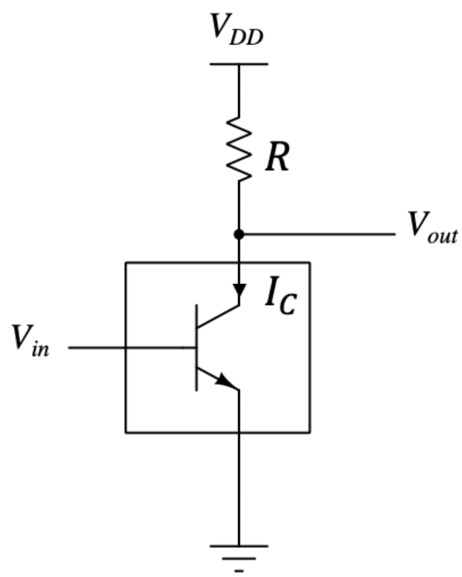
Solutions have been provided for all the problems in this Homework. You do not need to make a submission on gradescope. All the best for your midterm!

1 Linearizing for understanding amplification

Linearization isn't only important for control, robotics, machine learning, and optimization — it is one of the standard tools used across different areas, including thinking about circuits. The circuit below is a voltage amplifier, where the element inside the box is a bipolar junction transistor (BJT).



The bipolar transistor in the circuit can be modeled quite accurately as a nonlinear, voltage-controlled current source, where the collector current I_C is given by

$$I_C(V_{in}) = I_S e^{\frac{V_{in}}{V_{TH}}}, \tag{1}$$

where V_{TH} is the thermal voltage. We can assume $V_{TH} = 26$ mV at temperatures of 300K (close to room temperature). I_S is a constant whose exact value we are not giving you because we want you to find ways of eliminating it in favor of other quantities whenever possible.

With this amplifier, small variations in the input voltage V_{in} can turn into large variations in the output voltage V_{out} under the right conditions. We're going to investigate this amplification using linearization.

Let's consider the 2N3904 transistor, where the above expression for $I_C(V_{in})$ holds as long as $0.2\text{V} < V_{out} < 40\text{V}$, and $0.1\text{mA} < I_C < 10\text{mA}$.

(Note that the 2N3904 is a cheap transistor that people often use in personal projects. You can get them for 3 cents each if you buy in bulk.)

a) **Write a symbolic expression for V_{out} as a function of I_C.**

b) Now let's linearize I_C in the neighborhood of an input voltage V_{in}^* and a specific I_C^*. Assume that you have a found a particular pair of input voltage V_{in}^* and current I_C^* that satisfy the current equation (1).

We can look at nearby input voltages and see how much the current changes. We can write the linearized expression for the collector current around this point as:

$$I_C(V_{in}) = I_C(V_{in}^*) + \delta I_C \approx I_C^* + m(V_{in} - V_{in}^*) = I_C^* + m\ \delta V_{in} \tag{2}$$

where $\delta V_{in} = V_{in} - V_{in}^*$ is the change in input voltage and $\delta I_C = I_C - I_C^*$ is the change in collector current.

What is m here as a function of I_C^* and V_{TH}?

(If you take EE105, you will learn that this m is called the transconductance, which is usually written g_m, and is the single most important parameter in most analog circuit designs.)

(HINT: First just find m by taking the appropriate derivative and using the chain rule as needed. Then leverage the special properties of the exponential function to express it in terms of the desired quantities.)

c) We now have a linear relationship between small changes in current and voltage, $\delta I_C = m\ \delta V_{in}$ around a known solution (I_C^*, V_{in}^*). This is called a "bias point" in circuits terminology. (This is also why related things in neural nets are called bias terms — their function is to get the nonlinearity to behave the way we want it to.)

Going back to your equation from part (a), plug in your linearized equation for I_C. Define the appropriate V_{out}^* so that it makes sense to view $V_{out} = V_{out}^* + \delta V_{out}$ when we have $V_{in} = V_{in}^* + \delta V_{in}$, and **find the approximate linear relationship between δV_{out} and δV_{in}.**

The ratio $\frac{\delta V_{out}}{\delta V_{in}}$ is called the small-signal voltage gain of this amplifier around this bias point.

d) Assuming that $V_{DD} = 10V$, $R = 1k\Omega$, and $I_C^* = 1mA$ when $V_{in}^* = 0.65V$, **what is the small-signal voltage gain $\frac{\delta V_{out}}{\delta V_{in}}$, between the input and the output around this bias point?** (one or two digits of precision is plenty)

e) If $I_C^* = 9mA$ when $V_{in}^* = 0.7V$, **what is the small-signal voltage gain around this bias point?** (one or two digits is plenty)

This shows you how by appropriately biasing (choosing an operating point), we can adjust what our gain is for small signals. Although here, we just wanted to show you this as a simple application of linearization, these ideas are developed a lot further in 105, 140, and other courses to create things like op-amps and other analog information-processing systems.

2 Non-Linear Spring-Mass system

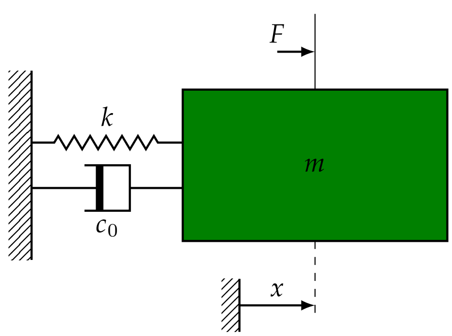


Figure 1: Schematic of a non-linear spring-mass system.

In this problem, we will analyze a non-linear spring-mass system shown in Figure 1. The dynamics of this system are given by

$$m\frac{d^2}{dt^2}x(t) = -kx(t) - c_0\frac{d}{dt}x(t) + F(t) \tag{3}$$

The spring constant in this case is not constant. Instead, it depends on the position x of the block. The spring constant is given by

$$k(x) = k_0\left(\frac{x^2}{a^2} - 1\right), \tag{4}$$

where a is a constant and $k_0 > 0$.

a) Write a state space model for this system using state variables $x_1 = x(t)$ and $x_2 = \frac{d}{dt}x(t)$. Is the system linear?

b) Find all the equilibrium points for this system in the absence of an external input, i.e., $F = 0$.

c) Linearize this system around each of the equilibrium points $\vec{x}^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$ and characterize whether those are stable or unstable. Can you comment on the stability using the spring constant $k(x)$.

d) Design a state-feedback controller for $F(t) = K\vec{x}(t)$ which stabilizes the linearized system at $x = 0, \frac{d}{dt}x = 0$.

e) We like the feedback controller that we have designed and decide to apply the feedback controller to the original non-linear system. We use $F(t) = k_1x_1(t) + k_2x_2(t)$. What are the equilibrium points for this new system?

3 Controllability in 2D

Consider the control of some two-dimensional linear discrete-time system

$$\vec{x}(k+1) = A\vec{x}(k) + Bu(k)$$

where A is a 2×2 real matrix and B is a 2×1 real vector.

a) Let $A = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ with $a, c, d \neq 0$, and $B = \begin{bmatrix} f \\ g \end{bmatrix}$. Find a B such that the system is controllable no matter what nonzero values a, c, d take on, and a B for which it is not controllable no matter what nonzero values are given for a, c, d. You can use the controllability rank test, but please explain your intuition as well.

b) Let $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ with $a, d \neq 0$. and $B = \begin{bmatrix} f \\ g \end{bmatrix}$ with $f, g \neq 0$. Is this system always controllable? If not, find configurations of nonzero a, d, f, g that make the system uncontrollable.

c) We want to see if controllability is preserved under changes of coordinates. To begin with, let $\vec{z}(k) = V^{-1}\vec{x}(k)$, please write out the system equation with respect to $\vec{z}$.

d) Now show that controllability is preserved under change of coordinates. (Hint: use the fact that rank(MA) = rank(A) for any invertible matrix M.)

4 System Identification

Consider a discrete-time system with unknown dynamics. Assume that starting from $x_0 = \begin{bmatrix} 1 & 2 \end{bmatrix}^\top$ we applied the following controls to the system, and observed the resulting states:

$$u_0 = 1, \quad u_1 = 2, \quad u_2 = 0, \quad u_3 = 1,$$
$$x_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, x_2 = \begin{bmatrix} 4 \\ 8 \end{bmatrix}, x_3 = \begin{bmatrix} 8 \\ 1 \end{bmatrix}, x_4 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

a) Set up a least-squares problem to recover $A \in \mathbb{R}^{2\times 2}$ and $B \in \mathbb{R}^{2\times 1}$ of a discrete-time model of this system

$$x_{k+1} = Ax_k + Bu_k$$

b) Could the estimates of A and B be uniqueley determined from less observations than those given? Explain. .

c) Now let's say that the matrix A was provided to us. Reformulate the problem to estimate the model parameters b_1 and b_2.

5 SVD

Find the singular value decomposition of the following matrix (leave all work in exact form, not decimal):

$$A = \begin{bmatrix} 1 & 0 & -\sqrt{3} \\ \sqrt{3} & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}$$

a) Find the eigenvalues of $A^\top A$ and order them from largest to smallest, $\lambda_1 > \lambda_2$.

b) Find orthonormal eigenvectors $\vec{v}_i$ of $A^\top A$ (all eigenvectors are mutually orthogonal and have unit length).

c) Find the singular values $\sigma_i = \sqrt{\lambda_i}$. Find the $\vec{u}_i$ vectors from:

$$A\vec{v}_i = \sigma_i \vec{u}_i$$

d) Write out A as a weighted sum of rank 1 matrices:

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^\top + \sigma_2 \vec{u}_2 \vec{v}_2^\top + \sigma_3 \vec{u}_3 \vec{v}_3^\top$$