1 Inductors: Introduction

So far in the class, we have learnt about capacitors. A capacitor typically consists of parallel metal plates separated by non-conducting material. As charge deposits on the metal plates, we have a resulting electric field and electric potential across the metal plates.

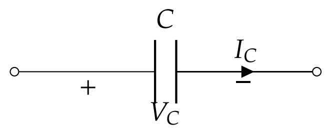


Figure 1: Capacitor element.

This charge-voltage relationship is what we have seen earlier.

$$Q_C \propto V_C \tag{1}$$

The proportionality constant for Equation 1 is a physical property of the capacitor and called its capacitance C. Specifically, we have $Q_C = CV_C$. Since current is defined as the rate of flow of charge, we can write

$$I_C = \frac{dQ_C}{dt} = C\frac{dV_C}{dt}. \tag{2}$$

An inductor converts electrical current into magnetic flux. We can construct an inductor by winding a wire into a coil and passing current through it.

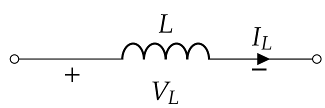


Figure 2: Inductor element.

The magnetic flux that develops as a result of the current flowing in a loop is proportional to the loop current.

$$\phi_L \propto I_L \tag{3}$$

Again the proportionality constant for Equation 3 is a physical property of the inductor and called its inductance L. Electric potential, or voltage, is related to magnetic flux as

$$V_L = \frac{d\phi_L}{dt} = L\frac{dI_L}{dt}. \tag{4}$$

From the point of view of current-voltage relationships, a capacitor and inductor are *duals* of each other.

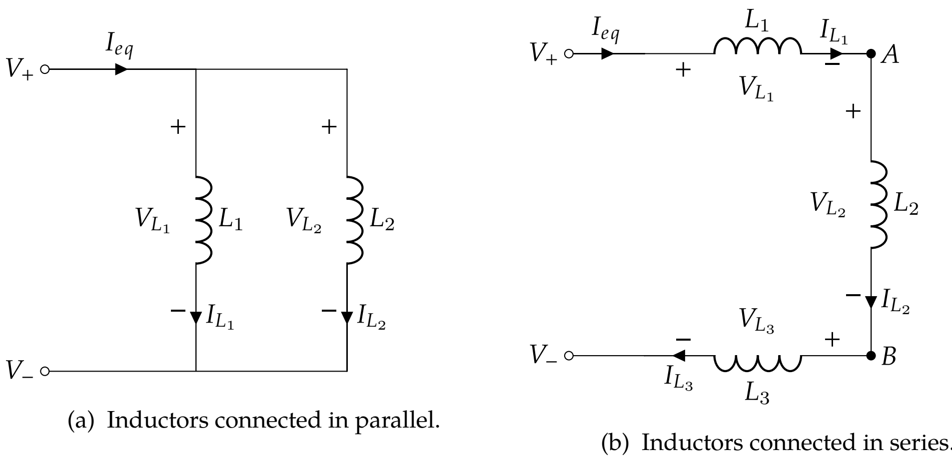


Figure 3: Finding equivalent inductance.

a) For the circuit shown in Figure 3a, find the equivalent inductance across the nodes V_+ and V_- for inductors connected in parallel.

Answer

For the circuit shown in Figure 3a, we know that

$$V_{L_1} = V_{L_2} = V_{eq} \tag{5}$$

Writing a KCL equation at the positive node of the inductors gives us

$$I_{eq} - I_{L_1} - I_{L_2} = 0 \tag{6}$$

Using the $I - V$ relationship in Equation 4, we have

$$V_{L_1} = L_1 \frac{d}{dt} I_{L_1},$$
$$V_{L_2} = L_2 \frac{d}{dt} I_{L_2}.$$

Combining the results from Equations 5 and 6 with the $I - V$ relations, we can write

$$\begin{aligned} I_{eq} &= I_{L_1} + I_{L_2} \\ \frac{d}{dt} I_{eq} &= \frac{d}{dt} I_{L_1} + \frac{d}{dt} I_{L_2} \\ &= \frac{V_{L_1}}{L_1} + \frac{V_{L_2}}{L_2} \\ &= V_{eq}\left(\frac{1}{L_1} + \frac{1}{L_2}\right) \end{aligned}$$

Therefore, inductors L_1 and L_2 when connected in parallel yield an equivalent inductance of

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} \tag{7}$$

b) For the circuit shown in Figure 3b, find the equivalent inductance across the nodes V_+ and V_- for inductors connected in series.

Answer

Since the inductors are connected in series we know that

$$I_{eq} = I_{L_1} = I_{L_2} = I_{L_3} \tag{8}$$

This can be verified by applying KCL and the different nodes in the circuit. For example, KCL at A tells us that $I_{L_1} = I_{L_2}$. For the voltages across the inductors, we can write

$$V_{eq} = V_+ - V_- = V_{L_1} + V_{L_2} + V_{L_3} \tag{9}$$

Substituting the $I - V$ relationship for an inductor into Equations 8 and 9, we have

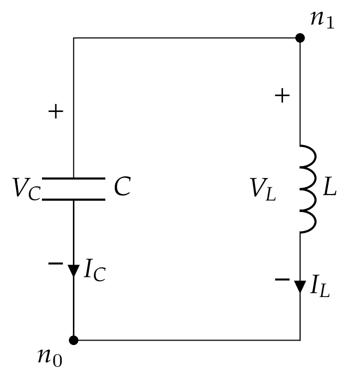
$$\begin{aligned} V_{eq} &= V_{L_1} + V_{L_2} + V_{L_3} \\ &= L_1 \frac{d}{dt} I_{L_1} + L_2 \frac{d}{dt} I_{L_2} + L_3 \frac{d}{dt} I_{L_3} \\ &= (L_1 + L_2 + L_3) \frac{d}{dt} I_{eq} \end{aligned}$$

Therefore, the equivalent inductance for the 3 inductors connected in series is

$$L_{eq} = L_1 + L_2 + L_3. \tag{10}$$

2 LC Tank: Oscillations

Consider the following circuit.



This is sometimes called an LC tank and we will look at its response in this problem. Assume at $t = 0$ we have $V_C(0) = V_S = 1$ V and $I_L(0) = 0$. For numerical calculations, use $C = 1uF$, $L = 10mH$.

a) **Write the system of differential equations in terms of state variables $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ that describes this circuit for $t \geq 0$.** Leave the system symbolic in terms of V_s, L, and C.

Answer

For this part, we need to find two differential equations, each including a derivative of one of the state variables.

First, let's consider the capacitor equation

$$I_C(t) = C\frac{d}{dt}V_C(t). \tag{11}$$

For the inductor, we have

$$V_L(t) = L\frac{d}{dt}I_L(t). \tag{12}$$

Since both the capacitor and the inductor are connected on node n_0 (as reference) and n_1 the voltage on them is shared

$$V_L(t) = V_C(t). \tag{13}$$

From KCL at node n_1,

$$I_C(t) = -I_L(t). \tag{14}$$

Rearranging the equations, we have

$$\frac{d}{dt}I_L(t) = \frac{1}{L}V_C(t) \tag{15}$$

$$\frac{d}{dt}V_C(t) = -\frac{1}{C}I_L(t) \tag{16}$$

b) In later problems, we will use diagonalization to solve for the inductor current $I_L(t)$ and the capacitor voltage $V_C(t)$. The diagonalization approach is more general and applicable to more complex circuits comprised of resistive elements. For this circuit, observe that the capacitor voltage and inductor current in this circuit obey

$$\frac{d^2}{dt^2}I_L(t) = \frac{-1}{LC}I_L(t) \tag{17}$$

$$\frac{d^2}{dt^2}V_C(t) = \frac{-1}{LC}V_C(t) \tag{18}$$

This expression describes a simple harmonic oscillator.

Verify that $V_C(t) = A\cos(\omega t + \theta)$, and $I_L(t) = B\sin(\omega t + \theta)$ is a solution to the system of differential equations originally derived in part (a). Determine the oscillation frequency ω, initial phase θ and scalar constants A and B.

Answer

We had previously derived the relationship between our variables $I_L(t)$ and $V_C(t)$ to be

$$\frac{d}{dt}I_L(t) = \frac{1}{L}V_C(t)$$
$$\frac{d}{dt}V_C(t) = -\frac{1}{C}I_L(t)$$

First, we use the simple harmonic oscillator formulation in Equation 17 to determine ω.

$$\begin{aligned}\frac{d^2}{dt^2}I_L(t) &= \frac{d^2}{dt^2}A\sin(\omega t + \theta)\\ &= -A\omega^2\sin(\omega t + \theta)\\ &= -\omega^2 I_L(t)\end{aligned}$$

Comparing the result that we have here with Equation 17, we see that $\omega^2 = \frac{1}{LC}$, or that $\omega = \frac{1}{\sqrt{LC}}$.

We can verify that following Equation 18, we get

$$\begin{aligned}\frac{d^2}{dt^2}V_C(t) &= \frac{d^2}{dt^2}B\cos(\omega t + \theta)\\ &= -B\omega^2\sin(\omega t + \theta)\\ &= -\omega^2 V_C(t)\end{aligned}$$

Evaluating the derivative of inductor current expression provided to us, we get

$$\frac{d}{dt}I_L(t) = \frac{d}{dt}B\sin(\omega t + \theta) = B\omega\cos(\omega t + \theta) \tag{19}$$

Combining Equation 19 with 15, we can write

$$\begin{aligned}B\omega\cos(\omega t + \theta) &= \frac{1}{L}A\cos(\omega t + \theta)\\ B\frac{1}{\sqrt{LC}} &= \frac{1}{L}A\\ B &= \sqrt{\frac{C}{L}}A\end{aligned}$$

Alternatively, evaluating the derivative of the capacitor voltage expression, we get

$$\frac{d}{dt}V_C(t) = \frac{d}{dt}A\cos(\omega t + \theta) = -A\omega\sin(\omega t + \theta) \tag{20}$$

Now, combining Equation 20 with 16, we get

$$-A\omega\sin(\omega t + \theta) = -\frac{1}{C}B\sin(\omega t + \theta)$$
$$A\sqrt{\frac{1}{LC}} = \frac{B}{C}$$
$$B = \sqrt{\frac{C}{L}}A$$

We can use our initial condition to determine the remaining quantities.

$$I_L(0) = B\sin(\theta) = 0 \tag{21}$$

This can be achieved with $B = 0$, or $\theta = 0$. Since $I_L(t)$ is not 0 at all times, we must have $B \neq 0$, and $\theta = 0$. Using this, we have

$$V_C(0) = A\cos(\theta) = A \tag{22}$$

Combining the results we have so far, we have

$$\omega = \sqrt{\frac{1}{LC}}$$
$$\theta = 0$$
$$A = V_s$$
$$B = \sqrt{\frac{C}{L}}V_s$$

c) For capacitance $C = 1\text{uF}$ and $L = 10\text{mH}$, **Sketch the capacitor voltage** $V_C(t)$ **and inductor current** $I_L(t)$. What is happening to the capacitor charge Q_C and inductor flux ϕ_L.

Answer

For the given values of L and C, we have $\omega = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{10^{-2}\cdot 10^{-6}}} = 10000$ rad/s.

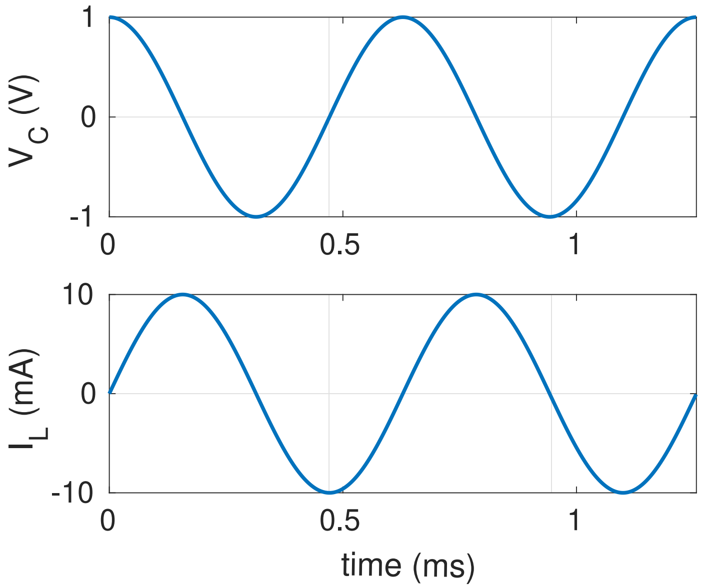


Figure 4: Capacitor voltage (V_C) and inductor current (I_L) over a period of 1.5ms

Figure 4 shows the capacitor voltage and inductor current over 2 complete cycles. Observe that when the capacitor voltage is at its maximum value (*i.e.* the capacitor is fully charged), the inductor current is 0. Similarly, when inductor current is at its maximum value (*i.e.* the inductor flux is at its peak, for example around $t = 0.5$ms.), the capacitor voltage is 0 (*i.e.* the capacitor is fully discharged).

d) The energy stored in the capacitor is given by $E_C = \frac{1}{2}CV_C^2$ and the energy stored in the inductor is given by $E_L = \frac{1}{2}LI_L^2$. **Evaluate how the total energy in the circuit is changing with time.**

Answer

Using the expressions derived above, we can evaluate the energy in the inductor to be

$$\begin{aligned} I_L(t) &= \sqrt{\frac{C}{L}} V_s \sin(\omega t) \\ E_L(t) &= \frac{1}{2} L I_L^2(t) \\ &= \frac{1}{2} L \cdot \frac{C}{L} V_s^2 \sin^2(\omega t) \\ &= \frac{1}{2} C V_s^2 \sin^2(\omega t) \end{aligned}$$

Similarly, the energy stored in the capacitor is

$$\begin{aligned} V_C(t) &= V_s \cos(\omega t) \\ E_C(t) &= \frac{1}{2} C V_C^2(t) \\ &= \frac{1}{2} C V_s^2 \cos^2(\omega t) \end{aligned}$$

The total energy in the circuit is

$$\begin{aligned} E_{total}(t) &= E_C(t) + E_L(t) \\ &= \frac{1}{2}CV_s\cos^2(\omega t) + \frac{1}{2}CV_s\sin^2(\omega t) \\ &= \frac{1}{2}CV_s^2 \end{aligned}$$

e) We will now use diagonalization to get to the same solution that we have analyzed so far. **Write the system of equations in vector/matrix form with the vector state variable $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.** This should be in the form $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ with a 2×2 matrix A.
Find the initial conditions $\vec{x}(0)$.

Answer

By inspection from the previous part, we have

$$\begin{bmatrix} \frac{d}{dt}x_1(t) \\ \frac{d}{dt}x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{23}$$

which is in the form $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$, with

$$A = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{bmatrix}. \tag{24}$$

We know that $V_C(0) = V_S$ and we know that $i_L(0) = 0$, thus $\vec{x}(0) = \begin{bmatrix} 0 \\ V_S \end{bmatrix}$

f) **Find the eigenvalues of the A matrix symbolically.**

Answer

To find the eigenvalues, we'll solve $\det(A - \lambda I) = 0$. In other words, we want to find λ such that

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & \frac{1}{L} \\ -\frac{1}{C} & -\lambda \end{bmatrix}\right) \tag{25}$$

$$= \lambda^2 + \frac{1}{LC} = 0. \tag{26}$$

Solving for λ we see

$$\lambda = \pm\frac{j}{\sqrt{LC}}. \tag{27}$$

g) Recall from our previous discussion that solutions for $x_i(t)$ will all be of the form

$$x_i(t) = \sum_k c_k e^{\lambda_k t}$$

where λ_k is an eigenvalue of our differential equation relation matrix A. Thus, we make the following guess for $\vec{x}(t)$:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_3 e^{\lambda_1 t} + c_4 e^{\lambda_2 t} \end{bmatrix}$$

where $c_1,\ c_2,\ c_3,\ c_4$ are all constants.

Evaluate $\vec{x}(t)$ and $\frac{d\vec{x}}{dt}(t)$ at time $t = 0$ in order to obtain four equations in four unknowns.

Answer

Evaluating $\vec{x}(0)$ and equating it to our result from part (b), we obtain

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_3 + c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ V_s \end{bmatrix}$$

which provides us two equations. Taking the derivative of our guess for $\vec{x}(t)$,

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 \lambda_1 e^{\lambda_1 t} + c_2 \lambda_2 e^{\lambda_2 t} \\ c_3 \lambda_1 e^{\lambda_1 t} + c_4 \lambda_2 e^{\lambda_2 t} \end{bmatrix}$$

Which becomes at $t = 0$

$$\frac{d}{dt}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} c_1 \lambda_1 + c_2 \lambda_2 \\ c_3 \lambda_1 + c_4 \lambda_2 \end{bmatrix}$$

We can equate this to $\frac{d}{dt}\vec{x}(0) = A\vec{x}(0)$ from part b:

$$\begin{bmatrix} c_1 \lambda_1 + c_2 \lambda_2 \\ c_3 \lambda_1 + c_4 \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} \frac{x_2(0)}{L} \\ -\frac{x_1(0)}{C} \end{bmatrix} = \begin{bmatrix} \frac{V_s}{L} \\ 0 \end{bmatrix}$$

thus providing our other two equations.

h) **Solve those equations for $c_1,\ c_2,\ c_3,\ c_4$ and plug them into your guess for $\vec{x}(t)$.** What do you notice about the solutions? Are they complex functions?

Answer

Solving the four equations, we obtain

$$c_1 = \sqrt{\frac{C}{L}}\frac{V_s}{2j} \qquad c_2 = -\sqrt{\frac{C}{L}}\frac{V_s}{2j}$$

$$c_3 = \frac{V_s}{2} \qquad c_4 = \frac{V_s}{2}$$

Plugging these into our guess,

$$\vec{x}(t) = \begin{bmatrix} \frac{V_s}{2j}\sqrt{\frac{C}{L}}\left(e^{\frac{j}{\sqrt{LC}}t} - e^{-\frac{j}{\sqrt{LC}}t}\right) \\ \frac{V_S}{2}\left(e^{\frac{j}{\sqrt{LC}}t} + e^{-\frac{j}{\sqrt{LC}}t}\right) \end{bmatrix}$$

$$= \begin{bmatrix} V_s\sqrt{\frac{C}{L}}\sin\left(\frac{t}{\sqrt{LC}}\right) \\ V_s\cos\left(\frac{t}{\sqrt{LC}}\right) \end{bmatrix} \qquad \text{Since} \qquad \begin{aligned} \sin(j\theta) &= \tfrac{1}{2j}(e^{j\theta} - e^{-j\theta}) \\ \cos(j\theta) &= \tfrac{1}{2}(e^{j\theta} + e^{-j\theta}) \end{aligned}$$

Using the provided values:

$$I_L(t) = I_{\max}\sin\left(\frac{t}{100\,\mu\text{s}}\right), \qquad I_{\max} = 10\,\text{mA}$$

$$V_C = V_S\cos\left(\frac{t}{100\,\mu\text{s}}\right), \qquad V_S = 1\,\text{V}$$