This homework is due on Thursday, December 3, 2020, at 10:59PM.
Self-grades are due on Thursday, December 10, 2020, at 10:59PM.

1 Sampling Theorem

Consider the following signal, $x(t)$ defined as,

$$x(t) = \cos(2\pi t) + \sin(4\pi t)$$

a) Find the maximum frequency, $\omega_{\max}$, of $x(t)$ in radians per second.

Solution

There are two distinct frequencies in this signal at $\omega = 2\pi$ and $\omega = 4\pi$. Therefore, $\omega_{\max} = 4\pi$ in radians per second, which is equal to $2\,\text{Hz}$.

b) If I sample every T seconds, what is the sampling frequency in radians per second?

Solution

$\omega_s = \frac{2\pi}{T}$.

c) What is the smallest sampling period T that may result in an imperfect reconstruction?

Solution

From the Nyquist Sampling Theorem, we must sample at $\omega_s > 2\omega_{\max} = 8\pi$ in order to perfectly reconstruct our signal. Therefore $T = \frac{2\pi}{\omega_s}$ must be strictly less than $\frac{1}{4}$ to guarantee a perfect reconstruction. Hence the smallest T for which we cannot reconstruct the signal is $T = \frac{1}{4}$.

2 Aliasing

Watch the following video: `https://www.youtube.com/watch?v=jQDjJRYmeWg`.

Assume the video camera running at 30 frames per second. That is to say, the camera takes 30 photos within a second, with the time between photos being constant.

a) Given that the main rotor has 5 blades, list *all* the possible rates at which the main rotor is spinning in revolutions per second assuming no physical limitations.
Hint: Your answer should depend on k where k can be any integer.

Solution

Let $\Delta = \frac{1}{30}s$ be the sample period. Since there are 5 blades, the rotor will look like itself after it finishes a fifth of a revolution.
This means that, in one Δ, the rotor could have completed $\frac{k}{5}$ revolutions, where k is an integer.
This means that the rotor could be spinning at $\frac{k}{5\Delta}$ revolutions per second, which is $(6 \times k)\, Hz$.

b) Given that the back rotor has 3 blades and completes 2 revolutions in 1 second **in the video**, list *all* the possible rates at which the back rotor is spinning in revolutions per second assuming no physical limitations.
Hint: Your answer should depend on k where k can be any integer.

Solution

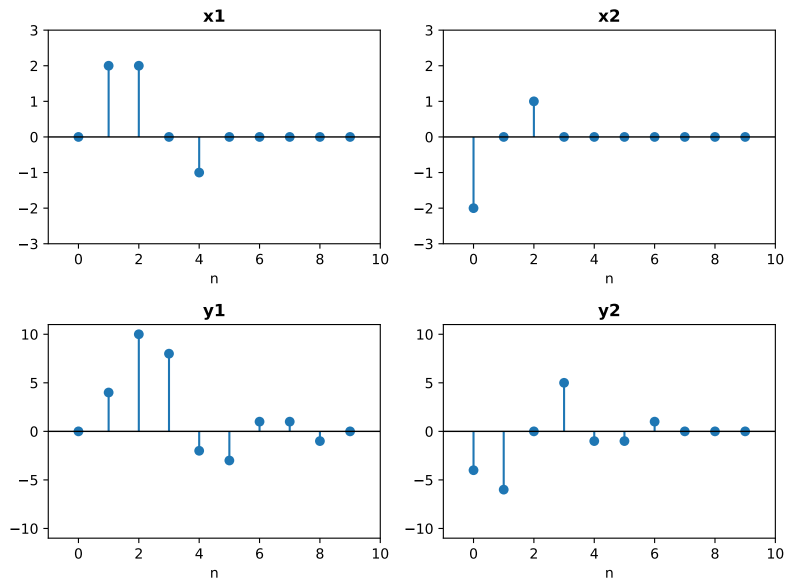
The video will always be sampling with sample period $\Delta = \frac{1}{30}s$. This means that the back rotor could have a period of Δ, 2Δ, 3Δ and so on and the camera would not be able to distinguish between them. Moreover, since a third of a rotation looks exactly like a complete rotation, the camera would not be able to tell the difference between periods of the form $k\frac{\Delta}{3}$ where k is an integer.
In the video, we see the back rotor moving roughly at the rate of 2 revolutions per second, which means that, in the video, it has a period of $\frac{1}{2}$ seconds.
This means that, in Δ seconds, it can finish $\frac{k}{3} + 2\Delta$ revolutions, where the $\frac{k}{3}$ revolutions gets hidden by the sample rate. This means that the rotor could be spinning at $\frac{k}{3\Delta} + 2$ revolutions per second, which is $(10k + 2)\, Hz$.

3 LTI Inputs

We have an LTI system whose exact characteristics we do not know. However, we know that it has a finite impulse response that is not longer than 5 samples. We also observed two sequences, x_1 and x_2, pass through the system and observed the system's responses y_1 and y_2.

x_1	0	2	2	0	−1	0	0	0	0	0
y_1	0	4	10	8	−2	−3	1	1	−1	0

x_2	−2	0	1	0	0	0	0	0	0	0
y_2	−4	−6	0	5	−1	−1	1	0	0	0



a) Given the above sequences, what would be the output for the input?

x_3	0	0	−2	0	1	0	0	0	0	0

Solution

This is a shifted version of x_2. Since the system is time-invariant, the output is:

y_3	0	0	−4	−6	0	5	−1	−1	1	0

b) What is the output of the system for the input $x_1 - x_2$?

Solution

Since the system is linear, the output is $y_1 - y_2$:

$y_1 - y_2$	4	10	10	3	−1	−2	0	1	−1	0

c) Given the above information, how could you find the impulse response of this system? What is the impulse response?

Solution

Since $\frac{1}{2}(x_1 + x_3)$ is an impulse at the second sample, $\frac{1}{2}(y_1 + y_3)$ is the system's impulse response starting from the second sample.

$y_1 + y_3$	0	4	6	2	−2	2	0	0	0	0

so

n	$n < 0$	0	1	2	3	4	$n > 4$
$h[n]$	0	2	3	1	−1	1	0

d) What is the output of this system for the following input:

x_4	1	1	1	0	0	0	0	0	0	0

Solution

The input x_4 can be written as a sum of three impulses: $x_4 = \delta[n] + \delta[n-1] + \delta[n-2]$. Since the system is LTI and we have computed the impulse response in the previous part, the output is going to be

$$y[n] = h[n] + h[n-1] + h[n-2].$$

y_4	2	5	6	3	1	0	1	0	0	0

4 LTI Low Pass Filters

Given a sequence of discrete samples with high frequency noise, we can de-noise our signal with a discrete low-pass filter. Two examples are given below:

$$y[n] = 0.5y[n-1] + x[n] \tag{1}$$
$$y[n] = 0.25x[n] + 0.25x[n-1] + 0.25x[n-2] + 0.25x[n-3] \tag{2}$$

a) Show that both systems (1) and (2) are LTI.

Solution

For (1),

(i) **Linearity:**

- Additivity:
  Let $x_1[n]$ and $x_2[n]$ be inputs with outputs $y_1[n]$ and $y_2[n]$.
  For input $\hat{x}[n] = (x_1 + x_2)[n]$, we see that

$$\begin{aligned}\hat{x}[n] = x_1[n] + x_2[n] &= y_1[n] - 0.5y_1[n-1] + y_2[n] - 0.5y_2[n-1] \\ &= y_1[n] + y_2[n] - 0.5(y_1[n-1] + y_2[n-1])\end{aligned}$$

  This shows that $\hat{y}[n] = y_1[n] + y_2[n]$ is the output.
- Scaling:
  Let $x[n]$ be an input with output $y[n]$. For a scaled input $\hat{x}[n] = \alpha x[n]$, we see that

$$\hat{x}[n] = \alpha x[n] = \alpha(y[n] - 0.5y[n-1]) = \alpha y[n] - \alpha 0.5y[n-1]$$

  This implies that $\hat{y}[n] = \alpha y[n]$.

(ii) **Time-Invariance**
Let $\hat{x}[n] = x[n-n_0]$ be a delayed input signal. We see that

$$\hat{x}[n] = x[n-n_0] = y[n-n_0] - 0.5y[n-n_0-1]$$

As a result, the output $\hat{y}[n]$ must be $\hat{y}[n] = y[n-n_0]$.

We could have also proven that this system is LTI by solving for $y[n]$ directly in terms of $x[n]$ and then showing that it is Linear and Time-Invariant.

For (2),

(i) **Linearity:**

- Additivity:
  Let $x_1[n]$ and $x_2[n]$ be inputs with outputs $y_1[n]$ and $y_2[n]$.
  For input $\hat{x}[n] = (x_1 + x_2)[n]$, we see that the output is

$$\begin{aligned}\hat{y}[n] &= 0.25\hat{x}[n] + 0.25\hat{x}[n-1] + 0.25\hat{x}[n-2] + 0.25\hat{x}[n-3] \\ &= 0.25(x_1+x_2)[n] + 0.25(x_1+x_2)[n-1] + 0.25(x_1+x_2)[n-2] + 0.25(x_1+x_2)[n-3] \\ &= 0.25x_1[n] + 0.25x_1[n-1] + 0.25x_1[n-2] + 0.25x_1[n-3] \\ &+ 0.25x_2[n] + 0.25x_2[n-1] + 0.25x_2[n-2] + 0.25x_2[n-3] \\ &= y_1[n] + y_2[n]\end{aligned}$$

  This shows that $\hat{y}[n] = y_1[n] + y_2[n]$ is the output.

- Scaling:
  Let $x[n]$ be an input with output $y[n]$. For a scaled input $\hat{x}[n] = \alpha x[n]$, we see that

$$\begin{aligned}\hat{y}[n] &= 0.25\hat{x}[n] + 0.25\hat{x}[n-1] + 0.25\hat{x}[n-2] + 0.25\hat{x}[n-3] \\ &= 0.25\alpha x[n] + 0.25\alpha x[n-1] + 0.25\alpha x[n-2] + 0.25\alpha x[n-3] \\ &= \alpha(0.25x[n] + 0.25x[n-1] + 0.25x[n-2] + 0.25x[n-3]) \\ &= \alpha y[n]\end{aligned}$$

  This implies that $\hat{y}[n] = \alpha y[n]$.

(ii) **Time-Invariance**
Let $\hat{x}[n] = x[n-n_0]$ be a delayed input signal. Then the output $\hat{y}[n]$ will be

$$\begin{aligned}\hat{y}[n] &= 0.25\hat{x}[n] + 0.25\hat{x}[n-1] + 0.25\hat{x}[n-2] + 0.25\hat{x}[n-3] \\ &= 0.25x[n-n_0] + 0.25x[n-n_0-1] + 0.25x[n-n_0-2] + 0.25x[n-n_0-3]\end{aligned}$$

On the other hand, $y[n-n_0]$ is equal to

$$y[n-n_0] = 0.25x[n-n_0] + 0.25x[n-n_0-1] + 0.25x[n-n_0-2] + 0.25x[n-n_0-3]$$

Therefore, we see that $\hat{y}[n] = y[n-n_0]$ and can conclude that the system is time-invariant.

b) Write the impulse responses $h[n]$ for (1) and (2). You may assume that $h[n] = 0$ for $n < 0$.

Solution

For (1),

$$h[n] = \begin{cases} 0.5^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

This has an infinite impulse response (IIR).

For (2),

$$h[n] = \begin{cases} 0.25, & n = 0, 1, 2, 3 \\ 0, & \text{otherwise} \end{cases}$$

This has a finite impulse response (FIR).

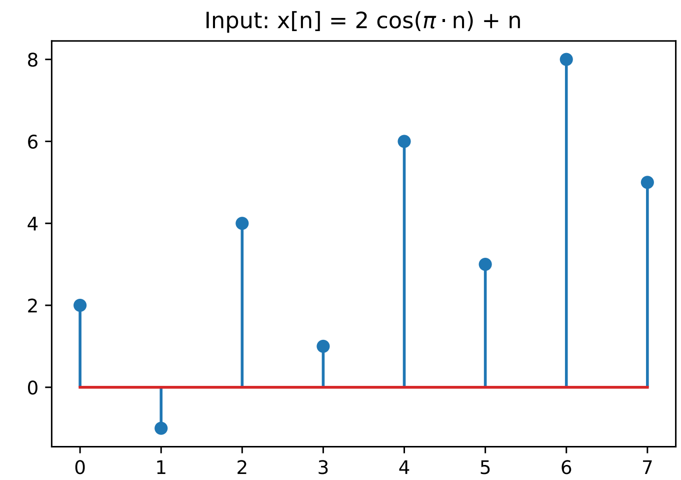
c) Are either of these systems causal? Are either of these systems stable?

Solution

They are both causal since $y[n]$ depends only on inputs from $x[n]$ and before. One could also argue that they are causal since the system is LTI and $h[n] = 0$ for $n < 0$.

Both filters are stable because the impulse responses are absolutely summable.

d) Given the input sequence $x[n] = 2\cos(\pi n) + n$ from $n = 0$ to $n = 7$, find the output y for each system from $n = 0$ to $n = 7$. Assume that $y[n] = 0$ for $n < 0$.

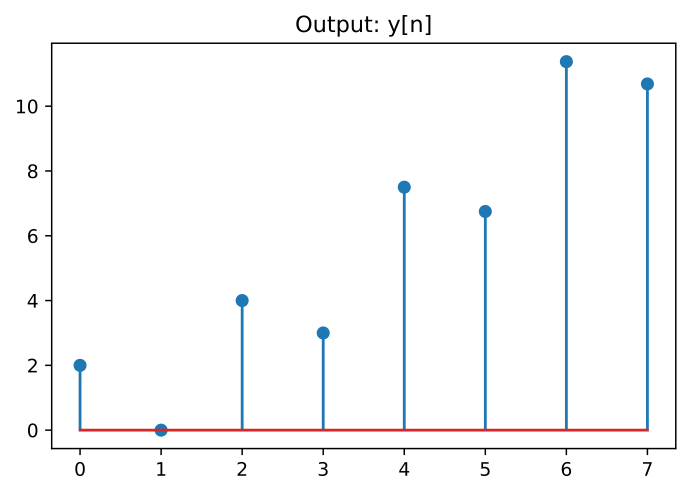


Solution

For both systems, we can compute the output $y[n] = 0.5y[n-1] + x[n]$ using the difference equation or through a convolution.

For (1), the output will be

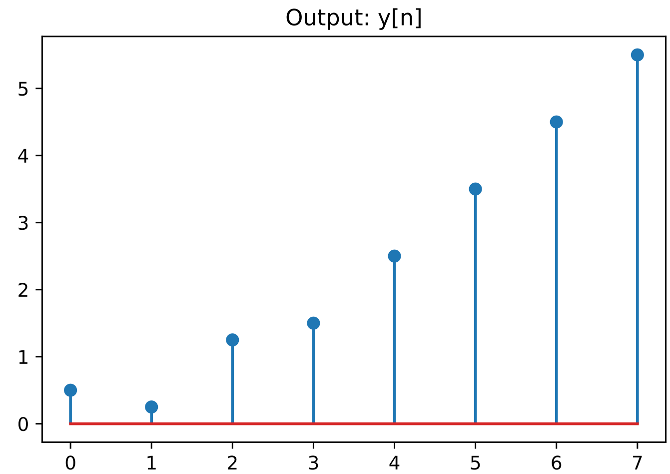
$$y = [2, 0, 4, 3, 7.5, 6.75, 11.375, 10.6875]$$



Notice how the oscillations have been reduced since this system is a low-pass filter.

For (2), the output will be

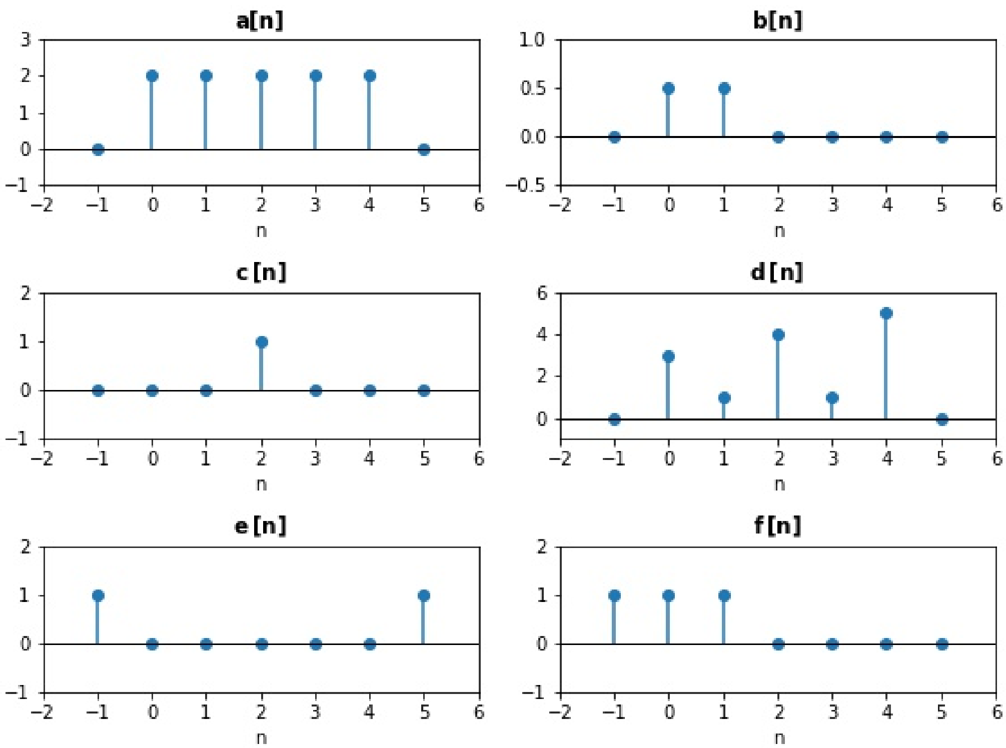
$$y = [0.5, 0.25, 1.25, 1.5, 2.5, 3.5, 4.5, 5.5]$$



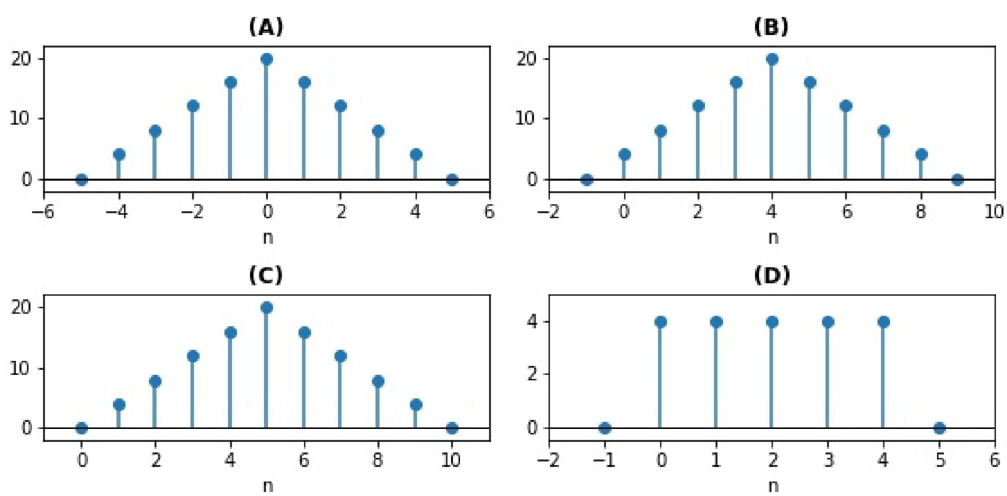
Again the oscillations $\cos(\pi n)$ have been significantly attenuated at the output. Notice how the output is approximately $y[n] \approx n$.

5 Convolution Matching

Consider the following discrete time signals:



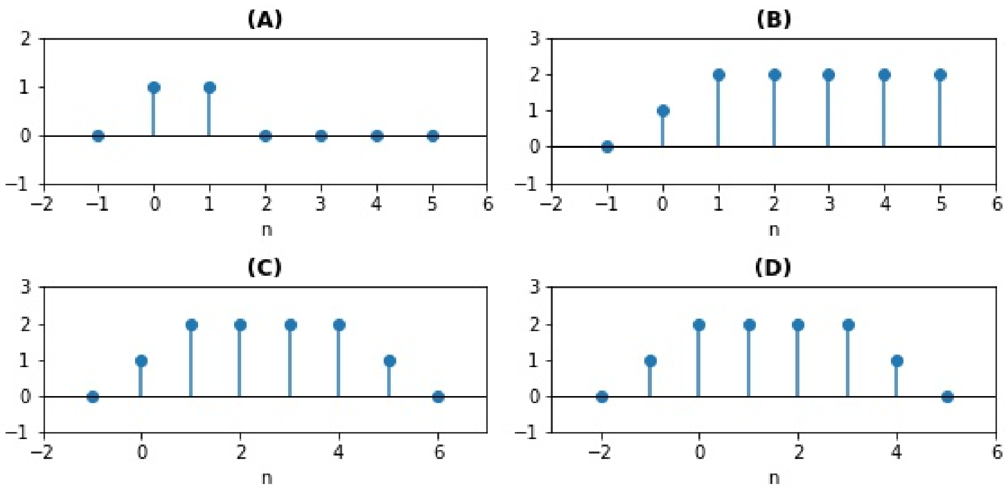
a) Which of the options below shows the correct plot for the convolution $a[n] * a[n]$?



Solution

The correct plot is B. A shows cross-correlation instead of convolution. C is close, but off by one timestep. D shows pointwise multiplication of the two signals.

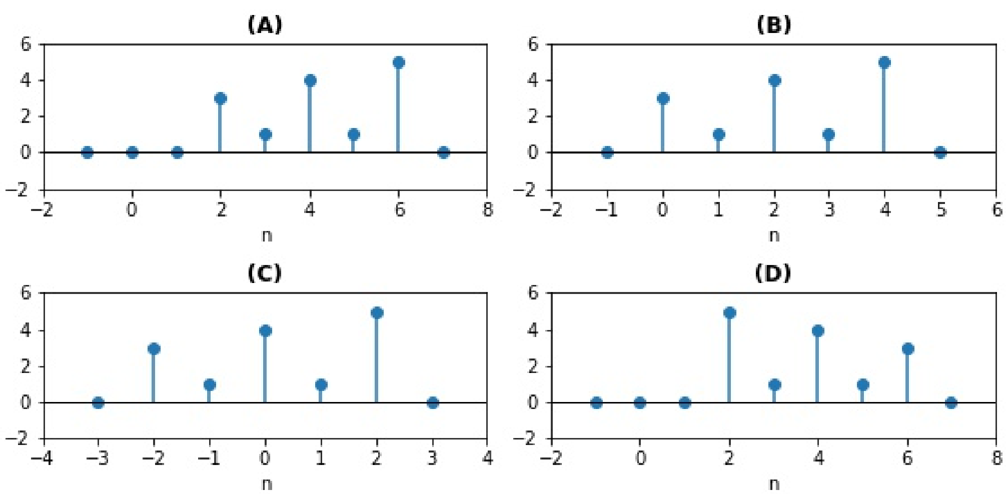
b) Which of the options below shows the correct plot for the convolution $a[n] * b[n]$?



Solution

The correct plot is C. A shows pointwise multiplication of the two signals. B would be the output if $a[n]$ was a unit step as opposed to a boxcar. D shows cross-correlation instead of convolution.

c) Which of the options below shows the correct plot for the convolution $c[n] * d[n]$?



Solution

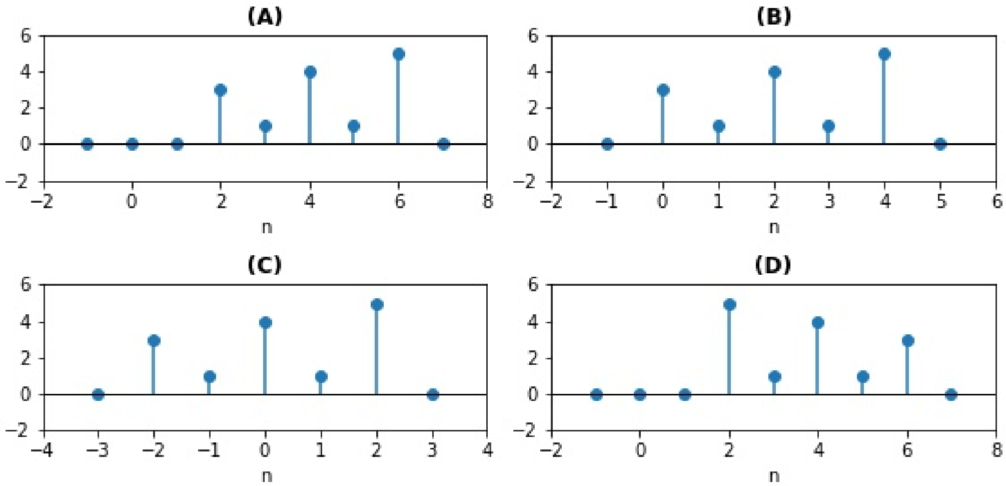
The correct plot is A. This demonstrates that convolution with a shifted impulse shifts the input signal! That is,

$$x[n] * \delta[n - n_0] = \sum_{m=-\infty}^{\infty} \delta[m - n_0]x[n - m] = x[n - n_0]$$

From this we can argue that the impulse response of a delay block would be a shifted unit impulse.

B shows the output if $d[n]$ were convolved with the identity element of convolution $\delta[n]$, i.e. a unit impulse with a delay of $n_0 = 0$. C shows cross-correlation instead of convolution. D shows convolution with $d[4-n]$.

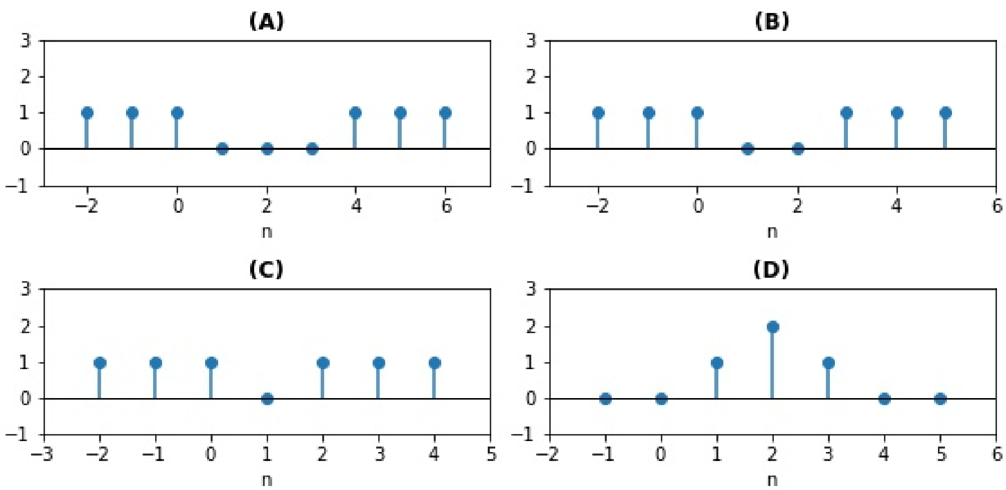
d) Which of the options below shows the correct plot for the convolution $d[n] * c[n]$?



Solution

The correct plot is still A. This is because convolution is commutative!

e) Which of the options below shows the correct plot for the convolution $e[n] * f[n]$?



Solution

The correct plot is A. We can either show this algebraically or use the convolution formula. Since $e[n] = \delta[n+1] + \delta[n-5]$ and convolution is distributive,

$$\begin{aligned}(e * f)[n] &= (f * e) = (f * (\delta[n+1] + \delta[n-5])) \\ &= f[n] * \delta[n+1] + f[n] * \delta[n-5] = f[n+1] + f[n-5]\end{aligned}$$

So it follows that the $(e * f)[n]$ is the sum of two copies of f. One shifted by 1 to the left and the other shifted 5 to the right.

6 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **What sources (if any) did you use as you worked through the homework?**

b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

c) **Roughly how many total hours did you work on this homework?**

d) **Do you have any feedback on this homework assignment?**