2 to obtain

$$\begin{aligned} I_{C_2} &= C_2\frac{d}{dt}V_{C_2} \\ \frac{d}{dt}V_{C_2}(0) &= -\frac{V_s}{4RC} \end{aligned}$$

5 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **What sources (if any) did you use as you worked through the homework?**

b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

c) **Roughly how many total hours did you work on this homework?**

d) **Do you have any feedback on this homework assignment?**