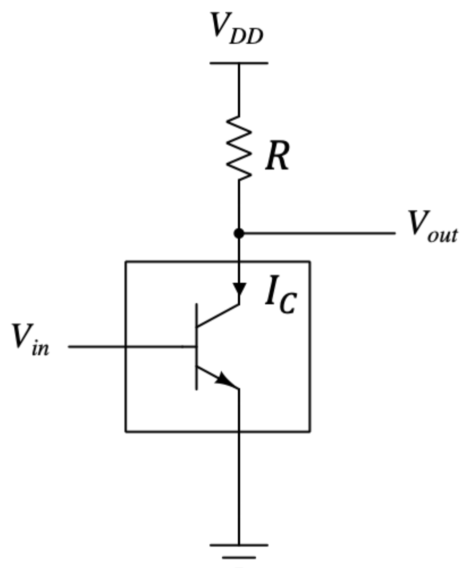
Solutions have been provided for all the problems in this Homework. You do not need to make a submission on gradescope. All the best for your midterm!

1 Linearizing for understanding amplification

Linearization isn't only important for control, robotics, machine learning, and optimization — it is one of the standard tools used across different areas, including thinking about circuits. The circuit below is a voltage amplifier, where the element inside the box is a bipolar junction transistor (BJT).



The bipolar transistor in the circuit can be modeled quite accurately as a nonlinear, voltage-controlled current source, where the collector current I_C is given by

$$I_C(V_{in}) = I_S e^{\frac{V_{in}}{V_{TH}}}, \tag{1}$$

where V_{TH} is the thermal voltage. We can assume $V_{TH} = 26$ mV at temperatures of 300K (close to room temperature). I_S is a constant whose exact value we are not giving you because we want you to find ways of eliminating it in favor of other quantities whenever possible.

With this amplifier, small variations in the input voltage V_{in} can turn into large variations in the output voltage V_{out} under the right conditions. We're going to investigate this amplification using linearization.

Let's consider the 2N3904 transistor, where the above expression for $I_C(V_{in})$ holds as long as $0.2\text{V} < V_{out} < 40\text{V}$, and $0.1\text{mA} < I_C < 10\text{mA}$.

(Note that the 2N3904 is a cheap transistor that people often use in personal projects. You can get them for 3 cents each if you buy in bulk.)

a) **Write a symbolic expression for V_{out} as a function of I_C.**

Solution

$V_{out} = V_{DD} - RI_C$ since we have a voltage drop of $I_C R$ across the resistor and the top voltage is V_{DD}.

b) Now let's linearize I_C in the neighborhood of an input voltage V_{in}^* and a specific I_C^*. Assume that you have a found a particular pair of input voltage V_{in}^* and current I_C^* that satisfy the current equation (1).

We can look at nearby input voltages and see how much the current changes. We can write the linearized expression for the collector current around this point as:

$$I_C(V_{in}) = I_C(V_{in}^*) + \delta I_C \approx I_C^* + m(V_{in} - V_{in}^*) = I_C^* + m\ \delta V_{in} \tag{2}$$

where $\delta V_{in} = V_{in} - V_{in}^*$ is the change in input voltage and $\delta I_C = I_C - I_C^*$ is the change in collector current.

What is m here as a function of I_C^* and V_{TH}?

(If you take EE105, you will learn that this m is called the transconductance, which is usually written g_m, and is the single most important parameter in most analog circuit designs.)

(HINT: First just find m by taking the appropriate derivative and using the chain rule as needed. Then leverage the special properties of the exponential function to express it in terms of the desired quantities.)

Solution

We start out by writing out the linearization form that we are looking for.

$$I_C(V_{in}) = I_C^* + \delta I_C = I_C(V_{in}^*) + m\ \delta V_{in}$$

Here, we can isolate the δI_C term by subtracting $I_C^* = I_C(V_{in}^*)$ from both sides.

$$\delta I_C = m\ \delta V_{in}$$

Now, the meaning of the m is the slope of the I_C curve at V_{in}^*.

$$m = \left.\frac{dI_C}{dV_{in}}\right|_{V_{in}^*} \tag{3}$$

$$= \left.\frac{1}{V_{TH}} I_S e^{\frac{V_{in}}{V_{TH}}}\right|_{V_{in}^*} \tag{4}$$

$$= \frac{I_C^*}{V_{TH}} \tag{5}$$

where in the last line, we recognize that the expression in the exponential with the I_S before it is just I_C itself. This is why the constant I_S does not need to be specified explicitly as long as I_C^* is known.

We can use these equations to linearize I_c at certain chosen values of V_{in}, such as values $V_{in}^* = 0.65$ V and $V_{in}^* = 0.65$ V given in parts (d) and (e) below. We plot these linearizations here to help visualize our results.

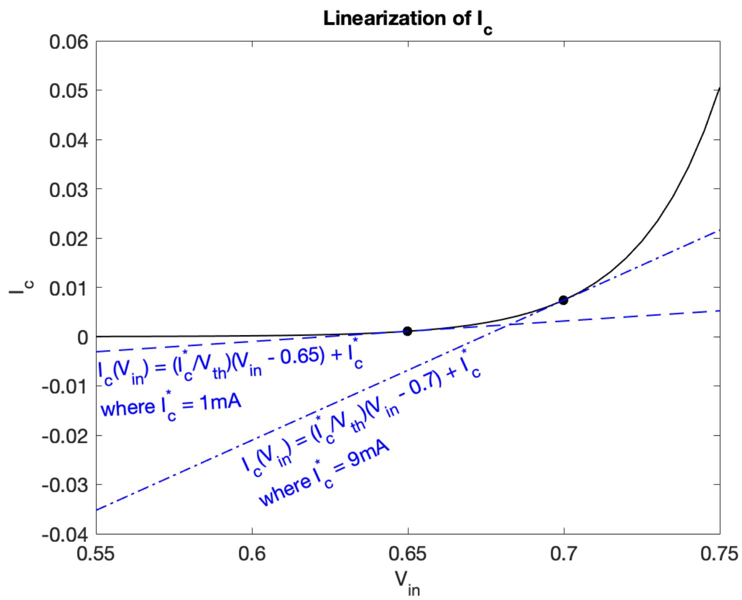


Figure 1: Linearization of I_c

c) We now have a linear relationship between small changes in current and voltage, $\delta I_C = m\ \delta V_{in}$ around a known solution (I_C^*, V_{in}^*). This is called a "bias point" in circuits terminology. (This is also why related things in neural nets are called bias terms — their function is to get the nonlinearity to behave the way we want it to.)

Going back to your equation from part (a), plug in your linearized equation for I_C. Define the appropriate V_{out}^* so that it makes sense to view $V_{out} = V_{out}^* + \delta V_{out}$ when we have $V_{in} = V_{in}^* + \delta V_{in}$, and **find the approximate linear relationship between δV_{out} and δV_{in}.**

The ratio $\frac{\delta V_{out}}{\delta V_{in}}$ is called the small-signal voltage gain of this amplifier around this bias point.

Solution

Expanding out and remembering the equation for V_{out} from above:

$$V_{out} = V_{out}^* + \delta V_{out} = V_{DD} - R(I_C^* + m\ \delta V_{in})$$

And so we should define $V_{out}^* = V_{DD} - RI_C^*$ and then have

$$\delta V_{out} = -R\ m\ \delta V_{in}$$

where the m is as it was above. Namely

$$\delta V_{out} = -\frac{I_C^* R}{V_{TH}}\ \delta V_{in} = -\frac{V_{DD} - V_{out}^*}{V_{TH}}\ \delta V_{in}$$

.

You don't have to simplify it to this point, but this form is useful because it shows you that the gap between the operating point V_{out^*} to the supply rail V_{DD} matters to understand the small-signal gain. We want as much current as possible to make the gain big, but there is a limit to how big the current can get.

We can use these equations to linearize V_{out} at certain chosen values of V_{in}, such as values $V_{in}^* = 0.65$ V and $V_{in}^* = 0.65$ V given in parts (d) and (e) below. We plot these linearizations here to help visualize our results. The slope of these lines are the small signal voltage gain $\frac{\delta V_{out}}{\delta V_{in}} = -\frac{I_C^* R}{V_{th}}$.

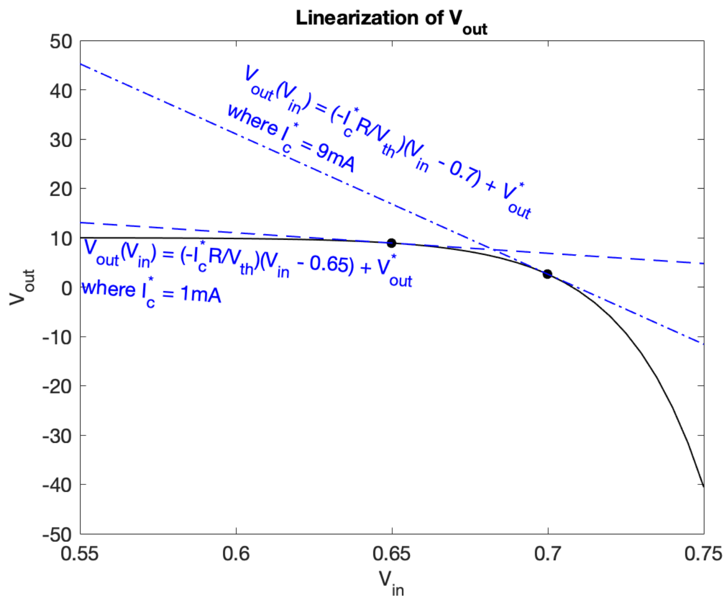


Figure 2: Linearization of V_{out}

d) Assuming that $V_{DD} = 10V$, $R = 1k\Omega$, and $I_C^* = 1mA$ when $V_{in}^* = 0.65V$, **what is the small-signal voltage gain $\frac{\delta V_{out}}{\delta V_{in}}$, between the input and the output around this bias point?** (one or two digits of precision is plenty)

Solution

Just plugging in using the current form:

$$-(1k\Omega)(1mA/26mV) = -1V/26mV \approx -38$$

e) If $I_C^* = 9mA$ when $V_{in}^* = 0.7V$, **what is the small-signal voltage gain around this bias point?** (one or two digits is plenty)

Solution

$$-(1k\Omega)(9mA/26mV) = -9V/26mV \approx -350$$

Notice here that we have V_{out}^* has already been pulled down around $1V$. So, this is close to as big as this gain can get. It is not obvious that it is actually V_{DD} and V_{TH} that provide the fundamental limit on the small-signal gain for such circuits, but the simple linearization analysis above reveals this. Before doing this analysis, it would be tempting to believe that it is the size of the resistance that matters a lot. These circuit insights and more are developed further in courses like 105 and then in 140 where ideas of feedback control and circuit design come together in interesting ways.

This shows you how by appropriately biasing (choosing an operating point), we can adjust what our gain is for small signals. Although here, we just wanted to show you this as a simple application of linearization, these ideas are developed a lot further in 105, 140, and other courses to create things like op-amps and other analog information-processing systems.

2 Non-Linear Spring-Mass system

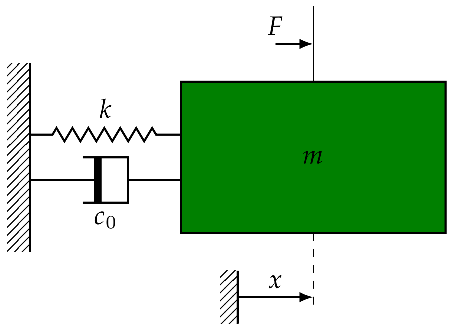


Figure 3: Schematic of a non-linear spring-mass system.

In this problem, we will analyze a non-linear spring-mass system shown in Figure 3. The dynamics of this system are given by

$$m\frac{d^2}{dt^2}x(t) = -kx(t) - c_0\frac{d}{dt}x(t) + F(t) \tag{6}$$

The spring constant in this case is not constant. Instead, it depends on the position x of the block. The spring constant is given by

$$k(x) = k_0\left(\frac{x^2}{a^2} - 1\right), \tag{7}$$

where a is a constant and $k_0 > 0$.

a) Write a state space model for this system using state variables $x_1 = x(t)$ and $x_2 = \frac{d}{dt}x(t)$. Is the system linear?

Solution

First, we can plug in the relationship given in Equation 7 into Equation 6 to get

$$m\frac{d^2}{dt^2}x(t) = k_0\left(1 - \frac{x(t)^2}{a^2}\right)x(t) - c_0\frac{d}{dt}x(t) + F(t) \tag{8}$$

$$\frac{d^2}{dt^2}x(t) = \frac{k_0}{m}\left(1 - \frac{x(t)^2}{a^2}\right)x(t) - \frac{c_0}{m}\frac{d}{dt}x(t) + \frac{1}{m}F(t) \tag{9}$$

Using state variables $x_1 = x(t)$ and $x_2 = \frac{dx(t)}{dt}$, we can then write

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}f_1\\f_2\end{bmatrix} = \begin{bmatrix}x_2\\ \frac{k_0}{m}x_1 - \frac{k_0}{ma^2}x_1^3 - \frac{c_0}{m}x_2 + \frac{1}{m}F\end{bmatrix} \tag{10}$$

For brevity, we will call $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to be $\vec{x}$ from here on. Because of the cubic term $\frac{k_0}{ma^2}x_1^3$, we cannot express this system as $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + BF(t)$. Our system is non-linear.

b) Find all the equilibrium points for this system in the absence of an external input, i.e., $F = 0$.

Solution

At equilibrium, for a continuous-time system, we must have $\frac{d}{dt}\vec{x} = 0$. In order to achieve this, we need

$$\frac{d}{dt}x_1 = x_2 = 0. \tag{11}$$

Simultaneously, we need

$$\frac{d}{dt}x_2 = \frac{k_0}{m}x_1 - \frac{k_0}{ma^2}x_1^3 - \frac{c_0}{m}x_2 + \frac{1}{m}F = 0 \tag{12}$$

Since $x_2 = 0$ and $F = 0$, Equation 12 reduces to

$$\frac{k_0}{m}\left(x_1 - \frac{x_1^3}{a^2}\right) = 0$$
$$x_1\left(1 - \frac{x_1^2}{a^2}\right) = 0$$

We have 3 different equilibrium positions associated with $x_2 = 0$. These are $x_1 = -a$, $x_1 = 0$, and $x_1 = a$.

We can also express these equilibria as $\vec{x}^* = \begin{bmatrix} -a \\ 0 \end{bmatrix}$, $\vec{x}^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $\vec{x}^* = \begin{bmatrix} a \\ 0 \end{bmatrix}$.

c) Linearize this system around each of the equilibrium points $\vec{x}^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$ and characterize whether those are stable or unstable. Can you comment on the stability using the spring constant $k(x)$.

Solution

Let's us start with the Jacobian for the system we have derived.

$$\nabla f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ \frac{k_0}{m} - \frac{3k_0}{ma^2}x_1^2 & -\frac{c_0}{m} \end{bmatrix}$$

Our linearized system can now be written using the Jacobian as

$$\frac{d}{dt}\tilde{x}(t) = \nabla f\,\tilde{x}(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F(t), \tag{13}$$

where $\tilde{x}(t)$ is the linearized variable, i.e., $\tilde{\vec{x}}(t) = \vec{x}(t) - \vec{x}^*$.

Since this is a continuous-time system, it is stable if all the eigenvalues of ∇f have negative real parts ($Re(\lambda_i) < 0\ \forall i$). The eigenvalues of the Jacobian matrix found above are given by

$$\begin{vmatrix} \lambda & -1 \\ -\frac{k_0}{m} + \frac{3k_0}{ma^2}x_1^2 & \lambda + \frac{c_0}{m} \end{vmatrix} = 0$$

$$\lambda(\lambda + \frac{c_0}{m}) - \left(\frac{k_0}{m} - \frac{3k_0}{ma^2}x_1^2\right) = 0$$

$$\lambda^2 + \frac{c_0}{m}\lambda - \left(\frac{k_0}{m} - \frac{3k_0}{ma^2}x_1^2\right) = 0$$

The roots for this polynomial can be found using the quadratic formula. These roots are

$$\lambda = \frac{-\frac{c_0}{m} \pm \sqrt{\left(\frac{c_0}{m}\right)^2 + 4\left(\frac{k_0}{m} - \frac{3k_0}{ma^2}x_1^2\right)}}{2} \tag{14}$$

Now, let us look at these eigenvalues for our 3 equilibrium points.

i) $x_1 = 0$

$$\lambda = \frac{-\frac{c_0}{m} \pm \sqrt{\left(\frac{c_0}{m}\right)^2 + \frac{4k_0}{m}}}{2}$$

The eigenvalue $\lambda = \frac{-\frac{c_0}{m} + \sqrt{\left(\frac{c_0}{m}\right)^2 + \frac{4k_0}{m}}}{2}$ is positive, making the linearized system unstable at this equilibrium.

ii) $x_1 = -a$ and $x_1 = a$. For these scenarios, we have the same eigenvalues for the linearized system. These are

$$\lambda = \frac{-\frac{c_0}{m} \pm \sqrt{\left(\frac{c_0}{m}\right)^2 + 4\left(\frac{k_0}{m} - \frac{3k_0a^2}{ma^2}\right)}}{2}$$

$$= \frac{-\frac{c_0}{m} \pm \sqrt{\left(\frac{c_0}{m}\right)^2 - \frac{8k_0}{m}}}{2}$$

Since $k_0 > 0$ and $m > 0$, we can either have $\left(\frac{c_0}{m}\right)^2 < \frac{8k_0}{m}$, which results in a pair of complex-conjugate eigenvalues with $Re(\lambda_i) = -\frac{c_0}{m} < 0$. In the other case, if $\left(\frac{c_0}{m}\right)^2 >= \frac{8k_0}{m}$, both our eigenvalues are real. However, $\frac{-\frac{c_0}{m} \pm \sqrt{\left(\frac{c_0}{m}\right)^2 - \frac{8k_0}{m}}}{2} < 0$, which gives us 2 negative eigenvalues.

In either of those scenarios, the linearized system is stable.

Let's us look at the spring constant to assess these equilibria. At $x = 0$, the spring constant $k(x) = -k_0 < 0$. In this scenario the force exerted by the spring is $-k(x)x(t) = k_0x(t)$. At the equilibrium position $x(t) = 0$, the spring exerts no force. However, if $x(t) > 0$, then the spring exerts a net positive (away from $x_1 = 0$) force. Similarly, when $x(t) < 0$, the spring exerts a net negative (also away from $x_1 = 0$) force. As a result, the equilibrium is unstable.

Let's look at the stable equilibrium $x(t) = a$. At $x(t) = a$, on the other hand, if $x(t) > a$, $k(x) > 0$, whereas if $x(t) < a$, $k(x) > 0$. As a result, if we nudge the mass outward, making $x(t) > a$, the spring pulls the block back. On the other hand, if we nudge the block inward, making $x(t) < a$, the spring pushes the block out. In either of the scenarios, the spring pushes the block back towards the equilibrium position making it stable.

d) Design a state-feedback controller for $F(t) = K\vec{x}(t)$ which stabilizes the linearized system at $x = 0, \frac{d}{dt}x = 0$.

Solution

Evaluating the linearized system from Equation 13 at $\vec{x}^* = 0$, we can write the state-feedback control with $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ as

$$\begin{aligned}
\frac{d}{dt}\tilde{x}(t) &= \nabla f \vec{x} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \tilde{x}(t) \\
&= \left(\nabla f + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right) \tilde{x}(t) \\
&= \left(\begin{bmatrix} 0 & 1 \\ \frac{k_0}{m} & -\frac{c_0}{m} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{k_1}{m} & \frac{k_2}{m} \end{bmatrix} \right) \vec{x}(t) \\
&= \begin{bmatrix} 0 & 1 \\ \frac{k_0+k_1}{m} & \frac{k_2-c_0}{m} \end{bmatrix} \vec{x}(t)
\end{aligned}$$

The eigenvalues for the system with state-feedback control are given by

$$\begin{aligned}
\begin{vmatrix} \lambda & -1 \\ -\left(\frac{k_0+k_1}{m}\right) & \lambda + \frac{c_0-k_2}{m} \end{vmatrix} &= 0 \\
\lambda \left(\lambda + \frac{c_0 - k_2}{m} \right) - \left(\frac{k_0 + k_1}{m} \right) &= 0 \\
\lambda^2 + \lambda \left(\frac{c_0 - k_2}{m} \right) - \left(\frac{k_0 + k_1}{m} \right) &= 0
\end{aligned}$$

Setting $k_2 = 0$ and $k_1 = -k_0 - \frac{c_0^2}{4m}$, gives us

$$\begin{aligned}
\lambda^2 + \frac{c_0}{m}\lambda + \frac{c_0^2}{4m^2} &= 0 \\
\left(\lambda + \frac{c_0}{2m} \right)^2 &= 0 \\
\lambda &= -\frac{c_0}{2m}
\end{aligned}$$

e) We like the feedback controller that we have designed and decide to apply the feedback controller to the original non-linear system. We use $F(t) = k_1 x_1(t) + k_2 x_2(t)$. What are the equilibrium points for this new system?

Solution

The new, non-linear system is given by

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix} &= \begin{bmatrix}x_2\\ \frac{k_0}{m}x_1-\frac{k_0}{ma^2}x_1^3-\frac{c_0}{m}x_2+\frac{k_1x_1+k_2x_2}{m}\end{bmatrix}\\ &= \begin{bmatrix}x_2\\ \frac{k_0}{m}x_1-\frac{k_0}{ma^2}x_1^3-\frac{c_0}{m}x_2+\frac{1}{m}\left(\left(-k_0-\frac{c_0^2}{4m}\right)x_1+0\cdot x_2\right)\end{bmatrix}\\ &= \begin{bmatrix}x_2\\ -\frac{k_0}{ma^2}x_1^3-\frac{c_0}{m}x_2-\frac{c_0^2}{4m^2}x_1\end{bmatrix}\end{aligned}$$

At equilibrium, we have

$$\begin{aligned}\frac{d}{dt}x_1 &= x_2 = 0\\ \frac{d}{dt}x_2 &= -\frac{k_0}{ma^2}x_1^3-\frac{c_0}{4m}x_2-\frac{c_0^2}{4m^2}x_1 = 0.\end{aligned}$$

Solving for x_1, we get

$$-\frac{x_1}{m}\left(\frac{c_0^2}{4m}+\frac{k_0}{a^2}x_1^2\right)=0 \tag{15}$$

Equation 15 has only 1 solution, $x_1=0$. If we choose to apply state-based feedback that we had previously derived to the system, we are left with a single equilibrium at $x_1=0, x_2=0$, which is stable. The previous equilibria at $x_1=\pm a, x_2=0$ are no longer equilibrium states.

3 Controllability in 2D

Consider the control of some two-dimensional linear discrete-time system

$$\vec{x}(k+1)=A\vec{x}(k)+Bu(k)$$

where A is a 2×2 real matrix and B is a 2×1 real vector.

a) Let $A=\begin{bmatrix}a&0\\c&d\end{bmatrix}$ with $a,c,d\neq 0$, and $B=\begin{bmatrix}f\\g\end{bmatrix}$. Find a B such that the system is controllable no matter what nonzero values a,c,d take on, and a B for which it is not controllable no matter what nonzero values are given for a,c,d. You can use the controllability rank test, but please explain your intuition as well.

Solution

With $B=\begin{bmatrix}1\\0\end{bmatrix}$, the system is controllable for all nonzeros a,c,d, because $[B, AB]=\begin{bmatrix}1&a\\0&c\end{bmatrix}$, which has full rank. With $B=\begin{bmatrix}0\\1\end{bmatrix}$ the system is not controllable because $[B, AB]=\begin{bmatrix}0&0\\1&d\end{bmatrix}$, which only has rank=1. The intuition is that, due to the zero entry in A, the state x_1 evolves autonomously, i.e., $\frac{d}{dt}x_1(t)=ax_1(t)$, hence it needs to be controlled by some input f. On the other hand we can control x_2 via controlling x_1, as $\frac{d}{dt}x_2(t)=cx_1(t)+dx_2(t)$, which implies that x_2 can be "tuned" by manipulating x_1.

b) Let $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ with $a, d \neq 0$. and $B = \begin{bmatrix} f \\ g \end{bmatrix}$ with $f, g \neq 0$. Is this system always controllable? If not, find configurations of nonzero a, d, f, g that make the system uncontrollable.

Solution

No. uncontrollable when $a = d$. In this case the matrix is just a constant a times the identity. So when you check with the controllability test, AB is just a scalar multiple of B and hence linearly dependent. The intuition is that the two states are inherently "coupled" as two eigenvalues are the same. Any control input can only move the states along a line hence the states cannot reach arbitrary points in $\mathbb{R}^2$.

c) We want to see if controllability is preserved under changes of coordinates. To begin with, let $\vec{z}(k) = V^{-1}\vec{x}(k)$, please write out the system equation with respect to $\vec{z}$.

Solution

$\vec{x}(k) = V\vec{z}(k)$, hence we have

$$V\vec{z}(k+1) = AV\vec{z}(k) + Bu(k)$$

$$\vec{z}(k+1) = V^{-1}AV\vec{z}(k) + V^{-1}Bu(k)$$

.

d) Now show that controllability is preserved under change of coordinates. (Hint: use the fact that $\text{rank}(MA) = \text{rank}(A)$ for any invertible matrix M.)

Solution

The matrix whose rank needs to be tested after the coordinate change is $[V^{-1}B, V^{-1}AVV^{-1}B] = [V^{-1}B, V^{-1}AB] = V^{-1}[B, AB]$ which has the same rank as $[B, AB]$, since V by assumption is full rank.

4 System Identification

Consider a discrete-time system with unknown dynamics. Assume that starting from $x_0 = \begin{bmatrix} 1 & 2 \end{bmatrix}^\top$ we applied the following controls to the system, and observed the resulting states:

$$u_0 = 1, \quad u_1 = 2, \quad u_2 = 0, \quad u_3 = 1,$$
$$x_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, x_2 = \begin{bmatrix} 4 \\ 8 \end{bmatrix}, x_3 = \begin{bmatrix} 8 \\ 1 \end{bmatrix}, x_4 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

a) Set up a least-squares problem to recover $A \in \mathbb{R}^{2\times 2}$ and $B \in \mathbb{R}^{2\times 1}$ of a discrete-time model of this system

$$x_{k+1} = Ax_k + Bu_k$$

Solution

Define the modeling error of these observations by

$$e = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} - \begin{bmatrix} x_0^\top & 0 & u_0 & 0 \\ 0 & x_0^\top & 0 & u_0 \\ x_1^\top & 0 & u_1 & 0 \\ 0 & x_1^\top & 0 & u_1 \\ x_2^\top & 0 & u_2 & 0 \\ 0 & x_2^\top & 0 & u_2 \\ x_3^\top & 0 & u_3 & 0 \\ 0 & x_3^\top & 0 & u_3 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \\ b_1 \\ b_2 \end{bmatrix}. \tag{16}$$

Note that $e \in \mathbb{R}^8$ and in the model definition in Equation 16, 0 represents a row vector $\begin{bmatrix} 0 & 0 \end{bmatrix}$. The optimization problem to solve is then

$$\min_{a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2} \quad \|e\|_2$$

Where e is defined as above.

b) Could the estimates of A and B be uniqueley determined from less observations than those given? Explain.

Solution

Only if the observation matrix in the least squares problem remains full-rank. Note that the data matrix

$$\begin{bmatrix} x_0^\top & u_0 \\ x_1^\top & u_1 \\ x_2^\top & u_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 4 & 8 & 0 \end{bmatrix}$$

has linearly dependent columns, and therefore matrix is not full rank. It follows that the matrix

$$\begin{bmatrix} x_0^\top & 0 & u_0 & 0 \\ 0 & x_0^\top & 0 & u_0 \\ x_1^\top & 0 & u_1 & 0 \\ 0 & x_1^\top & 0 & u_1 \\ x_2^\top & 0 & u_2 & 0 \\ 0 & x_2^\top & 0 & u_2 \end{bmatrix}$$

is also not full rank. Therefore the least-squares problem does not have a unique solution for this selection of observations. .

c) Now let's say that the matrix A was provided to us. Reformulate the problem to estimate the model parameters b_1 and b_2.

Solution

Since the matrix A is given to us, let us reformulate the least-squares problem for the remaining model parameters b_1 and b_2. Let us redefine $y_k = x_{k+1} - Ax_k$. This gives us a new system in y and u.

$$y_k = Bu_k \tag{17}$$

We can write the observations and inputs as

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} u_0 & 0 \\ 0 & u_0 \\ u_1 & 0 \\ 0 & u_1 \\ u_2 & 0 \\ 0 & u_2 \\ u_3 & 0 \\ 0 & u_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{18}$$

Again, in the least-squares solution for parameters b_1 and b_2 is given by

$$\underset{b_1, b_2}{\arg\min} \left\| \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} - \begin{bmatrix} u_0 & 0 \\ 0 & u_0 \\ u_1 & 0 \\ 0 & u_1 \\ u_2 & 0 \\ 0 & u_2 \\ u_3 & 0 \\ 0 & u_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\|^2 \tag{19}$$

5 SVD

Find the singular value decomposition of the following matrix (leave all work in exact form, not decimal):

$$A = \begin{bmatrix} 1 & 0 & -\sqrt{3} \\ \sqrt{3} & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}$$

a) Find the eigenvalues of $A^\top A$ and order them from largest to smallest, $\lambda_1 > \lambda_2$.

Solution

$$A^\top A = \begin{bmatrix} 1 & \sqrt{3} & 0 \\ 0 & 0 & 3 \\ -\sqrt{3} & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\sqrt{3} \\ \sqrt{3} & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\lambda_1 = 9, \quad \lambda_2 = 4, \quad \lambda_3 = 4$$

b) Find orthonormal eigenvectors $\vec{v}_i$ of $A^\top A$ (all eigenvectors are mutually orthogonal and have unit length).

Solution

$\lambda_1 = 9$:

$$\text{Null}(A^\top A - 9I) = \text{Null}\left(\begin{bmatrix} -5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$$

Alternatively:

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = \begin{bmatrix} 9v_{11} \\ 9v_{12} \\ 9v_{13} \end{bmatrix} \implies v_{11} = 0, v_{12} = 1, v_{13} = 0$$

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Since $\lambda_2 = \lambda_3$, any two mutually orthogonal unit vectors that are also orthogonal to $\vec{v}_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^\top$ will work. For example:

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

c) Find the singular values $\sigma_i = \sqrt{\lambda_i}$. Find the $\vec{u}_i$ vectors from:

$$A\vec{v}_i = \sigma_i \vec{u}_i$$

Solution

$$\sigma_1 = 3, \quad \sigma_2 = 2, \quad \sigma_3 = 2$$

$$\vec{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

d) Write out A as a weighted sum of rank 1 matrices:

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^\top + \sigma_2 \vec{u}_2 \vec{v}_2^\top + \sigma_3 \vec{u}_3 \vec{v}_3^\top$$

Solution

$$A = 3\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + 2\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$