1 Circulant Matrices

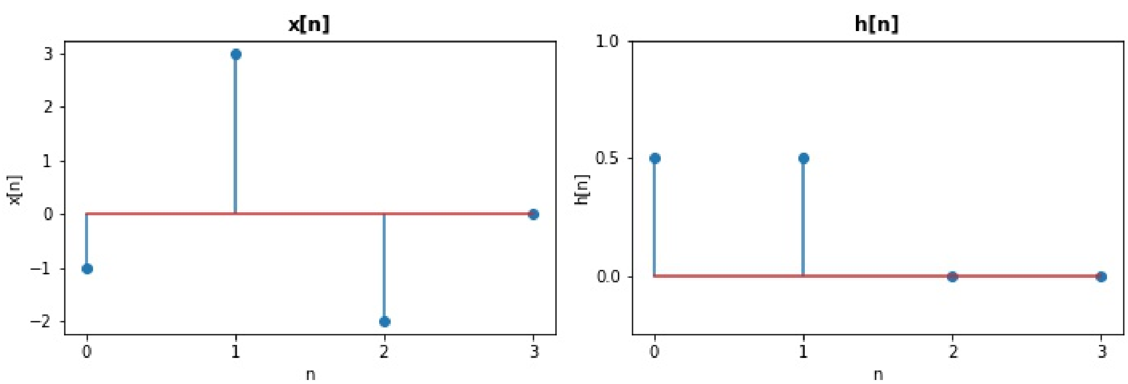
A square matrix C_h is circulant if each row vector is rotated one element to the right relative to the preceding row vector.

$$C_h = \begin{bmatrix} h_0 & h_{N-1} & \cdots & h_2 & h_1 \\ h_1 & h_0 & h_{N-1} & & h_2 \\ \vdots & h_1 & h_0 & \ddots & \vdots \\ h_{N-2} & \vdots & \ddots & \ddots & h_{N-1} \\ h_{N-1} & h_{N-2} & \cdots & h_1 & h_0 \end{bmatrix} \tag{1}$$

2 Circulant Matrices & Convolution

Consider the signal $x[n]$ of length 3 and an impulse response $h[n]$ of length 2. You may assume that they are zero everywhere else.

$$\vec{x} = \begin{bmatrix} -1 & 3 & -2 \end{bmatrix}^T \qquad \vec{h} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}^T \tag{2}$$



a) What is the convolution $y[n] = x[n] * h[n]$? Also what is the length of this output signal?

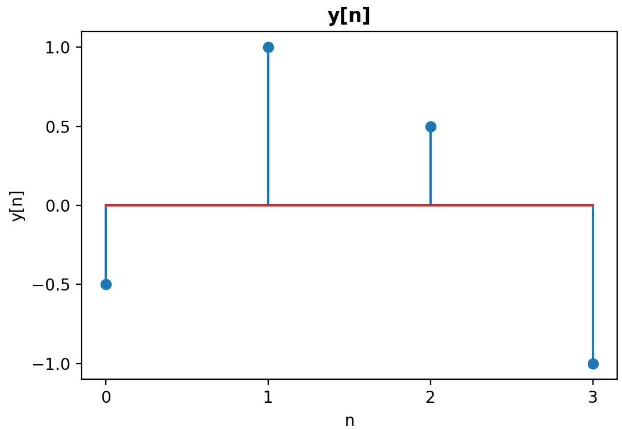
Answer

We can find the convolution by writing out the summation formula and the nonzero terms will remain

$$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=0}^{\infty} x[k]h[n-k] \tag{3}$$

$$\begin{aligned} y[0] &= x[0]h[0] = -0.5 \\ y[1] &= x[0]h[1] + x[1]h[0] = 1 \\ y[2] &= x[1]h[1] + x[2]h[0] = 0.5 \\ y[3] &= x[2]h[1] = -1 \end{aligned}$$

The length of the output is 4 and we show a visual of the result below



b) Now write each term of the output signal $y[n]$ as a sum using the convolution formula and set up a matrix equation $\vec{y} = A\vec{x}$. What is the size of this matrix?

Answer

$$\begin{aligned} y[0] &= x[0]h[0] = -0.5 \\ y[1] &= x[0]h[1] + x[1]h[0] = 1 \\ y[2] &= x[1]h[1] + x[2]h[0] = 0.5 \\ y[3] &= x[2]h[1] = -1 \end{aligned}$$

We can write this as the following matrix-vector equation

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 \\ h[1] & h[0] & 0 \\ 0 & h[1] & h[0] \\ 0 & 0 & h[1] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \end{bmatrix}$$

The matrix A is 4×3.

c) Add elements to the matrix A and zeros to the vector $\vec{x}$ to create a square matrix C_h that is circulant.

Answer

Note the first three rows of the matrix follow the pattern of a circulant matrix. Therefore, we will add one more cycle as columns and pad a zero to $\vec{x}$ to get

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 & h[1] \\ h[1] & h[0] & 0 & 0 \\ 0 & h[1] & h[0] & 0 \\ 0 & 0 & h[1] & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ 0 \end{bmatrix}$$

d) What is the importance behind this result? Compare the runtimes between convolution and the Fast Fourier Transform (FFT) which takes $O(N \log N)$ operations.

Answer

We can use circulant matrices to compute convolutions efficiently! The reason why is out of scope, but it is because circulant matrices are orthogonally diagonalized by the DFT basis. If $c[n]$ is a length N signal representing the first column of C, then $C = U\Lambda U^*$ where U is the DFT basis and Λ is a diagonal matrix with the DFT coefficients $C[k]$ of $c[n]$ as its entries.

If $x[n]$ and $h[n]$ are signals of length N, then convolution as matrix-vector multiplication takes $O(N^2)$ operations. On the other hand, the DFT can be computed using $O(N \log N)$ operations through the Fast Fourier Transform. This means we can find the output of any LTI system efficiently using the FFT.

As a reference for $N = 10^6$, convolution will take approximately 1 trillion operations while the FFT takes approximately 6 million operations. When ran in numpy for $N = 10^6$, convolution took 20 minutes while the FFT took 0.25 seconds.

3 Complex Inner Product

For the complex vector space $\mathbb{C}^n$, we can no longer use our conventional real dot product as a valid inner product for $\mathbb{C}^n$. This is because the real dot product is no longer positive-definite for complex vectors. For example, let $\vec{v} = \begin{bmatrix} j \\ j \end{bmatrix}$. Then, $\vec{v} \cdot \vec{v} = j^2 + j^2 = -2 < 0$.

Therefore, for two vectors $\vec{u}, \vec{v} \in \mathbb{C}^n$, we define the complex inner product to be:

$$\langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^{n} \overline{u}_i v_i = \overline{\vec{u}}^T \vec{v} = \vec{u}^* \vec{v}$$

where $\overline{\cdot}$ denotes the complex conjugate and $\vec{\cdot}^*$ denotes the complex conjugate transpose.

The complex inner product satisfies the following properties:

- Conjugate Symmetry: $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$
- Scaling: $\langle \vec{u}, c\vec{v} \rangle = c\langle \vec{u}, \vec{v} \rangle$ and $\langle c\vec{u}, \vec{v} \rangle = \overline{c}\langle \vec{u}, \vec{v} \rangle$
- Additivity: $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ and $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$
- Positive-definite: $\langle \vec{u}, \vec{u} \rangle \geq 0$ with $\langle \vec{u}, \vec{u} \rangle = 0$ if and only if $\vec{u} = \vec{0}$

Recall that the complex conjugate of a complex number $z = a + jb = re^{j\theta}$ is equal to $\overline{z} = a - jb = re^{-j\theta}$. The conjugate transpose of a vector $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is $\vec{v}^* = \begin{bmatrix} \overline{v}_1 & \cdots & \overline{v}_n \end{bmatrix}$.

Adjoint of a Matrix

The **adjoint** or **conjugate-transpose** of a matrix A is the matrix A^* such that $A^*_{ij} = \overline{A_{ji}}$. From the complex inner product, one can show that

$$\langle A\vec{x}, \vec{y} \rangle = \langle \vec{x}, A^*\vec{y} \rangle \tag{4}$$

A matrix is **self-adjoint** or **Hermitian** if $A = A^*$. Self-adjoint matrices are the complex extension of real-symmetric matrices. There is an equivalent version of the **Spectral Theorem** for such self-adjoint matrices.

Unitary Matrices

A **unitary** matrix is a square matrix whose columns are orthonormal with respect to the complex inner product.

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{bmatrix}, \qquad \vec{u}_i^* \vec{u}_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

Note that $U^*U = UU^* = I$, so the inverse of a unitary matrix is its conjugate transpose $U^{-1} = U^*$.

Since the columns of a unitary matrix are orthonormal vectors, we can interpret these matrices as "rotation" and "reflection" matrices of the standard axes. This also implies that $\|U\vec{v}\| = \|\vec{v}\|$ for any vector $\vec{v}$.

4 Everything is Complex

In this question, we explore the similarities and differences between real and complex vector spaces.

a) Suppose that we are given a matrix U that has orthornomal columns.

$$U = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{bmatrix} \tag{5}$$

We would like to change coordinates to express $\vec{x}$ as a linear combination of these orthonormal basis vectors. Find scalars α_i in terms of $\vec{x}$ and $\vec{u}_i$ so that

$$\vec{x} = \alpha_1 \vec{u}_1 + \ldots + \alpha_n \vec{u}_n \tag{6}$$

Answer

This question is HW 9 Q4b but now considers the case where U can have complex entries. However, you might notice that if we express everything through inner products, the solution will be identical.

- **Method 1: Change of Coordinates**
  Since we would like to express $\vec{x}$ using basis vectors from the matrix U, this reduces to finding scalars α_i of the vector $\vec{z}$.

$$\vec{x} = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{bmatrix} \underbrace{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}}_{\vec{z}} \tag{7}$$

  Therefore, we can multiply by U^* since $U^*U = I$.

$$U^*\vec{x} = \underbrace{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}}_{\vec{z}} \tag{8}$$

  We conclude by saying that $\alpha_i = \vec{u}_i^*\vec{x} = \langle \vec{u}_i, \vec{x} \rangle$.
- **Method 2: Orthogonality**
  To solve for scalars α_i, we can take the inner product of $\vec{x}$ with an arbitrary $\vec{u}_i$.

$$\langle \vec{x}, \vec{u}_i \rangle = \alpha_i \langle \vec{u}_i, \vec{u}_i \rangle \tag{9}$$

  Therefore, since $\alpha_i = \langle \vec{x}, \vec{u}_i \rangle = \vec{x}^*\vec{u}_i$, we can write out $\vec{x}$ as

$$\vec{x} = \sum_{i=1}^{n} (\vec{x}^*\vec{u}_i)\vec{u}_i \tag{10}$$

  It's important to note that the complex inner product is not symmetric so $\vec{x}^*\vec{u}_i \neq \vec{u}_i^*\vec{x}$.

b) Show that the eigenvalues λ of a unitary matrix U must always have magnitude 1.

Answer

Let $\vec{v}$ be an eigenvector of U with eigenvalue λ.

$$U\vec{v} = \lambda\vec{v} \tag{11}$$

If we look at the squared-norm of $U\vec{v}$, we see that

$$\|U\vec{v}\|^2 = \langle U\vec{v}, U\vec{v}\rangle = (U\vec{v})^*(U\vec{v}) = \vec{v}^*U^*U\vec{v} = \vec{v}^*\vec{v} = \|\vec{v}\|^2 \tag{12}$$

Lookign at the square-norm of the right-hand side $\lambda\vec{v}$ we see that

$$\|\lambda\vec{v}\|^2 = \langle \lambda\vec{v}, \lambda\vec{v}\rangle = \overline{\lambda}\langle \lambda\vec{v}, \vec{v}\rangle = \overline{\lambda}\lambda\langle \vec{v}, \vec{v}\rangle = |\lambda|^2\|\vec{v}\|^2 \tag{13}$$

Therefore, we see that $|\lambda| = 1$ meaning λ must be a complex number with magnitude 1.

c) Let A be an $m \times n$ matrix with complex entries. Show that A^*A must have non-negative eigenvalues. *Hint: Think back to what we did for the real case. Consider the quantity $\|A\vec{v}\|^2$ where $\vec{v}$ is an eigenvector of A^*A.*

Answer

Again, the proof should be near identical to the real case, but we are using the complex inner product now. If we consider the quantity $\|A\vec{v}\|^2$ where $\vec{v}$ is an eigenvector of A^*A, we see that

$$\|A\vec{v}\|^2 = \langle A\vec{v}, A\vec{v}\rangle = (A\vec{v})^*(A\vec{v}) = \vec{v}^*A^*A\vec{v} = \lambda\vec{v}^*\vec{v} = \lambda\|\vec{v}\|^2 \implies \lambda = \frac{\|A\vec{v}\|^2}{\|\vec{v}\|^2} \tag{14}$$

Since norms are non-negative and $\vec{v}$ is a eigenvector that cannot be $\vec{0}$, λ must be non-negative.